Categorical Duality Theory: With Applications to Domains, Convexity, and the Distribution Monad

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Abstract

Utilising and expanding concepts from categorical topology and algebra, we contrive a moderately general theory of dualities between algebraic, point-free spaces and set-theoretical, point-set spaces, which encompasses infinitary Stone dualities, such as the well-known duality between frames (aka. locales) and topological spaces, and a duality between σ-complete Boolean algebras and measurable spaces, as well as the classic finitary Stone, Gelfand, and Pontryagin dualities. Among different applications of our theory, we focus upon domain-convexity duality in particular: from the theory we derive a duality between Scott's continuous lattices and convexity spaces, and exploit the resulting insights to identify intrinsically the dual equivalence part of a dual adjunction for algebras of the distribution monad; the dual adjunction was uncovered by Bart Jacobs, but with no characterisation of the induced equivalence, which we do give here. In the Appendix, we place categorical duality in a wider context, and elucidate philosophical underpinnings of duality.

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1 Introduction

There are two conceptions of space: one comes from the ontic idea that the ultimate constituents of space are points with no extension; the other does not presuppose the concept of points in the first place, and starts with an epistemically more certain concept such as regions or observable properties. For instance, a topological space is an incarnation of the former idea of space, and a frame (or locale) is an embodiment of the latter. Duality often exists between point-free and point-set conceptions of space (to put it differently, between epistemology and ontology of space; see the Appendix as well), as exemplified by the well-known duality between frames and topological spaces (see, e.g., Johnstone [12]).

The most general duality theorem is this: any category $\mathbf{C}$ is dually equivalent to the opposite category $\mathbf{C}^{\mathrm{op}}$. It, of course, makes no substantial sense; however, note that it prescribes a generic form of duality in a non-obvious way (we say "non-obvious" because there may be different conceptions of a generic form of duality, some of which may not be based upon category theory at all). In this paper, we attempt to avoid such triviality by focusing upon a more specific context: we aim at developing a moderately general theory

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of dualities between point-free and point-set spaces, whilst having in mind applications to domain-convexity duality, where domains are seen as point-free convex structures.

Our general theory of dualities between point-free and point-set spaces builds upon the celebrated idea of a duality induced by a Janusian (aka. schizophrenic) object: "a potential duality arises when a single object lives in two different categories" (Lawvere's words quoted in Barr et al. [3]). Note that in this paper we mean by dualities dual adjunctions in general; dual equivalences are understood as special cases. There are different theories of dualities based upon the same idea (see, e.g., [3, 5, 12, 17]); some of them use universal algebra, whilst others are categorical. Our theory is in between universal algebra and category theory (although we use categorical terminology, nevertheless, everything can be recasted in terms of universal algebra and of general point-set spaces introduced in Maruyama [15]). More detailed comparison with related work is given below.

Our duality theory allows us to derive a number of concrete dualities, including infinitary Stone dualities, such as the aforementioned duality in point-free topology, and a certain duality between σ-complete Boolean algebras and measurable spaces, as well as the classic finitary Stone, Gelfand, and Pontryagin dualities (since the Pontryagin duality is a self-duality, how to treat it is slightly different from how to do the others as noted below).

In the present paper we focus, *inter alia*, upon dualities between point-free and point-set convex structures. On the one hand, we consider Scott's continuous lattices to represent point-free convex structures for certain reasons explained later, in Subsection 3.1. On the other hand, there are two kinds of point-set convex structures: i.e., convexity spaces (see van de Vel [18] or Coppel [6]; the definition is given in Preliminaries in Section 2) and algebras of the distribution monad (aka. barycentric algebras; see Fritz [8]).

Our general theory tells us that there is a duality between continuous lattices and convexity spaces. In contrast, Jacobs [11] shows a dual adjunction between preframes and algebras of the distribution monad, which can be reformulated as a dual adjunction between continuous lattices and algebras of the distribution monad. Although Jacobs [11] left it open to identify intrinsically the induced dual equivalence, in this paper, we give an intrinsic characterisation of the dual equivalence part of the dual adjunction for algebras of the distribution monad.

Technical Summary. In our duality theory, we mainly rely upon concrete category theory as in Adámek et al. [1], especially concepts from categorical topology (see also [2, 4]) and categorical algebra (in particular the theory of monads).

We start with a category $\mathbf{C}$ monadic over $\mathbf{Set}$ (which is equivalent to possibly infinitary varieties in terms of universal algebra) and with a topological category $\mathbf{D}$ of certain type, and then assume that there is a Janusian object Ω living in both $\mathbf{C}$ and $\mathbf{D}$. In passing, we introduce the new concept of classical topological axiom, and use it to identify a certain class of those full subcategories of a functor-structured category that represent categories of point-set spaces. Under the assumption of what we call the harmony condition, which basically means that algebraic operations are continuous in a suitable sense, we finally show that there is a dual adjunction between $\mathbf{C}$ and $\mathbf{D}$, given by homming into Ω.

The dual adjunction formally restricts to a dual equivalence via the standard method of taking those objects of $\mathbf{C}$ and $\mathbf{D}$ that are fixed under the unit and counit of the adjunction; to put it intuitively, the objects whose double duals are isomorphic to themselves. At the same time, however, it is often highly non-trivial to identify intrinsically the dual equivalence part of a dual adjunction in a concrete situation, as Porst-Tholen [17] remark, "This can be a very hard problem, and this is where categorical guidance comes to an end."[1]

[1] To exemplify what is meant here, consider the dual adjunction between frames and spaces, which

Using specialised, context-dependent methods, rather than the generic one mentioned above, we give intrinsic characterisations of dual equivalences induced by the dual adjunction for convexity spaces, and by the dual adjunction for algebras of the distribution monad. The concept of polytopes plays a crucial role in the characterisations, and in understanding how semilattices involve convex structures.

Comparison with Related Work. Our general theory of dualities may be compared with other duality theories as follows. Clark-Davey's theory of natural dualities [5] is based upon the same idea of a duality induced by a Janusian object. However, our theory is more comprehensive than natural duality theory, in that whilst natural duality theory specialises in dualities for finitary algebras, our theory is intended to encompass infinitary algebras as well (e.g., frames, σ-complete Boolean algebras, and continuous lattices). Our theory thus encompasses both finitary and infinitary Stone-type dualities.

Johnstone's general concrete duality [12, VI.4] and Porst-Tholen's natural dual adjunction [17] are more akin to ours.[2] A crucial difference is, however, that we stick to the practice of Stone-type dualities as far as possible. In their theories, there is no concrete concern with how to equip the "spectrum" of an "algebra" with a "topology" or how to equip the (collection of) "functions" on a "space" with an "algebraic" structure.

We consider that the processes of algebraisation and topologisation are essential in the practice of Stone-type dualities. In particular, algebraisation and topologisation are strikingly different processes in the practice; in spite of this, the two processes are treated in their theories as being in parallel and symmetric at a level of abstraction, which looks like an excessive abstraction from our perspective of the practice of duality.

We put a strong emphasis on the asymmetry between the two processes of algebraisation and topologisation in the practice of Stone-type dualities, and thus aim at simulating the processes within our theory, thereby representing the practice of duality in an adequate manner. In order to achieve this goal, our theory cannot and should not be so general as to symmetrise the asymmetry; this is the reason why we call our theory "moderately" general.

In comparison with Maruyama [15], which discusses a theory of T_1-type dualities based upon Chu spaces and a generic concept of closure conditions, the present paper aims at a theory of sober-type dualities; an example of duality of T_1-type is a (not very well known) duality between T_1 spaces and coatomistic frames (a subtlety is frame morphisms must be "maximal" to dualise continuous maps; see [15]). Sober-type dualities are based upon prime spectra, whilst T_1-type dualities are based upon maximal spectra. Affine varieties (except singletons) in $\mathbb{C}^n$ with Zariski topologies are non-sober T_1 spaces; they are homeomorphic to the maximal spectra of their coordinate rings. Both the former and the latter theories can be applied to different sorts of spaces, yielding T_1-type and sober-type dualities respectively.

Jacobs [11] and Maruyama [14] independently unveiled (different) dualities for convexity (convexity algebras in [14] are replaced in this paper by continuous lattices), and the present paper is meant to elucidate a precise link between them, which remained unclear so far. The two dualities turn out to be essentially the same in spite of their rather different outlooks.

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restricts to a dual equivalence between the frames and spaces whose double duals are isomorphic to themselves; this is trivial. Nevertheless, it is not trivial at all to notice that those frames are exactly the frames with enough points (i.e., spatial frames), and those spaces are precisely the spaces in which any non-empty irreducible closet set is the closure of a unique point (i.e., sober spaces).

[2] Johnstone's dual adjunction (Lemma VI.4.2) seems to be not very rigorous because he dares to say "we choose not to involve ourselves in giving a precise meaning to the word 'commute' in the last sentence" (p. 254), and the dual adjunction result actually relies upon the assumption of that commutativity. In this paper, we precisely formulate the concept of commutativity as what we call the harmony condition.

2 General Duality Theory

After preliminaries, we first review categorical topology, and then get into a general theory of dualities based upon the concepts of monad, functor-(co)structured category, and topological (co)axiom. Among other things, we introduce the new concept of classical topological axiom with the aim of treating different sorts of point-set spaces in a unified way.

Preliminaries. For a category $\mathbf{C}$ and a faithful functor $U : \mathbf{C} \to \mathbf{Set}$, a tuple $(\mathbf{C}, U)$ is called a concrete category, where $\mathbf{Set}$ denotes the category of sets and functions. U is called the underlying functor of the concrete category. For simplicity, we often omit and make implicit the functor U of a concrete category $(\mathbf{C}, U)$. We can also define the notion of a concrete category over a general category. For a category $\mathbf{C}$ and a faithful functor $U : \mathbf{C} \to \mathbf{D}$, $(\mathbf{C}, U)$ is called a concrete category over $\mathbf{D}$. A concrete category (over $\mathbf{Set}$) in this paper is called a construct in [1]. $\mathbf{Top}$ denotes the category of topological spaces and continuous functions. $\mathbf{Conv}$ denotes the category of convexity spaces and convexity preserving maps, where a convexity space is a tuple $(X, \mathcal{C})$ such that X is a set and $\mathcal{C}$ is a subset of the powerset of X that is closed under directed unions and arbitrary intersections; a convexity-preserving map is such that the inverse image of any convex set under it is again convex (see van de Vel [18] and Coppel [6], which develop substantial amount of convex geometry based upon this general concept of convexity space). $\mathbf{Meas}$ denotes the category of measurable spaces and measurable functions. $\mathbf{Frm}$ denotes the category of frames and their homomorphisms. $\mathbf{ContLat}$ denotes the category of continuous lattices and their homomorphisms (i.e., maps preserving directed joins and arbitrary meets). $\mathbf{BA}_\sigma$ denotes the category of σ-complete Boolean algebras with σ-distributivity and their homomorphisms, where σ-distributivity means that countable joins distribute over countable meets. $\mathcal{Q} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$ denotes the contravariant powerset functor.

2.1 A Categorical Conception of Point-Set Spaces

Here we introduce a general concept of space that encompasses topological spaces, convexity spaces, and measurable spaces. Our duality theory shall be developed based upon that concept of (generalised) space. For the fundamentals of functor-(co)structured category and topological (co)axiom, we refer to Adámek et al. [1]. We first review the notion of functor-structured category. Let $(\mathbf{C}, U : \mathbf{C} \to \mathbf{Set})$ be a concrete category in the following.

▶ **Definition 2.1** ([1]). A category $\mathbf{Spa}(U)$ is defined as follows.

1. An object of $\mathbf{Spa}(U)$ is a tuple $(C, \mathcal{O})$ where $C \in \mathbf{C}$ and $\mathcal{O} \subset U(C)$.
2. An arrow of $\mathbf{Spa}(U)$ from $(C, \mathcal{O})$ to $(C', \mathcal{O}')$ is an arrow $f : C \to C'$ of $\mathbf{C}$ such that $U(f)[\mathcal{O}] \subset \mathcal{O}'$.

A category of the form $\mathbf{Spa}(U)$ is called a functor-structured category. A category of the form $(\mathbf{Spa}(U))^{\mathrm{op}}$ is called a functor-costructured category.

We consider $\mathbf{Spa}(U)$ as a concrete category equipped with a faithful functor $U \circ F : \mathbf{Spa}(U) \to \mathbf{Set}$ where $F : \mathbf{Spa}(U) \to \mathbf{C}$ is the forgetful functor that maps $(C, \mathcal{O})$ to C.

Then we can show the following (for the definition of topological category, see [1]; although there are different notions of a topological category, we follow the terminology of [1]).

▶ Proposition 2.2 ([1]). Both a functor-structured category $\mathbf{Spa}(U)$ and a functor-costructured category $(\mathbf{Spa}(U))^{\mathrm{op}}$ are topological.

The concept of topological (co)axiom is defined as follows.

▶ **Definition 2.3** ([1]). A topological axiom in $(\mathbf{C}, U)$ is defined as an arrow $p : C \to C'$ of $\mathbf{C}$ such that

1. $U(C) = U(C')$;
2. $U(p) : U(C) \to U(C)$ is the identity morphism on $U(C)$.

An object C of $\mathbf{C}$ satisfies a topological axiom $p : D \to D'$ in $(\mathbf{C}, U)$ iff, for any arrow $f : D \to C$ of $\mathbf{C}$, there is an arrow $f' : D' \to C$ of $\mathbf{C}$ such that $U(f) = U(f')$.

A topological coaxiom is defined as a topological axiom with the following concept of satisfaction. An object C of $\mathbf{C}$ satisfies a topological coaxiom $p : D' \to D$ in $(\mathbf{C}, U)$ iff, for any arrow $f : C \to D$ of $\mathbf{C}$, there is an arrow $f : C \to D'$ of $\mathbf{C}$ such that $U(f) = U(f')$.

Topological axioms and coaxioms are the same, but the corresponding notions of satisfaction are dual to each other. For examples of topological (co)axiom, we refer to [1].

▶ **Definition 2.4** ([1])**.** Let X be a class of topological (co)axioms in a concrete category $\mathbf{C}$. A full subcategory $\mathbf{D}$ of $\mathbf{C}$ is definable by X in $\mathbf{C}$ iff the objects of $\mathbf{D}$ coincide with those objects of $\mathbf{C}$ that satisfy all the topological (co)axioms in X.

As in the following proposition, we can show a topological analogue of the Birkhoff theorem in universal algebra (for more details, see Theorem 22.3 and Corollary 22.4 in [1]).

▶ Proposition 2.5 ([1]). Let $\mathbf{C}$ be a concrete category. The following are equivalent:
1. $\mathbf{C}$ is fibre-small and topological;
2. $\mathbf{C}$ is isomorphic to a subcategory of a functor-structured category that is definable by a class of topological axioms in the functor-structured category.
3. $\mathbf{C}$ can be embedded into a functor-structured category as a full subcategory that is closed under the formation of products, initial subobjects, and indiscrete objects.

Now we introduce the new concept of classical topological (co)axiom, which shall play a crucial role in formulating our dual adjunction theorem.

▶ **Definition 2.6.** A classical topological axiom in $\mathbf{Spa}(U)$ is defined as a topological axiom $p : (C, \mathcal{O}) \to (C', \mathcal{O}')$ in $\mathbf{Spa}(U)$ such that
- Any element of $\mathcal{O}' \setminus \mathcal{O}$ can be expressed as a (possibly infinitary) Boolean combination of elements of $\mathcal{O}$.

A classical topological coaxiom in $(\mathbf{Spa}(U))^{\mathrm{op}}$ is defined as a topological coaxiom $p : (C, \mathcal{O}) \to (C', \mathcal{O}')$ in $(\mathbf{Spa}(U))^{\mathrm{op}}$ such that
- Any element of $\mathcal{O} \setminus \mathcal{O}'$ can be expressed as a (possibly infinitary) Boolean combination of elements of $\mathcal{O}'$.

Let $\mathcal{Q} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$ denote the contravariant powerset functor. Any of the category $\mathbf{Top}$ of topological spaces, the category $\mathbf{Conv}$ of convexity spaces, and the category $\mathbf{Meas}$ of measurable spaces is a full subcategory of $(\mathbf{Spa}(\mathcal{Q}))^{\mathrm{op}}$ that is definable by a class of classical topological coaxioms as in the following proposition, which can be shown just by spelling out the definitions involved.

▶ Proposition 2.7. $\mathbf{Top}$ is definable by the following class of classical topological coaxioms in $(\mathbf{Spa}(\mathcal{Q}))^{\mathrm{op}}$:

$$\begin{aligned} 1_S : (S, \{\emptyset, S\}) &\to (S, \emptyset) \\ 1_S : (S, \{X, Y, X \cap Y\}) &\to (S, \{X, Y\}) \\ 1_S : (S, \mathcal{O} \cup \{\bigcup \mathcal{O}\}) &\to (S, \mathcal{O}) \end{aligned}$$

for all sets S, all subsets X, Y of S and all subsets $\mathcal{O}$ of the powerset of S.

$\mathbf{Conv}$ is definable by the following class of classical topological coaxioms in $(\mathbf{Spa}(\mathcal{Q}))^{\mathrm{op}}$: (i) $1_S : (S, \{\emptyset, S\}) \to (S, \emptyset)$; (ii) $1_S : (S, \mathcal{C} \cup \{\bigcap \mathcal{C}\}) \to (S, \mathcal{C})$; (iii) $1_S : (S, \mathcal{C}' \cup \{\bigcup \mathcal{C}'\}) \to (S, \mathcal{C}')$

for all sets S, all subsets $\mathcal{C}$ of the powerset of S, and all those subsets $\mathcal{C}'$ of the powerset of S that are directed with respect to inclusion.

Meas is definable by the following class of classical topological coaxioms in $(\mathbf{Spa}(\mathcal{Q}))^{\mathrm{op}}$: (i) $1_S : (S, \{\emptyset, S\}) \to (S, \emptyset)$; (ii) $1_S : (S, \{X, X^c\}) \to (S, \{X\})$; (iii) $1_S : (S, \mathcal{B} \cup \{\bigcup \mathcal{B}\}) \to (S, \mathcal{B})$ for all sets S, all subsets X of S, and all those subsets $\mathcal{B}$ of the powerset of S that are of cardinality $\leq \omega$.

In order to develop a general duality theory, thus, we shall focus upon a full subcategory **Spa** of $(\mathbf{Spa}(\mathcal{Q}))^{\mathrm{op}}$ that is definable by a class of classical topological coaxioms in $(\mathbf{Spa}(\mathcal{Q}))^{\mathrm{op}}$.

We call $(S, \mathcal{O}) \in \mathbf{Spa}$ a generalised space and $\mathcal{O}$ a generalised topology.

Given a subset $\mathcal{P}$ of the powerset of a set S, we can generate a topology on S from $\mathcal{P}$, which is the weakest topology containing $\mathcal{P}$. We can also do the same thing in the case of generalised topology.

▶ Proposition 2.8. For a set S, let $\mathcal{P}$ be a subset of the powerset of S. Then, there is a weakest generalised topology on S containing $\mathcal{P}$ in **Spa**, i.e., there is $(S, \mathcal{O}) \in \mathbf{Spa}$ such that, if $\mathcal{P} \subset \mathcal{O}'$ for $(S, \mathcal{O}') \in \mathbf{Spa}$, then $\mathcal{O} \subset \mathcal{O}'$. We then say that $\mathcal{O}$ is generated in **Spa** by $\mathcal{P}$.

Proof. Define $\mathcal{O} = \bigcap\{\mathcal{X} \ ; \ \mathcal{P} \subset \mathcal{X} \text{ and } (S, \mathcal{X}) \in \mathbf{Spa}\}$. It is sufficient to show that $\mathcal{O}$ is a generalised topology on S in **Spa**, i.e., $(S, \mathcal{O})$ satisfies the class of topological coaxioms that define **Spa**. Assume that $p : (X, \mathcal{B}') \to (X, \mathcal{B})$ is one of such coaxioms and that $f : (S, \mathcal{O}) \to (X, \mathcal{B})$ is an arrow in $(\mathbf{Spa}(\mathcal{Q}))^{\mathrm{op}}$. For $B \in \mathcal{B}' \setminus \mathcal{B}$, we have $f^{-1}(B) \in \mathcal{X}$ for any $\mathcal{X}$ with $\mathcal{P} \subset \mathcal{X}$ and $(S, \mathcal{X}) \in \mathbf{Spa}$, which implies that $f^{-1}(B) \in \mathcal{O}$. ◀

2.2 Dual Adjunction via Harmony Condition

Throughout this subsection, let

- **Alg** denote a full subcategory of the Eilenberg-Moore category of a monad T on **Set**;
- **Spa** denote a full subcategory of $(\mathbf{Spa}(\mathcal{Q}))^{\mathrm{op}}$ that is definable by a class of classical topological coaxioms in $(\mathbf{Spa}(\mathcal{Q}))^{\mathrm{op}}$.

We aim at establishing a dual adjunction between **Alg** and **Spa** under the two assumptions:

- there is an object Ω living in both **Alg** and **Spa**, i.e., there is $\Omega \in \mathbf{Set}$ both with a structure map $h_\Omega : T(\Omega) \to \Omega$ such that $(\Omega, h_\Omega) \in \mathbf{Alg}$ and with a generalised topology $\mathcal{O}_\Omega \subset \mathcal{Q}(\Omega)$ such that $(\Omega, \mathcal{O}_\Omega) \in \mathbf{Spa}$;
- $(\mathbf{Alg}, \mathbf{Spa}, \Omega)$ satisfies the harmony condition in Definition 2.9 below.

Ω is intuitively a set of truth values, and shall work as a so-called dualising object (we do not use the term "schizophrenic", since it has a different technical meaning in a certain context). We simply write Ω instead of (Ω, h_Ω) or $(\Omega, \mathcal{O}_\Omega)$ when there is no confusion.

The harmony condition intuitively means that the algebraic structure of **Alg** and the geometric structure of **Spa** are in harmony via Ω. The precise definition is given below.

▶ **Definition 2.9.** $(\mathbf{Alg}, \mathbf{Spa}, \Omega)$ is said to satisfy the harmony condition iff, for each $S \in \mathbf{Spa}$,

$$(\ \mathrm{Hom}_{\mathbf{Spa}}(S, \Omega),\ h_S : T(\mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)) \to \mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)\ )$$

is an object in **Alg** such that, for any $s \in S$ (let p_s be the corresponding projection from $\mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)$ to Ω), the following diagram commutes:

$$\begin{array}{ccc}
T(\mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)) & \xrightarrow[h_S]{} & \mathrm{Hom}_{\mathbf{Spa}}(S, \Omega) \\
\downarrow{\scriptstyle T(p_s)} & & \downarrow{\scriptstyle p_s} \\
T(\Omega) & \xrightarrow[h_\Omega]{} & \Omega
\end{array} .$$

▶ Remark 2.10. The commutative diagram above means that the induced operations of $\mathrm{Hom}_{\mathbf{Spa}}(S,\Omega)$ are defined pointwise. The harmony condition then consists of the two parts:

(i) $\mathrm{Hom}_{\mathbf{Spa}}(S,\Omega)$ is closed under the pointwise operations;
(ii) $\mathrm{Hom}_{\mathbf{Spa}}(S,\Omega)$ with the pointwise operations is in **Alg**.

Here, (ii) is not so important for the reason that we can drop condition (ii) if **Alg** is the Eilenberg-Moore category of a monad on **Set**, rather than a full subcategory of it; this follows from the fact that, since **Alg** is then closed under products and subalgebras, we have a product Ω^S in **Alg**, and hence $\mathrm{Hom}_{\mathbf{Spa}}(S,\Omega)$ in **Alg** as a subalgebra of Ω^S (obviously, it actually suffices to assume that **Alg** is a quasi-variety or an implicational full subcategory of the Eilenberg-Moore category of a monad on **Set** in the sense of [1]). Regarding $\mathrm{Hom}_{\mathbf{Spa}}(S,\Omega)$ as the collection of generalised continuous functions on S, (i) above means that the continuous functions are closed under the algebraic operations defined pointwise, which is the most important part of the harmony condition, and after which the "harmony" condition is named.

We assume the harmony condition in the following part of this subsection.

The geometric structure of $\mathrm{Hom}_{\mathbf{Alg}}(A,\Omega)$ for $A \in \mathbf{Alg}$ can be provided as follows. By Proposition 2.8, equip $\mathrm{Hom}_{\mathbf{Alg}}(A,\Omega)$ with the generalised topology generated (in **Spa**) by

$$\{\langle a\rangle_O \ ; \ a \in A \text{ and } O \in \mathcal{O}_\Omega\}$$

where

$$\langle a\rangle_O := \{v \in \mathrm{Hom}_{\mathbf{Alg}}(A,\Omega) \ ; \ v(a) \in O\}.$$

The algebraic structure of $\mathrm{Hom}_{\mathbf{Spa}}(S,\Omega)$ is provided by h_S above.

The induced contravariant Hom-functors $\mathrm{Hom}_{\mathbf{Alg}}(\text{-},\Omega) : \mathbf{Alg} \to \mathbf{Spa}$ and $\mathrm{Hom}_{\mathbf{Spa}}(\text{-},\Omega) : \mathbf{Spa} \to \mathbf{Alg}$ can be shown to be well defined and form a dual adjunction between categories **Alg** and **Spa**, i.e., we have the following dual adjunction theorem:

▶ **Theorem 2.11.** There is a dual adjunction between **Alg** and **Spa**, given by contravariant functros $\mathrm{Hom}_{\mathbf{Alg}}(\text{-},\Omega)$ and $\mathrm{Hom}_{\mathbf{Spa}}(\text{-},\Omega)$. To be precise, $\mathrm{Hom}_{\mathbf{Alg}}(\text{-},\Omega)$ is left adjoint to $\mathrm{Hom}_{\mathbf{Spa}}(\text{-},\Omega)^{\mathrm{op}}$.

A proof of the theorem is given soon after the following remark.

▶ Remark 2.12. The theorem encompasses the well-known dual adjunction between frames and topological spaces; in this case, Ω is the two element frame with the Sierpinski topology, and the harmony condition boils down to the obvious fact that the collection of open sets is closed under the operations of arbitrary unions and finite intersections. The frame-space duality is thus an immediate corollary of the theorem above; this exhibits a sharp contrast to those general theories of dualities that require substantial work in deriving concrete results. Our theory is for duality in context, contrived to be effective in concrete situations.

In a similar way, we can derive a dual adjunction between σ-complete Boolean algebras and measurable spaces by letting Ω be the two element algebra with the discrete topology (in fact, any algebra with the discrete topology works as Ω), where σ-complete Boolean algebras may be seen as point-free measurable spaces. In Section 3, we discuss in detail a dual adjunction between continuous lattices and convexity spaces.

The most plain case is the dual adjunction between **Set** and **Set**, induced by the two element set as a dualising object Ω (any set actually works); the harmony condition is nothing in this case. The discrete Stone adjunction between Boolean algebras and **Set** is well known. The theorem above gives us a vast generalisation of it: there is a dual adjunction between any algebraic category (or variety in terms of universal algebra) and **Set**, induced by any

$\Omega \in \mathbf{Alg}$; the harmony condition is nothing in this case as well, thanks to the discrete nature of **Set** (i.e., the set of all functions $f : S \to \Omega$ are closed under arbitrary operations on it).

Furthermore, the theorem above encompasses the topological Stone adjunction between Boolean algebras and topological spaces, its diverse extensions for distributive lattices, MV-algebras ($[0, 1]$ works as a dualising object in this case), and algebras of substructural logics, and the Gelfand adjunction between commutative C^*-algebras with units 1 and topological spaces; note that the category of commutative C^*-algebras with 1 is monadic over **Set** (see [16]). Any dual adjunction automatically cuts down to a dual equivalence as explained below, and the method can be applied to all the dual adjunctions mentioned above in order to obtain dual equivalences (still it often is not that easy to give intrinsic characterisations of the resulting dual equivalences as already discussed in Section 1).

Let us think of the Pontryagin self-duality for locally compact Abelian groups. Although for simplicity we did not assume a topological structure on **Alg** and an algebraic structure on **Spa** in our set-up, this gets relevant in order to treat the Pontryagin duality within our framework. It is indeed straightforward: we start with topological **Alg** and algebraic **Spa**, and assume two harmony conditions; and the following proof can easily be adapted to that situation (just repeat the same arguments for the additional structures on **Alg** and **Spa**).

2.2.1 Proof of Dual Adjunction Theorem

We first show that the two Hom-functors are well defined.

▶ **Lemma 2.13.** The contravariant functor $\mathrm{Hom}_{\mathbf{Alg}}(\text{-}, \Omega) : \mathbf{Alg} \to \mathbf{Spa}$ is well defined.

Proof. The object part is well defined by Proposition 2.8. We show that the arrow part is well defined. Let $f : A \to A'$ be an arrow in **Alg**. We prove that $\mathrm{Hom}_{\mathbf{Alg}}(f, \Omega) : \mathrm{Hom}_{\mathbf{Alg}}(A', \Omega) \to \mathrm{Hom}_{\mathbf{Alg}}(A, \Omega)$ is an arrow in **Spa**. For $a \in A$ and $O \in \mathcal{O}_\Omega$, we have:

$$\begin{aligned}\mathrm{Hom}_{\mathbf{Alg}}(f, \Omega)^{-1}(\langle a \rangle_O) &= \{v \in \mathrm{Hom}_{\mathbf{Alg}}(A', \Omega)\ ;\ \mathrm{Hom}_{\mathbf{Alg}}(f, \Omega)(v) \in \langle a \rangle_O\} \\ &= \{v \in \mathrm{Hom}_{\mathbf{Alg}}(A', \Omega)\ ;\ v \circ f(a) \in O\} \\ &= \langle f(a) \rangle_O .\end{aligned}$$

Since **Spa** is definable by a class of Boolean topological coaxioms and since Boolean set operations are preserved by the inverse image function f^{-1}, this implies that $\mathrm{Hom}_{\mathbf{Alg}}(f, \Omega)$ is an arrow in **Spa**. ◀

▶ **Lemma 2.14.** The contravariant functor $\mathrm{Hom}_{\mathbf{Spa}}(\text{-}, \Omega) : \mathbf{Spa} \to \mathbf{Alg}$ is well defined.

Proof. The object part is well defined by the harmony condition (or can be verified as in (i) or (ii) in Remark 2.10 if we employ either of the other two definitions of **Alg**).

We show that the arrow part is well defined. Let $f : S \to S'$ be an arrow in **Spa**. We prove that $\mathrm{Hom}_{\mathbf{Spa}}(f, \Omega) : \mathrm{Hom}_{\mathbf{Spa}}(S', \Omega) \to \mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)$ is an arrow in **Alg**, i.e., the following diagram commutes:

$$\begin{array}{ccc} T(\mathrm{Hom}_{\mathbf{Spa}}(S', \Omega)) & \xrightarrow{h_{S'}} & \mathrm{Hom}_{\mathbf{Spa}}(S', \Omega) \\ \Big\downarrow {\scriptstyle T(\mathrm{Hom}_{\mathbf{Spa}}(f,\Omega))} & & \Big\downarrow {\scriptstyle \mathrm{Hom}_{\mathbf{Spa}}(f,\Omega)} \\ T(\mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)) & \xrightarrow[h_S]{} & \mathrm{Hom}_{\mathbf{Spa}}(S, \Omega) \end{array}$$

By the harmony condition applied to S (or the commutativity of the lower square in the figure below), this is equivalent to the commutativity of the outermost square in the following diagram for any $s \in S$:

$$\begin{array}{ccc}
T(\mathrm{Hom}_{\mathbf{Spa}}(S', \Omega)) & \xrightarrow{h_{S'}} & \mathrm{Hom}_{\mathbf{Spa}}(S', \Omega) \\
\downarrow{\scriptstyle T(\mathrm{Hom}_{\mathbf{Spa}}(f,\Omega))} & & \downarrow{\scriptstyle \mathrm{Hom}_{\mathbf{Spa}}(f,\Omega)} \\
T(\mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)) & \xrightarrow[h_S]{} & \mathrm{Hom}_{\mathbf{Spa}}(S, \Omega) \\
\downarrow{\scriptstyle T(p_s)} & & \downarrow{\scriptstyle p_s} \\
T(\Omega) & \xrightarrow[h_\Omega]{} & \Omega
\end{array}$$

where recall that p_s denotes the corresponding projection. By the harmony condition applied to S', we have: for any $s' \in S'$, $h_\Omega \circ T(p_{s'}) = p_{s'} \circ h_{S'}$. By taking $s' = f(s)$ in this equation, we have $h_\Omega \circ T(p_{f(s)}) = p_{f(s)} \circ h_{S'}$. It is straightforward to verify that $p_{f(s)} = p_s \circ \mathrm{Hom}_{\mathbf{Spa}}(f, \Omega)$. Thus we obtain $h_\Omega \circ T(p_s \circ \mathrm{Hom}_{\mathbf{Spa}}(f, \Omega)) = p_s \circ \mathrm{Hom}_{\mathbf{Spa}}(f, \Omega) \circ h_{S'}$. Since T is a functor, this yields the commutativity of the outermost square above. Hence, the arrow part is well defined. ◀

Now we define two natural transformations in order to show the dual adjunction.

▶ **Definition 2.15.** Natural transformations

$$\Phi : 1_{\mathbf{Alg}} \to \mathrm{Hom}_{\mathbf{Spa}}(\mathrm{Hom}_{\mathbf{Alg}}(\text{-}, \Omega), \Omega)$$

and

$$\Psi : 1_{\mathbf{Spa}} \to \mathrm{Hom}_{\mathbf{Alg}}(\mathrm{Hom}_{\mathbf{Spa}}(\text{-}, \Omega), \Omega)$$

are defined as follows. For $A \in \mathbf{Alg}$, define Φ_A by $\Phi_A(a)(v) = v(a)$ where $a \in A$ and $v \in \mathrm{Hom}_{\mathbf{Alg}}(A, \Omega)$. For $S \in \mathbf{Spa}$, define Ψ_S by $\Psi_S(x)(f) = f(x)$ where $x \in S$ and $f \in \mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)$.

We have to show that Φ and Ψ are well defined.

▶ **Lemma 2.16.** For $A \in \mathbf{Alg}$ and $a \in A$, $\Phi_A(a)$ is an arrow in **Spa**.

Proof. For $O \in \mathcal{O}_\Omega$, we have

$$\Phi_A(a)^{-1}(O) = \{v \in \mathrm{Hom}_{\mathbf{Alg}}(A, \Omega)\ ;\ \Phi_A(a)(v) \in O\} = \langle a \rangle_O.$$

Thus, $\Phi_A(a)$ is an arrow in **Spa**. ◀

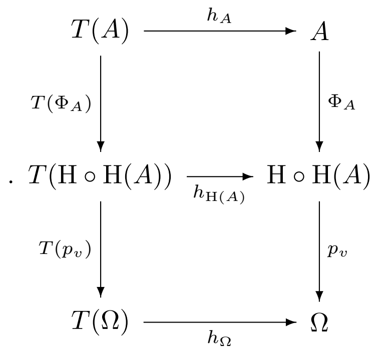
▶ **Lemma 2.17.** For $S \in \mathbf{Spa}$ and $x \in S$, $\Psi_S(x)$ is an arrow in **Alg**.

Proof. This lemma follows immediately from the harmony condition applied to $\mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)$ together with the fact that $p_x = \Psi_S(x)$. ◀

We also have to show that Φ_A is an arrow in **Alg** and that Ψ_S is an arrow in **Spa**.

▶ **Lemma 2.18.** For $A \in \mathbf{Alg}$, Φ_A is an arrow in $\mathbf{Alg}$.

Proof. Let $h_A : T(A) \to A$ denote the structure map of A. For the simplicity of description, let $\mathrm{H}(A)$ denote $\mathrm{Hom}_{\mathbf{Alg}}(A, \Omega)$ and $\mathrm{H} \circ \mathrm{H}(A)$ denote $\mathrm{Hom}_{\mathbf{Spa}}(\mathrm{Hom}_{\mathbf{Alg}}(A, \Omega), \Omega)$. In order to show the commutativity of the upper square in the diagram below, it is sufficient to prove that the outermost square is commutative for any $v \in \mathrm{H}(A)$, since the lower square is commutative because of the harmony condition applied to $\mathrm{H} \circ \mathrm{H}(A)$.



It is straightforward to verify that $p_v \circ \Phi_A = v$. Then, it suffices to show that $v \circ h_A = h_\Omega \circ T(v)$. This is nothing but the fact that $v \in \mathrm{H}(A)$. ◀

▶ **Lemma 2.19.** For $S \in \mathbf{Spa}$, Ψ_S is an arrow in $\mathbf{Spa}$.

Proof. For $f \in \mathrm{Hom}_{\mathbf{Alg}}(\mathrm{Hom}_{\mathbf{Spa}}(\text{-}, \Omega), \Omega)$ and $O \in \mathcal{O}_\Omega$, we have

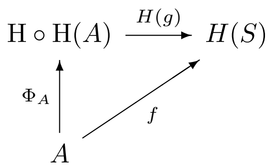
$$\Psi_S^{-1}(\langle f \rangle_O) = \{x \in S \ ; \ \Psi_S(x) \in \langle f \rangle_O\} = f^{-1}(O).$$

Since $\mathbf{Spa}$ is definable by a class of Boolean topological coaxioms and since Boolean set operations are preserved by the inverse image function Ψ_S^{-1}, this implies that Ψ_S is an arrow in $\mathbf{Spa}$. ◀

Now it is straightforward to verify that Φ and Ψ are actually natural transformations.

We finally give a proof of the dual adjunction theorem, Theorem 2.11: $\mathrm{Hom}_{\mathbf{Alg}}(\text{-}, \Omega)$ is left adjoint to $\mathrm{Hom}_{\mathbf{Spa}}(\text{-}, \Omega)^{\mathrm{op}}$ with Φ the unit and Ψ^{op} the counit of the adjunction.

Proof. Let $A \in \mathbf{Alg}$ and $S \in \mathbf{Spa}$. It is enough to show that, for any $f : A \to \mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)$ in $\mathbf{Alg}$, there is a unique $g : S \to \mathrm{Hom}_{\mathbf{Alg}}(A, \Omega)$ in $\mathbf{Spa}$ such that the following diagram commutes:



where $\mathrm{H}(S)$ denotes $\mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)$, $\mathrm{H}(g)$ denotes $\mathrm{Hom}_{\mathbf{Spa}}(g, \Omega)$, $\mathrm{H}(A)$ denotes $\mathrm{Hom}_{\mathbf{Alg}}(A, \Omega)$ and $\mathrm{H} \circ \mathrm{H}(A)$ denotes $\mathrm{Hom}_{\mathbf{Spa}}(\mathrm{Hom}_{\mathbf{Alg}}(A, \Omega), \Omega)$. We first show that such g exists. Define $g : S \to \mathrm{Hom}_{\mathbf{Alg}}(A, \Omega)$ by $g(x)(a) = \Psi_S(x)(f(a))$ where $x \in S$ and $a \in A$. Then we have

$$\begin{aligned} (\mathrm{Hom}_{\mathbf{Spa}}(g, \Omega) \circ \Phi_A(a))(x) &= (\Phi_A(a) \circ g)(x) = g(x)(a) \\ &= \Psi_S(x)(f(a)) = f(a)(x). \end{aligned}$$

Thus, the above diagram commutes for this g. It remains to show that g is an arrow in **Spa**. For $a \in A$ and $O \in \mathcal{O}_\Omega$, we have

$$\begin{aligned} g^{-1}(\langle a\rangle_O) &= \{x \in S \ ;\ g(x) \in \langle a\rangle_O\} \\ &= \{x \in S \ ;\ g(x)(a) \in O\} \\ &= \{x \in S \ ;\ f(a)(x) \in O\} \\ &= f(a)^{-1}(O). \end{aligned}$$

Since $f(a) \in \mathrm{Hom}_{\mathbf{Spa}}(S, \Omega)$ and since **Spa** is definable by a class of Boolean topological coaxioms, this implies that g is an arrow in **Spa**.

Finally, in order to show the uniqueness of such g, we assume that $g' : S \to \mathrm{Hom}_{\mathbf{Alg}}(A, \Omega)$ in **Spa** makes the above diagram commute. Then we have

$$f(a)(x) = (\mathrm{Hom}_{\mathbf{Spa}}(g', \Omega) \circ \Phi_A(a))(x) = (\Phi_A(a) \circ g')(x) = g'(x)(a).$$

Since we also have $f(a)(x) = g(x)(a)$, it follows that $g = g'$. This completes the proof. ◄

2.2.2 Deriving Equivalence from Adjunction

We briefly review standard methods to derive a (dual) equivalence from a (dual) adjunction. Assume that $F : \mathbf{C} \to \mathbf{D}$ is left adjoint to $G : \mathbf{D} \to \mathbf{C}$ with Φ and Ψ the unit and the counit of the adjunction, respectively.

► **Definition 2.20.** $\mathrm{Fix}(\mathbf{C})$ is a full subcategory of $\mathbf{C}$ such that $C \in \mathrm{Fix}(\mathbf{C})$ iff Φ_C is an isomorphism in $\mathbf{C}$. $\mathrm{Fix}(\mathbf{D})$ is a full subcategory of $\mathbf{D}$ such that $D \in \mathrm{Fix}(\mathbf{D})$ iff Ψ_D is an isomorphism in $\mathbf{D}$.

► Proposition 2.21. $\mathrm{Fix}(\mathbf{C})$ and $\mathrm{Fix}(\mathbf{D})$ are categorically equivalent. Moreover, this equivalence is the maximal one that can be derived from the adjunction between $\mathbf{C}$ and $\mathbf{D}$.

If we require a condition about the original adjunction, we have another way to describe $\mathrm{Fix}(\mathbf{C})$ and $\mathrm{Fix}(\mathbf{D})$. We first introduce the following notations.

► **Definition 2.22.** $\mathrm{Img}(\mathbf{C})$ is a full subcategory of $\mathbf{C}$ such that $C \in \mathrm{Img}(\mathbf{C})$ iff $C \simeq G(D)$ for some $D \in \mathbf{D}$. $\mathrm{Img}(\mathbf{D})$ is a full subcategory of $\mathbf{D}$ such that $D \in \mathrm{Img}(\mathbf{D})$ iff $D \simeq F(C)$ for some $C \in \mathbf{C}$.

► Proposition 2.23. Assume that $F(C) \in \mathrm{Fix}(\mathbf{D})$ for any $C \in \mathbf{C}$ and that $G(D) \in \mathrm{Fix}(\mathbf{C})$ for any $D \in \mathbf{D}$. It then holds that $\mathrm{Img}(\mathbf{C}) = \mathrm{Fix}(\mathbf{C})$ and $\mathrm{Img}(\mathbf{D}) = \mathrm{Fix}(\mathbf{D})$. Hence, $\mathrm{Img}(\mathbf{C})$ and $\mathrm{Img}(\mathbf{D})$ are categorically equivalent.

Note that the above assumption is satisfied in the case of the duality between spatial frames and sober topological spaces, and also in the case of a duality between spatial continuous lattices and sober convexity spaces, which is presented below.

3 Domain-Convexity Duality

In this section, we apply the general theory to obtain a dual adjunction between continuous lattices and convexity spaces, and then refine the dual adjunction into a dual equivalence between algebraic lattices and sober convexity spaces, which allows us to characterise the dual equivalence part of Jacobs' dual adjunction for algebras of the distribution monad, with the help of the notion of idempotency for those algebras.

3.1 Convexity-Theoretical Duality for Scott's Continuous Lattices

The concept of a continuous lattice is usually defined in terms of way-below relations: i.e., a continuous lattice is a complete lattice in which any element can be expressed as the join of those elements that are way-below it. From our perspective of duality between point-free and point-set spaces, another characterisation of continuous lattices is helpful:

▶ Proposition 3.1 (Theorem I-2.7 in [7]). A poset is a continuous lattice iff it satisfies the following: (i) it has directed joins including 0; (ii) it has arbitrary meets including 1; (iii) arbitrary meets distribute over directed joins.

The proposition above suggests that continuous lattices may be considered to be point-free convexity spaces; recall that a convexity space is a tuple $(S, \mathcal{C})$ where S is a set, and $\mathcal{C}$ is a subset of $\mathcal{P}(S)$ that is closed under directed unions and arbitrary intersections; $\mathcal{C}$ is called the convexity of the space. Many theorems in convex geometry such as Helly-type theorems (see [9]) can be treated in terms of convexity spaces with suitable conditions (see [6, 18]).

Just as a frame is a point-free abstraction of a topological space, so a continuous lattice is a point-free abstraction of a convexity space; this is what the proposition above tells us. Note that item 1 above is mathematically redundant, but suggests the definition of a homomorphism, which preserves directed joins and arbitrary meets.

This idea in turn suggests that there is a duality between **ContLat** and **Conv**. To apply our duality theory, recall that the continuous lattices are the algebras of the filter monad on **Set** (see [7]), and that the convexity spaces can be expressed as a full subcategory of $(\mathbf{Spa}(\mathcal{Q}))^{\mathrm{op}}$ that is definable by a class of classical topological coaxioms.

We can see $\mathbf{2}$ (i.e., $\{0, 1\}$) as a continuous lattice by its natural ordering $0 < 1$ and also as a convexity space by equipping it with the Sierpinski convexity $\{\emptyset, \{1\}, \mathbf{2}\}$. In order to show that homming into $\mathbf{2}$ gives us a dual adjunction between continuous lattices and convexity spaces, it suffices to verify the harmony condition. It is immediate because the harmony condition in this case boils down to the fact that $\mathrm{Hom}_{\mathbf{Conv}}(S, \mathbf{2})$, which can be seen as the set of convex sets in S, forms a continuous lattice. We then obtain the following theorem.

▶ **Theorem 3.2.** There is a dual adjunction between **ContLat** and **Conv**, given by contravariant functors $\mathrm{Hom}_{\mathbf{Conv}}(\text{-}, \mathbf{2})$ and $\mathrm{Hom}_{\mathbf{ContLat}}(\text{-}, \mathbf{2})$.

We can formally refine the dual adjunction into a dual equivalence in the canonical way as already discussed. It is non-trivial, however, to find an intrinsic description of the induced dual equivalence. We shall achieve it in the following. Although we do not have space to give proofs in this subsection, relevant proofs can be found in Maruyama [14]; note that it causes no essential change in proofs to replace convexity algebras in [14] with continuous lattices.

$\mathrm{Hom}_{\mathbf{Conv}}(X, \mathbf{2})$ can be seen as the collection of convex sets in X, so we write Conv(-) for $\mathrm{Hom}_{\mathbf{Conv}}(\text{-}, \mathbf{2})$. Likewise, we write Spec(-) for $\mathrm{Hom}_{\mathbf{ContLat}}(\text{-}, \mathbf{2})$, for the reason that $\mathrm{Hom}_{\mathbf{ContLat}}(L, \mathbf{2})$ can be seen as the collection of Scott-open meet-complete filters of L where meet-completeness is defined as closedness under arbitrary meets.

▶ **Definition 3.3.** We denote by $\Phi : \mathrm{Id}_{\mathbf{ContLat}} \to \mathrm{Conv} \circ \mathrm{Spec}$ and $\Psi : \mathrm{Id}_{\mathbf{Conv}} \to \mathrm{Spec} \circ \mathrm{Conv}$ the unit and counit of the dual adjunction between **ContLat** and **Conv**, respectively.

The question is when the unit Φ and the counit Ψ give isomorphisms.

We define the notion of spatiality of continuous lattices as the existence of enough Scott-open meet-complete filters:

▶ **Definition 3.4.** A continuous lattice L is spatial iff, for any $a, b \in L$ with $a \not\leq b$, there is a Scott-open meet-complete filter P of L such that $a \in P$ and $b \notin P$.

The following proposition is crucial.

▶ Proposition 3.5. A continuous lattice L is spatial iff $\Phi_L : L \to \mathrm{Conv} \circ \mathrm{Spec}(L)$ is an isomorphism.

Spatiality is characterised as algebraicity.

▶ Proposition 3.6. Let L be a continuous lattice. Then, L is spatial iff L is algebraic (i.e., every element can be expressed as the join of a directed set of compact elements).

Sober convexity spaces are defined in terms of polytopes, which make sense in general convexity spaces as follows.

▶ **Definition 3.7.** The convex hull $\mathrm{ch}(Y)$ of a subset Y of a convexity space $(X, \mathcal{C})$ is defined as $\bigcap\{Z \mid Z \in \mathcal{C} \text{ and } Y \subset Z\}$. Then, a polytope in a convexity space is defined as the convex hull of a set of finitely many points in it.

A convex set C in $\mathcal{C}$ is said to be directed-irreducible iff if $C = \bigcup_{i \in I} C_i$ for a directed subset $\{C_i \,;\, i \in I\}$ of $\mathcal{C}$ then there exists $i \in I$ such that $C = C_i$.

▶ Proposition 3.8. A convex subset of a convexity space is directed-irreducible iff it is a polytope.

Polytopes form a canonical basis for any convexity (if we assume the axiom of choice):

▶ Proposition 3.9. Any convex set in a convexity space can be expressed as the union of a directed set of polytopes.

Sobriety is defined as follows (polytopes may be replaced with directed-irreducible sets).

▶ **Definition 3.10.** A convexity space is sober iff every polytope in it is the convex hull of a unique point.

In contrast to the first impression, sobriety is a natural concept. Let us see examples.

▶ **Definition 3.11.** Given a convexity space S, we can equip the set of polytopes in S with the ideal convexity: i.e., a convex set is an ideal of the lattice of polytopes.

The space of polytopes is then sober, and gives the soberification of the original space. The space of polytopes corresponds to the space of irreducible varieties in algebraic geometry; to put it differently, "a unique point" above plays, in convex geometry, the role of "a generic point" in terms of algebraic geometry.

In algebraic geometry, we soberify a variety by adding irreducible varieties as additional generic points (in other words, the prime spectrum of the coordinate ring of a variety gives the soberification). In convex geometry, we soberify a space by adding polytopes as generic points (in other words, the prime spectrum of the lattice of convex sets gives the soberification). Here recall that polytopes can be characterised by directed-irreducibility.

▶ Proposition 3.12. For a convexity space S, S is sober iff Ψ_S is an isomorphism in **Conv**.

Let **SobConv** denote the category of sober convexity spaces and convexity preserving maps, **AlgLat** the category of algebraic lattices and homomorphisms, and **SpaContLat** the category of spatial continuous lattices and homomorphisms. We finally obtain the following.

▶ **Theorem 3.13.** **AlgLat** (= **SpaContLat**) and **SobConv** are dually equivalent.

3.2 Jacobs Duality for Algebras of the Distribution Monad

Let $\mathcal{D} : \mathbf{Set} \to \mathbf{Set}$ be the distribution monad on **Set**. The object part is defined by:

$$\mathcal{D}(X) := \left\{ f : X \to [0,1] \mid \sum_{x \in X} f(x) = 1 \text{ and } f \text{ has a finite support} \right\}.$$

The arrow part is defined by:

$$\mathcal{D}(f : X \to Y)(g : X \to [0,1])(y) = \sum_{f(x)=y} g(x).$$

As in [8, 11], algebras of $\mathcal{D}$ can concretely be described as barycentric algebras, which are basically sets with convex combination operations; the precise definition is given below.

Jacobs [11] shows a dual adjunction between preframes and algebras of $\mathcal{D}$. We first observe that we can restrict the category of preframes into the category of continuous lattices, since the dual of an algebra of $\mathcal{D}$ (i.e., $\mathrm{PF}(X)$ below) is actually a continuous lattice. And then we characterise the induced dual equivalence via the concept of idempotent algebras of $\mathcal{D}$.

▶ **Definition 3.14.** A $\mathcal{D}$-algebra (aka. barycentric algebra) is a set X with a ternary function

$$\langle \text{-}, \text{-}, \text{-} \rangle : [0,1] \times X \times X \to X$$

such that
1. $\langle r, x, x \rangle = x$;
2. $\langle 0, x, y \rangle = y$;
3. $\langle r, x, y \rangle = \langle 1-r, y, x \rangle$;
4. $\langle r, x, \langle s, y, z \rangle \rangle = \langle r + (1-r)s, \langle r/(r+(1-r)s), x, y \rangle, z \rangle$.

Morphisms of $\mathcal{D}$-algebras are affine maps, i.e., maps f preserving $\langle \text{-}, \text{-}, \text{-} \rangle$ in the following way:

$$f(\langle r, x, y \rangle) = \langle r, f(x), f(y) \rangle.$$

$\mathbf{Alg}(\mathcal{D})$ denotes the category of $\mathcal{D}$-algebras and affine maps.

$\mathbf{Alg}(\mathcal{D})$ in the sense above is equivalent to the Eilenberg-Moore category of the distribution monad (see [11, 8]).

Semilattices can be regarded as $\mathcal{D}$-algebras in a canonical way.

▶ Proposition 3.15. Any meet-semilattice L forms a $\mathcal{D}$-algebra: define $\langle \text{-}, \text{-}, \text{-} \rangle : [0,1] \times L \times L \to L$ by

$$\langle r, x, y \rangle = x \wedge y$$

if $r \in (0,1)$; otherwise, define $\langle r, x, y \rangle = x$ if $r = 1$, and $\langle r, x, y \rangle = y$ if $r = 0$. Similarly, any join-semilattice forms a $\mathcal{D}$-algebra (by replacing $\wedge$ above with $\vee$).

In the following, we suppose any semilattice is equipped with the convex structure defined in the proposition above. We review the following concepts from Jacobs [11].

▶ **Definition 3.16.** For a $\mathcal{D}$-algebra $(X, \langle \text{-}, \text{-}, \text{-} \rangle)$, a subset Y of X is defined as
- a subalgebra iff $y_1, y_2 \in Y$ implies that for any $r \in [0,1]$, $\langle r, y_1, y_2 \rangle \in Y$;
- a filter iff $\langle r, x_1, x_2 \rangle \in Y$ and $r \neq 0, 1$ together imply both $x_1 \in Y$ and $x_2 \in Y$;
- a prime filter iff it is both a subalgebra and a filter.

Let us define a contravariant functor

$$\mathrm{PF}(\text{-}) : \mathbf{Alg}(\mathcal{D})^{\mathrm{op}} \to \mathbf{ContLat}.$$

For a $\mathcal{D}$-algebra X, $\mathrm{PF}(X)$ is the lattice of prime filters of X. For an affine map f, we let $\mathrm{PF}(f) = f^{-1}$.

We define a contravariant functor

$$\mathrm{Sp}(\text{-}) : \mathbf{ContLat}^{\mathrm{op}} \to \mathbf{Alg}(\mathcal{D})$$

as follows. For a continuous lattice L, define $\mathrm{Sp}(L)$ as the set of Scott-open meet-complete filters of L, equipped with a meet-semilattice structure by finite intersections, and hence with a $\mathcal{D}$-algebra structure (see Proposition 3.15). For a homomorphism f, we let $\mathrm{Sp}(f) = f^{-1}$.

Since $\mathrm{PF}(X)$ always forms a continuous lattice, the methods of Jacobs [11] completely work in the present situation, thus yielding the following dual adjunction theorem.

▶ **Theorem 3.17.** There is a dual adjunction between **ContLat** and $\mathbf{Alg}(\mathcal{D})$, given by $\mathrm{Sp} : \mathbf{ContLat}^{\mathrm{op}} \to \mathbf{Alg}(\mathcal{D})$ and $\mathrm{PF} : \mathbf{Alg}(\mathcal{D})^{\mathrm{op}} \to \mathbf{ContLat}$.

In the following, we aim at identifying the dual equivalence induced by the dual adjunction above. Towards this end, we introduce the concept of idempotent $\mathcal{D}$-algebras.

▶ **Definition 3.18.** A $\mathcal{D}$-algebra $(X, \langle \text{-}, \text{-}, \text{-} \rangle)$ is idempotent iff for any $x, y \in X$, and for any $r, s \in (0, 1)$ (i.e., the open unit interval),

$$\langle r, x, y \rangle = \langle s, x, y \rangle.$$

It is straightforward to see the following.

▶ Proposition 3.19. Any meet-semilattice and join-semilattice form an idempotent $\mathcal{D}$-algebra. In particular, $\mathrm{Sp}(L)$ is an idempotent $\mathcal{D}$-algebra.

▶ Proposition 3.20. For a $\mathcal{D}$-algebra X, $\mathrm{PF}(X)$ is an algebraic lattice.

Proof. This follows from the fact that $\mathrm{PF}(X)$ is a subalgebra of the powerset algebraic lattice $\mathcal{P}(X)$ with respect to directed unions and arbitrary intersections, and that the class of all algebraic lattices is closed under subalgebras. ◀

▶ Proposition 3.21. If L is an algebraic lattice, then L is isomorphic to $\mathrm{PF} \circ \mathrm{Sp}(L)$.

Proof. Firstly, $\mathrm{Sp}(L)$ can be regarded as $\mathrm{Hom}_{\mathbf{ContLat}}(L, \mathbf{2})$ by identifying subsets with their characteristic functions. Likewise, $\mathrm{PF} \circ \mathrm{Sp}(L)$ can be thought of as $\mathrm{Hom}_{\mathbf{Conv}}(\mathrm{Sp}(L), \mathbf{2})$. Then, the previously obtained duality between algebraic lattices and sober convexity spaces immediately tells us that L is indeed isomorphic to $\mathrm{PF} \circ \mathrm{Sp}(L)$. ◀

▶ Proposition 3.22. If X is an idempotent $\mathcal{D}$-algebra, then X is isomorphic to $\mathrm{Sp} \circ \mathrm{PF}(X)$.

Proof. For $x, y \in X$, define $x \wedge y$ by $\langle 1/2, x, y \rangle$. By idempotency, X with $\wedge$ forms a meet-semilattice. Since $\mathrm{Sp} \circ \mathrm{PF}(X)$ is an idempotent $\mathcal{D}$-algebra by Proposition 3.19, $\mathrm{Sp} \circ \mathrm{PF}(X)$ also forms a meet-semilattice in the same way. It holds that if the meet-semilattices of two idempotent $\mathcal{D}$-algebras are isomorphic, then the original $\mathcal{D}$-algebras are isomorphic as well. Thus, it suffices to prove that X is isomorphic to $\mathrm{Sp} \circ \mathrm{PF}(X)$ as a meet-semilattice.

Now, X can in turn be equipped with the ideal convexity: the convex sets are defined as the ideals of X. Then, X is a sober convexity space (with the convex sets of X forming an algebraic lattice). The polytopes of X, denoted $\mathrm{Poly}(X)$, form a join-semilattice: for two polytopes $\mathrm{ch}(X)$ and $\mathrm{ch}(Y)$ with X, Y finite, their join is defined as $\mathrm{ch}(X \cup Y)$, where recall ch(-) denotes the convex hull operation. And then $\mathrm{Poly}(X)^{\mathrm{op}}$, the order dual of the polytope join-semilattice $\mathrm{Poly}(X)$, is actually isomorphic to X as a meet-semilattice; this holds for any meet-semilattice X by an equivalence between the categories of join-semilattices and of sober convexity spaces as remarked in Maruyama [14]. Since $\mathrm{PF}(X)$ is the lattice of ideals of X, it turns out that $\mathrm{Sp} \circ \mathrm{PF}(X)$ is the meet-semilattice of compact (aka. directed-irreducible) elements of the ideal lattice, which is precisely $\mathrm{Poly}(X)^{\mathrm{op}}$ (see Proposition 3.8); recall $\mathrm{Poly}(X)^{\mathrm{op}}$ is isomorphic to X, and the proof is done. ◀

Let **IdemAlg**($\mathcal{D}$) denote the category of idempotent $\mathcal{D}$-algebras. Propositions 3.21 and 3.22 above finally give us the following theorem identifying the dual equivalence part of the dual adjunction between **ContLat** and **Alg**($\mathcal{D}$).

▶ **Theorem 3.23.** The dual adjunction between **ContLat** and **Alg**($\mathcal{D}$) restricts to a dual equivalence between **AlgLat** and **IdemAlg**($\mathcal{D}$). This is the largest dual equivalence induced by the dual adjunction.

Since the converses of Propositions 3.21 and 3.22 hold by Propositions 3.19 and 3.20 respectively, the theorem above gives the maximal dual equivalence that can result from restricting the dual adjunction between **ContLat** and **Alg**($\mathcal{D}$). Although we do not have space to work out details, the duality above is closely related to the classic Hofmann-Mislove-Stralka duality [10]; indeed, the duality above reveals a convexity-theoretical aspect of the Hofmann-Mislove-Stralka duality.

Summing up, we have obtained the following dualities in this section:

$$\mathbf{IdemAlg}(\mathcal{D}) \simeq \mathbf{AlgLat}^{\mathrm{op}} \simeq \mathbf{SobConv}.$$

It thus follows that **IdemAlg**($\mathcal{D}$) $\simeq$ **SobConv**; behind the equivalence, we actually have an adjunction between **IdemAlg**($\mathcal{D}$) and **Conv**. And the functor from **Conv** to **IdemAlg**($\mathcal{D}$) has clear meaning in terms of polytopes: it maps a convexity space S to the $\mathcal{D}$-algebra of $\mathrm{Poly}(S)^{\mathrm{op}}$, i.e., the order dual of the polytope join-semilattice of S. These domain-convexity dualities tell us that domain theory and convex geometry are naturally intertwined in the (sometimes beautiful, sometimes insane) universe of mathematics.

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A Categorical Duality as Philosophy of Space

We finally speculate on categorical duality in a broader context, and attempt to elucidate conceptual foundations of categorical duality, especially in relation to philosophy of space. Let us start with the following (rough) picture of categorical dualities in diverse disciplines, almost all of which may be conceived of as arising between the epistemological and the ontological. The concept of duality between ontology and epistemology, we think, yields a unifying perspective on categorical dualities in wide-ranging fields; it is like "duality between the conceptual and the formal" in Lawvere's terms in his seminal hyperdoctrine paper "Adjointness in Foundations" (more links with other thinkers shall be pursued afterwards).

	Ontological	Epistemological	Duality
Logic	Model	Theory	Stone
Algebraic Logic	Algebraic Semantics	Logical System	Tarski
Complex Geometry	Riemann Surface	Algebraic Function Field	Riemann
Classical Alg. Geom.	Variety over k	k-Algebra	Hilbert
Modern Alg. Geom.	Scheme	Ring	Grothendieck
Representation Theory	Group	Representation	Pontryagin
Topology	Point	Open Set	Isbell, Papert
Convex Geometry	Point	Convex Set	Maruyama
Galois Theory	(Profinite) G-set	Algebra Extension	Galois
Program Semantics	Denotation of Program	Observable Property	Abramsky
System Science	Computer System	Its Behaviour	Coalg|Alg
General Relativity	Spacetime Manifold	Field	Weyl
Quantum Physics	State	Observable	Gelfand

The aforementioned duality between denotations and observable properties of programs basically amounts to domain-theoretical variants of infinitary Stone dualities, such as the Isbell-Papert one. The duality theory of the present paper is relevant to finitary and infinitary Stone, Gelfand, Pontryagin, and even Hilbert dualities (because Hilbert and Stone dualities are closely related as discussed below). The above duality between computer systems and their behaviours boils down to algebra-coalgebra duality in mathematical terms. The most basic case is the Abramksy duality between modal algebras and coalgebras of the Vietoris endofunctor on the category of Stone spaces, which was later rediscovered and explicated by Kupke-Kurz-Venema. A universal-algebraic general theory of such algebra-coalgebra dualities is developed in the author's previous paper "Natural Duality, Modality, and Coalgebra" in *Journal of Pure and Applied Algebra*.

Some of the dualities above are tightly intertwined as a matter of fact. The Stone duality for classical logic is precisely equivalent to a Hilbert duality for geometry over $\mathbb{GF}(2)$ (i.e., the prime field of two elements). Furthermore, logical completeness for classical logic corresponds to Nullstellensatz for geometry over $\mathbb{GF}(2)$, in a mathematically rigorous manner; note that this is different from the model-theoretic correspondence between logic and algebraic geometry. Here it should be noted that logical completeness tells us a poset duality between models and theories, and Nullstellensatz a poset duality between affine varieties and radical ideals, which can be upgraded into the corresponding categorical dualities, namely the Stone duality and the Hilbert duality, respectively (note that the Stone duality is a generalisation of completeness, the syntax-semantics equivalence, in a mathematically precise sense). In this sense, completeness and Nullstellensatz may be said to be "predualities." The correspondence between logic and algebraic geometry may be summarised as follows:

	Logic	Algebraic Geometry
Algebra	Formulae	Polnomials
Spectrum	Models	Variety
Poset Duality	Completeness	Nullstellensatz
Categorical Duality	Stone Duality	Hilbert Duality

The author's recent investigation shows that this correspondence between logic and algebraic geometry extends to $\mathbb{GF}(p^n)$-valued logic and geometry over $\mathbb{GF}(p^n)$ where p is a prime number, and n is an integer more than 1 (and $\mathbb{GF}(p^n)$ is the Galois field of order p^n).

The concept of space has undergone a revolution in the modernisation of mathematics, shifting the emphasis from underlying point-set spaces to algebraic structures upon them, to put it more concretely, from topological spaces to locales (or formal topology as predicate locales), toposes, schemes (i.e., sheaves of rings), and non-commutative point-free spaces (such as C^* and von Neumann algebras). Categorical duality has supported and eased this shift from point-set to point-free space, since it basically tells us the algebraic point-free structure on a point-set space keeps the same amount of information as the original point-set space, allowing us to recover the points as the spectrum of the algebraic structure.

Having seen different categorical dualities seemingly share certain conceptual essence, it would be natural to ask where the (mathematical) origin of those dualities lie, even though there may be no single origin, and the concept of origin *per se* may be misguided. Since duality allows us to regard algebra itself as (point-free) space, another relevant question is where the origin of the shift from point-set to point-free space is.

The first mathematician who elucidated the point could be Riemann, who proved (what is now called) a Riemann surface can be recovered from its function field. At the same time, however, we may think of several serious contenders, especially Kronecker and Dedekind-Weber on the one hand, who are considered (e.g., by Harold Edwards) to be precursors of arithmetic geometry, and Brouwer on the other. Whilst it seems Riemann did not take algebra itself to be space, Kronecker and Dedekind-Weber indeed algebraised complex geometry (e.g., the Riemann-Roch theorem), considering algebraic function fields *per se* to be (equivalents of) spaces (and uncovering a grand link with algebraic number theory, the crucial analogy between algebraic number fields and function fields). Brouwer, even though coming into the scene later than them, vigorously formulated and articulated, in terms of so-called spreads and choice sequences, the notion of continuums that does not presuppose point, transforming a bare, speculative idea into a full-fledged, mathematically substantial enterprise.

Comparable shifts seem to have been caused in philosophy as well. Whitehead's process philosophy puts more emphasis on dynamic *processes* than static *substances*. His philosophy of space is, in its spirit, very akin to the idea of point-free topology:

> Whitehead's basic thought was that we obtain the abstract idea of a spatial point by considering the limit of a real-life series of volumes extending over each other, for example in much the same way that we might consider a nested series of Russian dolls or a nested series of pots and pans. However, it would be a mistake to think of a spatial point as being anything more than an abstraction.

This is from Irvine's *Stanford Encyclopedia of Philosophy* article on Whitehead, and may indeed be read as a brilliant illustration of the idea of prime ideals (or filter) as points in duality theory and algebraic geometry: recall that the open neigbourhoods of a point in a topological space form a completely prime filter of its open set locale, with the complement yielding a prime ideal. Although Whitehead's philosophy of space tends to be discussed in the context of mereology, which is sort of peculiar mathematics, nevertheless, it is indeed highly relevant to the core idea of modern geometry in mainstream mathematics; the idea of points as prime ideals is particularly important in algebraic and non-commutative geometry.

Whitehead's process philosophy would be relevant to category theory in general: for example, John Baez asserts "a category is the simplest framework where we can talk about systems (objects) and processes (morphisms)" in his paper "Physics, Topology, Logic and Computation: A Rosetta Stone." Abramsky-Coecke's categorical quantum mechanics follows a similar line of idea, regarding a †-compact category as a "universe" of quantum processes expressed in an intuitively meaningful graphical language; this is Bob Coecke's quantum picturalism. At the same time, however, we must be aware of the possibility that formalisation distorts or misses a crucial point of an original philosophical idea. Indeed, Whitehead's concept of process would ultimately be unformalisable by its nature. This remark is applicable throughout the whole discussion here, and we have to be cautious of distortion via formalisation, a common mistake the mathematician or logician tends to make.

Yet another point-free philosopher of space is Wittgenstein: "What makes it apparent that space is not a collection of points, but the realization of a law?" (*Philosophical Remarks*, p. 216). Wittgenstein's intensional view on space is a compelling consequence of his persistent disagreement with the set-theoretical extensional view of mathematics:

> Mathematics is ridden through and through with the pernicious idioms of set theory. One example of this is the way people speak of a line as composed of points. A line is a law and isn't composed of anything at all (*Philosophical Grammar*, p. 211).

What does he mean by "law"? Brouwer defined his concept of a spread as a certain law to approximate a "point", and this could possibly be a particular case of Brouwer's influence on Wittgenstein's philosophy. More detailed discussion is in my paper: "Wittgenstein's Conception of Space and the Modernist Transformation of Geometry via Duality", *Papers of 36th International Wittgenstein Symposium*, Austrian Wittgenstein Society, 2013.

Where is the philosophical origin of such a mode of thinking? Just as remarked in the case of the mathematical origin, there may be no single origin, and it might even be wrong to seek an origin at all. Certain postmodern philosophers assert that the idea of the original tends to be invented through a number of copies: after all, there may only be copies having no origin or essence in common. Anyway, we could just envisage a bunch of family-resemblant copies (possibly sharing no genuine feature in common at all) in the form of a series of dichotomies:

Cassirer Shift	Substance	Function
Whitehead Shift	Material	Process
Brouwer Shift	Point	Choice Sequence
Wittgenstein Shift	Tractatus	Investigations
Bohr Shift	Classical Realism	Complementarity
Gödel Shift	Right	Left
Lawvere Duality	Conceptual	Formal
Granger Duality	Object	Operation
Zeno Paradox	Continuous	Discrete
Aristotle	Matter	Form
Natural Philosophy	Newton	Leibniz
Kant	Thing Itself	Appearance
Phenomenology	Object	Subject
Theory of Meaning	Davidson	Dummett

"Shift" means that each thinker emphasises in his dichotomy the shift from a concept on the left-hand side to that on the right-hand side. Cassirer could possibly be a philosophical origin of the modernist shift discussed so far. In contrast with "shift", "duality" does not imply anything on which concept is prior to the other; rather, it does suggest equivalence between two views concerned. Finally, no uniform relationships are intended to hold between two concepts in each of the rest of dichotomies, which do not particularly focus upon shifting from one concept to the other. It would be of conceptual significance to reflect upon the table of categorical dualities in the light of these philosophical dichotomies.

Duality is more than dualism, just as categorical duality in point-free geometry starts with the dualism of space and then tells us that the two conceptions of space are equivalent via functors (in a sense reducing dualism to monism; or it could be called monism on the top of dualism). Category theory often goes beyond dualism. Other sorts of dualism include "geometry vs. algebra", and "model-theoretic vs. proof-theoretic semantics." For instance, the concept of algebras of monads even encompasses geometric structures such as topological spaces and convex structures. Categorical logic tells us model-theoretic semantics amounts to interpreting logic in set-based categories, and proof-theoretic semantics to interpreting logic in so-called syntactic categories. We may thus say category theory transcends dualism.

Gödel's shift from "right" to "left" would need to be explicated. In his "The modern development of the foundations of mathematics in the light of philosophy", Gödel says:

> [T]he development of philosophy since the Renaissance has by and large gone from right to left [...] Particularly in physics, this development has reached a peak in our own time, in that, to a large extent, the possibility of knowledge of the objectivisable states of affairs is denied, and it is asserted that we must be content to predict results of observations. This is really the end of all theoretical science in the usual sense [...]

In the physical context, thus, Gödel's "right" means the emphasis of reality, substance, and the like, and "left" something like observational phenomena. Turning into other contexts, Gödel says metaphysics is "right", and formal logic is "left" in his terminology.

We finally articulate three senses of foundations of mathematics, thereby arguing that philosophy of space counts as foundations mathematics in one of the three senses. A popular, prevailing conception of foundations of mathematics is what may be called a "Reductive Absolute Foundation", which reduces everything to one framework, giving an absolute, domain-independent context to work in. The most popular one is (currently) set-theoretical foundations, but category theory (e.g., Lawvere's ETCS and toposes) can do the job as well.

Category theory can give another sense of foundation. That is a "Structural Relative Foundation", which changes a framework according to our structural focus (and see what remains *invariant*, and what does not), and gives a relative, domain-specific context to work in: e.g., ribbon categories for foundations of knot theory and †-compact categories for foundations of quantum mechanics and information (in these two cases, certain monoidal or linear logical structures are shared and invariant). Recall Grothendieck's relative point of view, and that change of base is a fundamental idea of category theory. The reductive-structural distinction is taken from Prawitz' notions of reductive and structural proof theory.

Philosophy of space as discussed above, we believe, counts as a "Conceptual Foundation of Mathematics", which aims at elucidating the nature of fundamental concepts in mathematics, and, presumably, are compelling for the working mathematician as well (where we basically mean mathematical space rather than physical or intuitive space). In this strand, the author's Categorical Universal Logic proposes a logical universal concept of space to unify toposes and quantum space categories in terms of monad-relativised Lawvere hyperdoctrines, allowing us to reconcile Abramsky-Coecke's categorical quantum mechanics and Birkhoff-von Neumann's traditional quantum logic, which have (slightly misleadingly) been claimed to be in conflict with each other (several papers on CUL are available on the author's webpage).