Annotation-Free Sequent Calculi for Full Intuitionistic Linear Logic*

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Abstract

Full Intuitionistic Linear Logic (FILL) is multiplicative intuitionistic linear logic extended with par. Its proof theory has been notoriously difficult to get right, and existing sequent calculi all involve inference rules with complex annotations to guarantee soundness and cut-elimination. We give a simple and annotation-free display calculus for FILL which satisfies Belnap's generic cut-elimination theorem. To do so, our display calculus actually handles an extension of FILL, called Bi-Intuitionistic Linear Logic (BiILL), with an 'exclusion' connective defined via an adjunction with par. We refine our display calculus for BiILL into a cut-free nested sequent calculus with deep inference in which the explicit structural rules of the display calculus become admissible. A separation property guarantees that proofs of FILL formulae in the deep inference calculus contain no trace of exclusion. Each such rule is sound for the semantics of FILL, thus our deep inference calculus and display calculus are conservative over FILL. The deep inference calculus also enjoys the subformula property and terminating backward proof search, which gives the NP-completeness of BiILL and FILL.

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1 Introduction

Multiplicative Intuitionistic Linear Logic (MILL) contains as connectives only tensor $\otimes$, its unit I, and its residual $\multimap$, where we use I rather than the usual 1 to avoid a clash with the categorical notation for terminal object. The connective par $\parr$ and its unit $\perp$ are traditionally only introduced when we move to classical Multiplicative Linear Logic (MLL), but Hyland and de Paiva's Full Intuitionistic Linear Logic (FILL) [20] shows that a sensible notion of par can be added to MILL without collapse to classicality. FILL's semantics are categorical, with the interaction between the $(\otimes, I, \multimap)$ and $(\parr, \perp)$ fragments entirely described by the equivalent formulae shown below:

$$(p \otimes (q \parr r)) \multimap ((p \otimes q) \parr r) \qquad\qquad ((p \multimap q) \parr r) \multimap (p \multimap (q \parr r)) \tag{1}$$

The first formula is variously called *weak distributivity* [20, 11], *linear distributivity* [12], and *dissociativity* [14]. The second we call Grishin (b) [16]. Its converse, called Grishin (a), is not FILL-valid, and indeed adding it to FILL recovers MLL.

From a traditional sequent calculus perspective, FILL is the logic specified by taking a two-sided sequent calculus for MLL, which enjoys cut-elimination, and restricting its $(\multimap R_2)$

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rule to apply only to "singletons on the right", giving $(\multimap R_1)$, as shown below:

$$(\multimap R_1)\ \frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B} \qquad\qquad (\multimap R_2)\ \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \multimap B, \Delta}$$

Since exactly this restriction converts Gentzen's LK for ordinary classical logic to Gentzen's LJ for intuitionistic logic, FILL arises very naturally. Unfortunately the resulting calculus fails cut-elimination [26]. (Note that there is also work on natural deduction and proof nets for FILL [12, 1, 24, 13]. In this setting the problems of cut-elimination are side-stepped; see the discussion of "essential cuts" in [12] in particular.)

Hyland and de Paiva [20] therefore sought a middle ground between the too weak $(\multimap R_1)$ and the unsound $(\multimap R_2)$ by annotating formulae with *term assignments*, and using them to restrict the application of $(\multimap R_2)$ - the restriction requires that the variable typed by A not appear free in the terms typed by Δ. Reasoning with freeness in the presence of variable binders is notoriously tricky, and a bug was subsequently found by Bierman [4] which meant that the proof of the sequent below requires a cut that is not eliminable:

$$(a \parr b) \parr c \vdash a, (b \parr c \multimap d) \parr e \multimap d \parr e \tag{2}$$

Bierman [4] presented two possible corrections to the term assignment system, one due to Bellin. These were subsequently refined by Bräuner and de Paiva [6] to replace the term assignments by rules annotated with a binary relation between formulae on the left and on the right of the turnstile, which effectively trace variable occurrence. The only existing annotation-free sequent calculi for FILL [15, 16] are incorrect. The first [15] uses $(\multimap R_2)$ without the required annotations, making it unsound, and also contains other transcription errors. The second [16] identifies FILL with 'Bi-Linear Logic', which fails weak distributivity and has an extra connective called 'exclusion', of which more shortly.

The existing correct annotated sequent calculi [4, 6] have some weaknesses. First, the introduction rules for a connective do not define that connective in isolation, as was Gentzen's ideal. Instead, they introduce $\multimap$ on the right only when the context in which the rule sits obeys the rule's side-condition. A consequence is that they cannot be used for naive backward proof search since we must apply the rule upwards blindly, and then check the side-conditions once we have a putative derivation. Second, the term-calculus that results from the annotations has not been shown to have any computational content since its sole purpose is to block unsound inferences by tracking variable occurrence [6]. Thus, FILL's close relationship with other logics is obscured by these complex annotational devices, leading to it being described as proof-theoretically "curious" [12], and leading others to conclude that FILL "does not have a satisfactory proof theory" [9].

We believe these difficulties arise because efforts have focused on an 'unbalanced' logic. We show that adding an 'exclusion' connective $\prec$, dual to $\multimap$, gives a fully 'balanced' logic, which we call Bi-Intuitionistic Linear Logic (BiILL). The beauty of BiILL is that it has a simple *display calculus* [3, 16] BiILL$_{dc}$ that inherits Belnap's general cut-elimination theorem "for free". A similar situation has already been observed in classical modal logic, where it has proved impossible to extend traditional Gentzen sequents to a uniform and general proof-theory encompassing the numerous extensions of normal modal logic K. Display calculi capture a large class of such modal extensions uniformly and modularly [27, 22] by viewing them as fragments of (the display calculi for) tense logics, which conservatively extend modal logic with two modalities ⧫ and ■, respectively adjoint to the original □ and ◇.

In tense logics, the conservativity result is trivial since both modal and tense logics are defined with respect to the same Kripke semantics. With BiILL and FILL, however, there is no such existing conservativity result via semantics. The conservativity of BiILL over FILL would follow if we could show that a derivation of a FILL formula in BiILL$_{dc}$ preserved FILL-validity downwards: unfortunately, this does not hold, as explained next.

Belnap's generic cut-elimination procedure applies to BiILL$_{dc}$ because of the "display property", whereby any substructure of a sequent can be displayed as the whole of either the antecedent or succedent. The display property for BiILL$_{dc}$ is obtained via certain reversible structural rules, called *display rules*, which encode the various adjunctions between the connectives, such as the one between par and exclusion. Any BiILL$_{dc}$-derivation of a FILL formula that uses this adjunction to display a substructure contains occurrences of a structural connective which is an exact proxy for exclusion. That is, a BiILL$_{dc}$-derivation of a FILL formula may require inference steps that have no meaning in FILL, thus we cannot use our display calculus BiILL$_{dc}$ directly to show conservativity of BiILL over FILL. We circumvent this problem by showing that the structural rules to maintain the display property become *admissible*, provided one uses *deep inference*.

Following a methodology established for bi-intuitionistic and tense logics [17, 18], we show that the display calculus for BiILL can be refined to a *nested sequent calculus* [21, 7], called BiILL$_{dn}$, which contains no explicit structural rules, and hence no cut rule, as long as its introduction rules can act "deeply" on any substructure in a given structure. To prove that BiILL$_{dn}$ is sound and complete for BiILL, we use an intermediate nested sequent calculus called BiILL$_{sn}$ which, similar to our display calculus, has explicit structural rules, including cut, and uses *shallow* inference rules that apply only to the topmost sequent in a nested sequent. Our shallow inference calculus BiILL$_{sn}$ can simulate cut-free proofs of our display calculus BiILL$_{dc}$, and vice versa. It enjoys cut-elimination, the display property and coincides with the deep-inference calculus BiILL$_{dn}$ with respect to (cut-free) derivability. Together these imply that BiILL$_{dn}$ is sound and (cut-free) complete for BiILL. Our deep nested sequent calculus BiILL$_{dn}$ also enjoys a *separation property*: a BiILL$_{dn}$-derivation of a formula A uses only introduction rules for the connectives appearing in A. By selecting from BiILL$_{dn}$ only the introduction rules for the connectives in FILL, we obtain a nested (cut-free and deep inference) calculus FILL$_{dn}$ which is complete for FILL. We then show that the rules of FILL$_{dn}$ are also sound for the semantics of FILL. The conservativity of BiILL over FILL follows since a FILL formula A which is valid in BiILL will be cut-free derivable in BiILL$_{dc}$, and hence in BiILL$_{dn}$, and hence in FILL$_{dn}$, and hence valid in FILL.

Viewed upwards, introduction rules for display calculi use shallow inference and can require disassembling structures into an appropriate form using the display rules, meaning that display calculi do not enjoy a "substructure property". The modularity of display calculi also demands explicit structural rules for associativity, commutativity and weak-distributivity. These necessary aspects of display calculi make them unsuitable for proof search since the various structural rules and reversible rules can be applied indiscriminately. As structural rules are admissible in the nested deep inference calculus BiILL$_{dn}$, proof search in it is easier to manage than in the display calculus. Using BiILL$_{dn}$, we show that the tautology problem for BiILL and FILL are in fact NP-complete.

For full proof details we refer readers to the extended version of this paper [10].

2 Display Calculi

2.1 Syntax

▶ **Definition 1.** BiILL-*formulae* are defined using the grammar below where p is from some fixed set of propositional variables

$$A ::= p \mid I \mid \bot \mid A \otimes A \mid A \parr A \mid A \multimap A \mid A \prec A$$

Antecedent and *succedent* BiILL-*structures* (also known as antecedent and succedent parts) are defined by mutual induction, where Φ is a structural constant and A is a BiILL-formula:

$$X_a ::= A \mid \Phi \mid X_a, X_a \mid X_a < X_s \qquad\qquad X_s ::= A \mid \Phi \mid X_s, X_s \mid X_a > X_s$$

FILL-formulae are BiILL-formulae with no occurrence of the exclusion connective $\prec$. FILL-structures are BiILL-structures with no occurrence of $<$, and containing only FILL-formulae. We stipulate that $\otimes$ and $\parr$ bind tighter than $\multimap$ and $\prec$, that comma binds tighter than $>$ and $<$, and resolve $A \multimap B \multimap C$ as $A \multimap (B \multimap C)$. A BiILL- (resp. FILL-) *sequent* is a pair comprising an antecedent and a succedent BiILL- (resp. FILL-) structure, written $X_a \vdash X_s$.

▶ **Definition 2.** We can translate sequents $X \vdash Y$ into formulae as $\tau^a(X) \multimap \tau^s(Y)$, given the mutually inductively defined antecedent and succedent τ-*translations*:

	A	Φ	X, Y	$X > Y$	$X < Y$
τ^a	A	I	$\tau^a(X) \otimes \tau^a(Y)$		$\tau^a(X) \prec \tau^s(Y)$
τ^s	A	$\bot$	$\tau^s(X) \parr \tau^s(Y)$	$\tau^a(X) \multimap \tau^s(Y)$	

Hence Φ and comma are *overloaded* to be translated into different connectives depending on their position. By uniformly replacing our structural connective $<$ with $>$, we could have also overloaded $>$ to stand for $\multimap$ and $\prec$, which would have avoided the blank spaces in the above table, but we have opted to use different connectives to help visually emphasise whether a given structure lives in BiILL or its fragment FILL.

The display calculi for FILL and BiILL are given in Fig. 1.

▶ Remark. For conciseness, we treat comma-separated structures as multisets and usually omit explicit use of (Ass $\vdash$), ($\vdash$ Ass), (Com $\vdash$) and ($\vdash$ Com). The *residuated pair* and *dual residuated pair* rules (rp) and (drp) are the *display postulates* which give Thm. 3 below. Our display postulates build in commutativity of comma, so the two (Com) rules are derivable. If we wanted to drop commutativity [12], we would have to use the more general display postulates from [16]. Note that (drp) may create the structure $<$ which has no meaning in FILL, so we will return to this issue. For now, observe that proofs of even apparently trivial FILL-sequents such as $(p \parr q) \parr r \vdash p, (q \parr r)$ require (drp) to 'move p out the way' so ($\vdash \parr$) can be applied. Another (drp) then eliminates the $<$ to restore p to the right. The rule ($\vdash$ Grnb) is the structural version of Grishin (b), the right hand formula of (1); the rule (Grnb $\vdash$) is equivalent. Fig. 2 gives a cut-free proof of the example from Bierman (2).

▶ **Theorem 3** (Display Property). *For every structure Z which is an antecedent (resp. succedent) part of the sequent $X \vdash Y$, there is a sequent $Z \vdash Y'$ (resp. $X' \vdash Z$) obtainable from $X \vdash Y$ using only (rp) and (drp), thereby displaying the Z as the whole of one side.*

▶ **Theorem 4** (Cut-Admissibility). *From cut-free* BiILL$_{dc}$*-derivations of $X \vdash A$ and $A \vdash Y$ there is an effective procedure to obtain a cut-free* BiILL$_{dc}$*-derivation of $X \vdash Y$.*

Proof. BiILL$_{dc}$ obeys Belnap's conditions for cut-admissibility [3]: see App. A. ◀

Cut and identity:

$$(\mathrm{id})\quad p \vdash p \qquad\qquad (\mathrm{cut})\ \frac{X \vdash A \qquad A \vdash Y}{X \vdash Y}$$

Logical rules:

$$(I \vdash)\ \frac{\Phi \vdash X}{I \vdash X} \qquad\qquad (\vdash I)\quad \Phi \vdash I$$

$$(\bot \vdash)\quad \bot \vdash \Phi \qquad\qquad (\vdash \bot)\ \frac{X \vdash \Phi}{X \vdash \bot}$$

$$(\otimes \vdash)\ \frac{A, B \vdash X}{A \otimes B \vdash X} \qquad\qquad (\vdash \otimes)\ \frac{X \vdash A \qquad Y \vdash B}{X, Y \vdash A \otimes B}$$

$$(\parr \vdash)\ \frac{A \vdash X \qquad B \vdash Y}{A \parr B \vdash X, Y} \qquad\qquad (\vdash \parr)\ \frac{X \vdash A, B}{X \vdash A \parr B}$$

$$(\multimap\vdash)\ \frac{X \vdash A \qquad B \vdash Y}{A \multimap B \vdash X > Y} \qquad\qquad (\vdash\multimap)\ \frac{X \vdash A > B}{X \vdash A \multimap B}$$

Structural rules:

$$(\mathrm{rp})\ \frac{X \vdash Y > Z}{X, Y \vdash Z} \qquad (\mathrm{rp})\ \frac{X, Y \vdash Z}{Y \vdash X > Z} \qquad (\mathrm{drp})\ \frac{X < Y \vdash Z}{X \vdash Y, Z} \qquad (\mathrm{drp})\ \frac{X \vdash Y, Z}{X < Z \vdash Y}$$

$$(\Phi \vdash)\ \frac{X, \Phi \vdash Y}{X \vdash Y} \qquad\qquad (\vdash \Phi)\ \frac{X \vdash \Phi, Y}{X \vdash Y}$$

$$(\mathrm{Ass} \vdash)\ \frac{W, (X, Y) \vdash Z}{(W, X), Y \vdash Z} \qquad\qquad (\vdash \mathrm{Ass})\ \frac{W \vdash (X, Y), Z}{W \vdash X, (Y, Z)}$$

$$(\mathrm{Com} \vdash)\ \frac{X, Y \vdash Z}{Y, X \vdash Z} \qquad\qquad (\vdash \mathrm{Com})\ \frac{X \vdash Y, Z}{X \vdash Z, Y}$$

$$(\mathrm{Grnb} \vdash)\ \frac{W, (X < Y) \vdash Z}{(W, X) < Y \vdash Z} \qquad\qquad (\vdash \mathrm{Grnb})\ \frac{W \vdash (X > Y), Z}{W \vdash X > (Y, Z)}$$

Further logical rules for BiILLdc:

$$(\prec \vdash)\ \frac{A < B \vdash X}{A \prec B \vdash X} \qquad\qquad (\vdash \prec)\ \frac{X \vdash A \qquad B \vdash Y}{X < Y \vdash A \prec B}$$

Figure 1 FILLdc and BiILLdc: display calculi for FILL and BiILL.

2.2 Semantics

▶ **Definition 5.** A *FILL-category* is a category equipped with

- a symmetric monoidal closed structure $(\otimes, I, \multimap)$
- a symmetric monoidal structure $(\parr, \bot)$
- a natural family of *weak distributivity* arrows $A \otimes (B \parr C) \to (A \otimes B) \parr C$.

A *BiILL-category* is a FILL-category where the $\parr$ bifunctor has a *co-closure* $\prec$, so there is a natural isomorphism between arrows $A \to B \parr C$ and $A \prec B \to C$.

▶ **Definition 6.** The *free* FILL- (resp. BiILL-) category has FILL- (resp. BiILL-) formulae as objects and the following arrows (quotiented by certain equations) where we are given objects A, A', A'', B, B' and arrows $f : A \to A', f' : A' \to A'', g : B \to B'$, $(\heartsuit, K) \in \{(\otimes, I), (\parr, \bot)\}$, and where the co-closure arrows exist in the free BiILL-category only:

Category: $A \xrightarrow{id} A \qquad A \xrightarrow{f' \circ f} A''$

Symmetric Monoidal: $A \heartsuit B \xrightarrow{f \heartsuit g} A' \heartsuit B' \qquad (A \heartsuit B) \heartsuit C \underset{\alpha^{-1}}{\overset{\alpha}{\rightleftarrows}} A \heartsuit (B \heartsuit C)$

$$K \heartsuit A \underset{\lambda^{-1}}{\overset{\lambda}{\rightleftarrows}} A \qquad A \heartsuit K \underset{\rho^{-1}}{\overset{\rho}{\rightleftarrows}} A \qquad A \heartsuit B \xrightarrow{\gamma} B \heartsuit A$$

$$(\parr\vdash)\ \dfrac{a \vdash a \qquad b \vdash b}{a \parr b \vdash a, b}$$

$$(\parr\vdash)\ \dfrac{a \parr b \vdash a, b \qquad c \vdash c}{(a \parr b) \parr c \vdash a, b, c}$$

$$\textbf{(drp)}\ \dfrac{(a \parr b) \parr c \vdash a, b, c}{(a \parr b) \parr c < a \vdash b, c}$$

$$(\vdash\parr)\ \dfrac{(a \parr b) \parr c < a \vdash b, c}{(a \parr b) \parr c < a \vdash b \parr c}$$

$$(\multimap\vdash)\ \dfrac{(a \parr b) \parr c < a \vdash b \parr c \qquad d \vdash d}{b \parr c \multimap d \vdash ((a \parr b) \parr c < a) > d}$$

$$(\parr\vdash)\ \dfrac{b \parr c \multimap d \vdash ((a \parr b) \parr c < a) > d \qquad e \vdash e}{(b \parr c \multimap d) \parr e \vdash (((a \parr b) \parr c < a) > d), e}$$

$$(\vdash \mathrm{Grnb})\ \dfrac{(b \parr c \multimap d) \parr e \vdash (((a \parr b) \parr c < a) > d), e}{(b \parr c \multimap d) \parr e \vdash ((a \parr b) \parr c < a) > d, e}$$

$$(\mathrm{rp})\ \dfrac{(b \parr c \multimap d) \parr e \vdash ((a \parr b) \parr c < a) > d, e}{(b \parr c \multimap d) \parr e, ((a \parr b) \parr c < a) \vdash d, e}$$

$$(\vdash\parr)\ \dfrac{(b \parr c \multimap d) \parr e, ((a \parr b) \parr c < a) \vdash d, e}{(b \parr c \multimap d) \parr e, ((a \parr b) \parr c < a) \vdash d \parr e}$$

$$(\mathrm{rp})\ \dfrac{(b \parr c \multimap d) \parr e, ((a \parr b) \parr c < a) \vdash d \parr e}{(a \parr b) \parr c < a \vdash (b \parr c \multimap d) \parr e > d \parr e}$$

$$(\vdash\multimap)\ \dfrac{(a \parr b) \parr c < a \vdash (b \parr c \multimap d) \parr e > d \parr e}{(a \parr b) \parr c < a \vdash (b \parr c \multimap d) \parr e \multimap d \parr e}$$

$$\textbf{(drp)}\ \dfrac{(a \parr b) \parr c < a \vdash (b \parr c \multimap d) \parr e \multimap d \parr e}{(a \parr b) \parr c \vdash a, (b \parr c \multimap d) \parr e \multimap d \parr e}$$

Figure 2 The cut-free FILL$_{dc}$-derivation of the example from Bierman.

Closed: $A \multimap B \xrightarrow{A \multimap g} A \multimap B'$ $\qquad (A \multimap B) \otimes A \xrightarrow{\varepsilon} B$ $\qquad A \xrightarrow{\eta} B \multimap A \otimes B$

Weak Distributivity: $A \otimes (A' \parr A'') \xrightarrow{\omega} (A \otimes A') \parr A''$

Co-Closed: $A \prec B \xrightarrow{f \prec B} A' \prec B$ $\qquad A \parr B \prec A \xrightarrow{\varepsilon} B$ $\qquad A \xrightarrow{\eta} B \parr (A \prec B)$

We will suppress explicit reference to the associativity and symmetry arrows.

▶ **Definition 7.** A FILL- (resp. BiILL-) sequent $X \vdash Y$ is *satisfied* by a FILL- (resp. BiILL-) category if, given any valuation of its propositional variables as objects, there exists an arrow $I \to \tau^a(X) \multimap \tau^s(Y)$. It is FILL- (resp. BiILL-) *valid* if it is satisfied by all such categories. In fact, we only need to check the free categories under their generic valuations.

▶ Remark. Those familiar with categorical logic will note that our use of category theory here is rather shallow, looking only at whether hom-sets are populated, and not at the rich structure of equivalences between proofs that categorical logic supports. This is an adequate basis for this work because the question of FILL-validity alone has proved so vexed.

▶ **Theorem 8.** BiILL$_{dc}$ *(Fig. 1) is* sound *and* cut-free complete *for BiILL-validity.*

Proof. BiILL$_{dc}$-proof rules and the arrows of the free BiILL-category are interdefinable. ◀

▶ **Corollary 9.** *The display calculus* FILL$_{dc}$ *is cut-free complete for FILL-validity.*

Proof. Because BiILL-categories are FILL-categories, and BiILL$_{dc}$ proofs of FILL-sequents are FILL$_{dc}$ proofs. ◀

We will return to the question of *soundness* for FILL$_{dc}$ in Sec. 4.

3 Deep Inference and Proof Search

We now present a refinement of the display calculus BiILL$_{dc}$, in the form of a nested sequent calculus, that is more suitable for proof search. A nested sequent is essentially just a structure in display calculus, but presented in a more sequent-like notation. This change of notation allows us to present the proof systems much more concisely. The proof system we are interested in is the deep inference system in Sec. 3.2, but we shall first present an intermediate system, BiILL$_{sn}$, which is closer to display calculus, and which eases the proof of correspondence between the deep inference calculus and the display calculus for BiILL.

3.1 The Shallow Inference Calculus

The syntax of nested sequents is given by the grammar below where A_i and B_j are formulae.

$$S \quad T ::= S_1, \dots, S_k, A_1, \dots, A_m \Rightarrow B_1, \dots, B_n, T_1, \dots, T_l$$

We use Γ and Δ for multisets of formulae and use P, Q, S, T, X, Y, etc., for sequents, and $\mathcal{S}$, $\mathcal{X}$, etc., for multisets of sequents and formulae. The empty multiset is $\cdot$ ('dot').

A nested sequent can naturally be represented as a tree structure as follows. The nodes of the tree are traditional two-sided sequents (i.e., pairs of multisets). The edges between nodes are labelled with either a $-$, denoting nesting to the left of the sequent arrow, or a $+$, denoting nesting to the right of the sequent arrow. For example, the nested sequent below can be visualised as the tree in Fig. 3 (i):

$$(e, f \Rightarrow g), (p, (u, v \Rightarrow x, y) \Rightarrow q, r), a, b \Rightarrow c, d, (\cdot \Rightarrow s) \tag{3}$$

A display sequent can be seen as a nested sequent, where $\vdash$, $>$ and $<$ are all replaced by $\Rightarrow$ and the unit Φ is represented by the empty multiset. The definition of a nested sequent incorporates implicitly the associativity and commutativity of comma, and the effects of its unit, via the multiset structure.

▶ **Definition 10.** Following Def. 2, we can translate nested sequents into equivalence classes of BiILL-formulae (modulo associativity, commutativity, and unit laws) via *τ-translations*:

$$\tau^a(S_1, \dots, S_k, A_1, \dots, A_m \Rightarrow B_1, \dots, B_n, T_1, \dots, T_l)$$
$$= (\tau^a(S_1) \otimes \cdots \otimes \tau^a(S_k) \otimes A_1 \otimes \cdots \otimes A_m) \prec (B_1 \parr \cdots \parr B_n \parr \tau^s(T_1) \parr \cdots \parr \tau^s(T_l))$$

$$\tau^s(S_1, \dots, S_k, A_1, \dots, A_m \Rightarrow B_1, \dots, B_n, T_1, \dots, T_l)$$
$$= (\tau^a(S_1) \otimes \cdots \otimes \tau^a(S_k) \otimes A_1 \otimes \cdots \otimes A_m) \multimap (B_1 \parr \cdots \parr B_n \parr \tau^s(T_1) \parr \cdots \parr \tau^s(T_l)).$$

The translations τ^a and τ^s differ only in their translation of the sequent symbol $\Rightarrow$ to $\multimap$ and $\prec$ respectively. Where $m = 0$, $A_1 \otimes \cdots \otimes A_m$ translates to I, and similarly $B_1 \parr \cdots \parr B_n$ translates to $\perp$ when $n = 0$. These translations each extend to a map from multisets of nested sequents and formulae to formulae: τ^a (resp. τ^s) acts on each sequent as above, leaves formulae unchanged, and connects the resulting formulae with $\otimes$ (resp. $\parr$). Empty multisets are mapped to I (resp. $\perp$).

A *context* is either a 'hole' [ ], called the *empty context*, or a sequent where exactly one node has been replaced by a hole [ ]. Contexts are denoted by $X[\ ]$. We write $X[S]$ to denote a sequent resulting from replacing the hole [ ] in $X[\ ]$ with the sequent S. A non-empty context $X[\ ]$ is *positive* if the hole [ ] occurs immediately to the right of a sequent arrow $\Rightarrow$, and *negative* otherwise. This simple definition of polarities of a context is made possible by the use of the same symbol $\Rightarrow$ to denote the structural counterparts of $\multimap$ and $\prec$. As we shall see in Sec. 3.2, this overloading of $\Rightarrow$ allows a presentation of deep inference rules that ignores context polarity.

The shallow inference system BiILL$_{sn}$ for BiILL is given in Fig. 4. The main difference from BiILL$_{dc}$ is that we allow multiple-conclusion logical rules. This implicitly builds the Grishin (b) rules into the logical rules (see [10]).

▶ **Theorem 11.** *A formula is cut-free* BiILL$_{sn}$*-provable iff it is cut-free* BiILL$_{dc}$*-provable.*

▶ **Corollary 12.** *The cut rule is admissible in* BiILL$_{sn}$*.*

Just as in display calculus (Thm. 3), the display property holds for BiILL$_{sn}$.

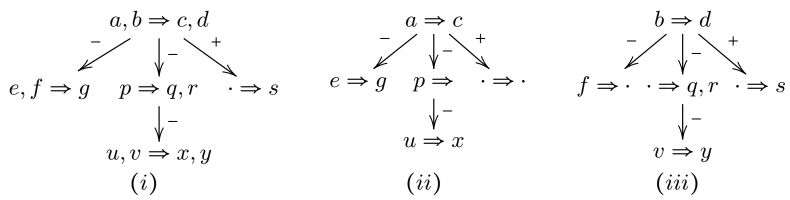

Figure 3 A tree representation of a nested sequent (i), and its partitions (ii and iii).

Cut and identity:

$$\frac{}{p \Rightarrow p}\ id \qquad \frac{\mathcal{S} \Rightarrow \mathcal{S}', A \quad A, \mathcal{T} \Rightarrow \mathcal{T}'}{\mathcal{S}, \mathcal{T} \Rightarrow \mathcal{S}', \mathcal{T}'}\ cut$$

Structural rules:

$$\frac{\mathcal{S} \Rightarrow \mathcal{T}, \mathcal{T}'}{(\mathcal{S} \Rightarrow \mathcal{T}) \Rightarrow \mathcal{T}'}\ drp_1 \qquad \frac{\mathcal{S}, \mathcal{T} \Rightarrow \mathcal{T}'}{\mathcal{S} \Rightarrow (\mathcal{T} \Rightarrow \mathcal{T}')}\ rp_1 \qquad \frac{(\mathcal{S} \Rightarrow \mathcal{S}'), \mathcal{T} \Rightarrow \mathcal{T}'}{(\mathcal{S}, \mathcal{T} \Rightarrow \mathcal{S}') \Rightarrow \mathcal{T}'}\ gl$$

$$\frac{(\mathcal{S} \Rightarrow \mathcal{T}) \Rightarrow \mathcal{T}'}{\mathcal{S} \Rightarrow \mathcal{T}, \mathcal{T}'}\ drp_2 \qquad \frac{\mathcal{S} \Rightarrow (\mathcal{T} \Rightarrow \mathcal{T}')}{\mathcal{S}, \mathcal{T} \Rightarrow \mathcal{T}'}\ rp_2 \qquad \frac{\mathcal{S} \Rightarrow (\mathcal{S}' \Rightarrow \mathcal{T}'), \mathcal{T}}{\mathcal{S} \Rightarrow (\mathcal{S}' \Rightarrow \mathcal{T}', \mathcal{T})}\ gr$$

Logical rules:

$$\frac{}{\bot \Rightarrow \cdot}\ \bot_l \qquad \frac{\mathcal{S} \Rightarrow \mathcal{T}}{\mathcal{S} \Rightarrow \mathcal{T}, \bot}\ \bot_r \qquad \frac{\mathcal{S} \Rightarrow \mathcal{T}}{\mathcal{S}, \mathrm{I} \Rightarrow \mathcal{T}}\ \mathrm{I}_l \qquad \frac{}{\cdot \Rightarrow \mathrm{I}}\ \mathrm{I}_r$$

$$\frac{\mathcal{S}, A, B \Rightarrow \mathcal{T}}{\mathcal{S}, A \otimes B \Rightarrow \mathcal{T}}\ \otimes_l \qquad \frac{\mathcal{S} \Rightarrow A, \mathcal{T} \quad \mathcal{S}' \Rightarrow B, \mathcal{T}'}{\mathcal{S}, \mathcal{S}' \Rightarrow A \otimes B, \mathcal{T}, \mathcal{T}'}\ \otimes_r$$

$$\frac{\mathcal{S}, A \Rightarrow \mathcal{T} \quad \mathcal{S}', B \Rightarrow \mathcal{T}'}{\mathcal{S}, \mathcal{S}', A \parr B \Rightarrow \mathcal{T}, \mathcal{T}'}\ \parr_l \qquad \frac{\mathcal{S} \Rightarrow A, B, \mathcal{T}}{\mathcal{S} \Rightarrow A \parr B, \mathcal{T}}\ \parr_r$$

$$\frac{\mathcal{S} \Rightarrow A, \mathcal{T} \quad \mathcal{S}', B \Rightarrow \mathcal{T}'}{\mathcal{S}, \mathcal{S}', A \multimap B \Rightarrow \mathcal{T}, \mathcal{T}'}\ \multimap_l \qquad \frac{\mathcal{S} \Rightarrow \mathcal{T}, (A \Rightarrow B)}{\mathcal{S} \Rightarrow \mathcal{T}, A \multimap B}\ \multimap_r$$

$$\frac{\mathcal{S}, (A \Rightarrow B) \Rightarrow \mathcal{T}}{\mathcal{S}, A \prec B \Rightarrow \mathcal{T}}\ \prec_l \qquad \frac{\mathcal{S} \Rightarrow A, \mathcal{T} \quad \mathcal{S}', B \Rightarrow \mathcal{T}'}{\mathcal{S}, \mathcal{S}' \Rightarrow A \prec B, \mathcal{T}, \mathcal{T}'}\ \prec_r$$

Figure 4 The shallow inference system BiILL$_{sn}$, where gl and gr capture Grishin (b).

▶ **Proposition 13** (Display property). *Let $X[\ ]$ be a positive (negative) context. For every $\mathcal{S}$, there exists $\mathcal{T}$ such that $\mathcal{T} \Rightarrow \mathcal{S}$ (respectively $\mathcal{S} \Rightarrow \mathcal{T}$) is derivable from $X[\mathcal{S}]$ using only the structural rules from $\{drp_1, drp_2, rp_1, rp_2\}$. Thus $\mathcal{S}$ is "displayed" in $\mathcal{T} \Rightarrow \mathcal{S}$ ($\mathcal{S} \Rightarrow \mathcal{T}$).*

3.2 The Deep Inference Calculus

A deep inference rule can be applied to any sequent within a nested sequent. This poses a problem in formalising context splitting rules, e.g., $\otimes$ on the right. To be sound, we need to consider a context splitting that splits an entire tree of sequents, as formalised next.

Given two sequents X_1 and X_2, their *merge set* $X_1 \bullet X_2$ is defined inductively as:

$$\begin{aligned} X_1 \bullet X_2 = \{\ (\Gamma_1, \Gamma_2, Y_1, \ldots, Y_m \Rightarrow \Delta_1, \Delta_2, Z_1, \ldots, Z_n)\ \mid\ & \\ & X_1 = (\Gamma_1, P_1, \ldots, P_m \Rightarrow \Delta_1, Q_1, \ldots, Q_n) \text{ and} \\ & X_2 = (\Gamma_2, S_1, \ldots, S_m \Rightarrow \Delta_2, T_1, \ldots, T_n) \text{ and} \\ & Y_i \in P_i \bullet S_i \text{ for } 1 \le i \le m \text{ and } Z_j \in Q_j \bullet T_j \text{ for } 1 \le j \le n\ \ \} \end{aligned}$$

Note that the merge set of two sequents may not always be defined since mergeable sequents need to have the same structure. Note also that, because there can be more than

Propagation rules:

$$\frac{X[\mathcal{S} \Rightarrow (A, \mathcal{S}' \Rightarrow \mathcal{T}'), \mathcal{T}]}{X[\mathcal{S}, A \Rightarrow (\mathcal{S}' \Rightarrow \mathcal{T}'), \mathcal{T}]}\ pl_1 \qquad \frac{X[(\mathcal{S} \Rightarrow \mathcal{T}, A), \mathcal{S}' \Rightarrow \mathcal{T}']}{X[(\mathcal{S} \Rightarrow \mathcal{T}), \mathcal{S}' \Rightarrow A, \mathcal{T}']}\ pr_1$$

$$\frac{X[\mathcal{S}, A, (\mathcal{S}' \Rightarrow \mathcal{T}') \Rightarrow \mathcal{T}]}{X[\mathcal{S}, (\mathcal{S}', A \Rightarrow \mathcal{T}') \Rightarrow \mathcal{T}]}\ pl_2 \qquad \frac{X[\mathcal{S} \Rightarrow \mathcal{T}, A, (\mathcal{S}' \Rightarrow \mathcal{T}')]}{X[\mathcal{S} \Rightarrow \mathcal{T}, (\mathcal{S}' \Rightarrow \mathcal{T}', A)]}\ pr_2$$

Identity and logical rules: In branching rules, $X[\ ] \in X_1[\ ] \bullet X_2[\ ]$, $\mathcal{S} \in \mathcal{S}_1 \bullet \mathcal{S}_2$ and $\mathcal{T} \in \mathcal{T}_1 \bullet \mathcal{T}_2$.

$$\frac{X[\ ], \mathcal{U} \text{ and } \mathcal{V} \text{ are hollow.}}{X[\mathcal{U}, p \Rightarrow p, \mathcal{V}]}\ id^d \qquad \frac{X[\ ], \mathcal{U} \text{ and } \mathcal{V} \text{ are hollow.}}{X[\bot, \mathcal{U} \Rightarrow \mathcal{V}]}\ \bot_l^d \qquad \frac{X[\mathcal{S} \Rightarrow \mathcal{T}]}{X[\mathcal{S} \Rightarrow \mathcal{T}, \bot]}\ \bot_r^d$$

$$\frac{X[\mathcal{S} \Rightarrow \mathcal{T}]}{X[\mathcal{S}, \mathrm{I} \Rightarrow \mathcal{T}]}\ \mathrm{I}_l^d \qquad \frac{X[\ ], \mathcal{U} \text{ and } \mathcal{V} \text{ are hollow.}}{X[\mathcal{U} \Rightarrow \mathrm{I}, \mathcal{V}]}\ \mathrm{I}_r^d$$

$$\frac{X[\mathcal{S}, A, B \Rightarrow \mathcal{T}]}{X[\mathcal{S}, A \otimes B \Rightarrow \mathcal{T}]}\ \otimes_l^d \qquad \frac{X_1[\mathcal{S}_1 \Rightarrow A, \mathcal{T}_1] \quad X_2[\mathcal{S}_2 \Rightarrow B, \mathcal{T}_2]}{X[\mathcal{S} \Rightarrow A \otimes B, \mathcal{T}]}\ \otimes_r^d$$

$$\frac{X_1[\mathcal{S}_1 \Rightarrow A, \mathcal{T}_1] \quad X_2[\mathcal{S}_2, B \Rightarrow \mathcal{T}_2]}{X[\mathcal{S}, A \multimap B \Rightarrow \mathcal{T}]}\ \multimap_l^d \qquad \frac{X[\mathcal{S} \Rightarrow \mathcal{T}, (A \Rightarrow B)]}{X[\mathcal{S} \Rightarrow \mathcal{T}, A \multimap B]}\ \multimap_r^d$$

$$\frac{X_1[\mathcal{S}_1, A \Rightarrow \mathcal{T}_1] \quad X_2[\mathcal{S}_2, B \Rightarrow \mathcal{T}_2]}{X[\mathcal{S}, A \parr B \Rightarrow \mathcal{T}]}\ \parr_l^d \qquad \frac{X[\mathcal{S} \Rightarrow A, B, \mathcal{T}]}{X[\mathcal{S} \Rightarrow A \parr B, \mathcal{T}]}\ \parr_r^d$$

$$\frac{X[\mathcal{S}, (A \Rightarrow B) \Rightarrow \mathcal{T}]}{X[\mathcal{S}, A \prec B \Rightarrow \mathcal{T}]}\ \prec_l^d \qquad \frac{X_1[\mathcal{S}_1 \Rightarrow A, \mathcal{T}_1] \quad X_2[\mathcal{S}_2, B \Rightarrow \mathcal{T}_2]}{X[\mathcal{S} \Rightarrow A \prec B, \mathcal{T}]}\ \prec_r^d$$

Figure 5 The deep inference system BiILL$_{dn}$.

one way to enumerate elements of a multiset in the left/right hand side of a sequent, the result of the merging of two nested sequents is a set, rather than a single nested sequent. When $X \in X_1 \bullet X_2$, we say that X_1 and X_2 are a *partition* of X. Fig. 3 (ii) and (iii) show a partitioning of the nested sequent (3) in the tree representation. Note that the partitions (ii) and (iii) must have the same tree structure as the original sequent (i).

Given two contexts $X_1[\ ]$ and $X_2[\ ]$ their merge set $X_1[\ ] \bullet X_2[\ ]$ is defined as follows:

If $X_1[\ ] = [\ ]$ and $X_2[\ ] = [\ ]$ then $X_1[\ ] \bullet X_2[\ ] = \{[\ ]\}$

$$\begin{array}{lcl}
\text{If } X_1[\ ] & = & (\Gamma_1, Y_1[\ ], P_1, \ldots, P_m \Rightarrow \Delta_1, Q_1, \ldots, Q_n) \text{ and} \\
X_2[\ ] & = & (\Gamma_2, Y_2[\ ], S_1, \ldots, S_m \Rightarrow \Delta_2, T_1, \ldots, T_n) \text{ then} \\
X_1[\ ] \bullet X_2[\ ] & = & \{\ (\Gamma_1, \Gamma_2, Y[\ ], U_1, \ldots, U_m \Rightarrow \Delta_1, \Delta_2, V_1, \ldots, V_n) \ \mid \\
& & \quad Y[\ ] \in Y_1[\ ] \bullet Y_2[\ ] \text{ and } U_i \in P_i \bullet S_i \text{ for } 1 \le i \le m \text{ and} \\
& & \quad V_j \in Q_j \bullet T_j \text{ for } 1 \le j \le n\ \} \\
\text{If } X_1[\ ] & = & (\Gamma_1, P_1, \ldots, P_m \Rightarrow \Delta_1, Y_1[\ ], Q_1, \ldots, Q_n) \text{ and} \\
X_2[\ ] & = & (\Gamma_2, S_1, \ldots, S_m \Rightarrow \Delta_2, Y_2[\ ], T_1, \ldots, T_n) \text{ then} \\
X_1[\ ] \bullet X_2[\ ] & = & \{\ (\Gamma_1, \Gamma_2, U_1, \ldots, U_m \Rightarrow \Delta_1, \Delta_2, Y[\ ], V_1, \ldots, V_n) \mid \\
& & \quad Y[\ ] \in Y_1[\ ] \bullet Y_2[\ ] \text{ and } U_i \in P_i \bullet S_i \text{ for } 1 \le i \le m \text{ and} \\
& & \quad V_j \in Q_j \bullet T_j \text{ for } 1 \le j \le n\ \}
\end{array}$$

If $X[\ ] = X_1[\ ] \bullet X_2[\ ]$ we say $X_1[\ ]$ and $X_2[\ ]$ are a *partition* of $X[\ ]$.

We extend the notion of a merge set between multisets of formulae and sequents as follows. Given $\mathcal{X} = \Gamma \cup \{X_1, \ldots, X_n\}$ and $\mathcal{Y} = \Delta \cup \{Y_1, \ldots, Y_n\}$ their merge set contains all multisets of the form: $\Gamma \cup \Delta \cup \{Z_1, \ldots, Z_n\}$ where $Z_i \in X_i \bullet Y_i$.

A nested sequent X (resp. a context $X[\ ]$) is said to be *hollow* iff it contains no occurrences of formulae. For example, $(\cdot \Rightarrow \cdot) \Rightarrow (\cdot \Rightarrow [\ ]), (\cdot \Rightarrow \cdot)$ is a hollow context.

The deep inference system for BiILL, called BiILL$_{dn}$, is given in Fig. 5. Fig. 6 shows a cut-free derivation of Bierman's example in BiILL$_{dn}$.

$$\dfrac{\dfrac{}{a \Rightarrow a, (\cdot \Rightarrow \cdot)}\,id^d \quad \dfrac{\dfrac{}{\cdot \Rightarrow (b \Rightarrow b)}\,id^d}{b \Rightarrow (\cdot \Rightarrow b)}\,pl_1}{\mathbf{a} \parr \mathbf{b} \Rightarrow a, (\cdot \Rightarrow b)}\,\parr^d_l$$

$$\dfrac{\mathbf{a} \parr \mathbf{b} \Rightarrow a, (\cdot \Rightarrow b) \quad \dfrac{\dfrac{}{\cdot \Rightarrow (c \Rightarrow c)}\,id^d}{c \Rightarrow (\cdot \Rightarrow c)}\,pl_1}{\dfrac{(\mathbf{a} \parr \mathbf{b}) \parr \mathbf{c} \Rightarrow a, (\cdot \Rightarrow b, c)}{(a \parr b) \parr c \Rightarrow a, (\cdot \Rightarrow \mathbf{b} \parr \mathbf{c})}\,\parr^d_r}\,\parr^d_l$$

$$\dfrac{\dfrac{(a \parr b) \parr c \Rightarrow a, (\cdot \Rightarrow \mathbf{b} \parr \mathbf{c}) \quad \dfrac{}{\cdot \Rightarrow (d \Rightarrow d)}\,id^d}{(a \parr b) \parr c \Rightarrow a, (\mathbf{b} \parr \mathbf{c} \multimap \mathbf{d} \Rightarrow d)}\,\multimap^d_l \quad \dfrac{}{\cdot \Rightarrow (e \Rightarrow e)}\,id^d}{\dfrac{\dfrac{(a \parr b) \parr c \Rightarrow a, ((\mathbf{b} \parr \mathbf{c} \multimap \mathbf{d}) \parr \mathbf{e} \Rightarrow d, e)}{(a \parr b) \parr c \Rightarrow a, ((b \parr c \multimap d) \parr e \Rightarrow \mathbf{d} \parr \mathbf{e})}\,\parr^d_r}{(a \parr b) \parr c \Rightarrow a, (\mathbf{b} \parr \mathbf{c} \multimap \mathbf{d}) \parr \mathbf{e} \multimap \mathbf{d} \parr \mathbf{e}}\,\multimap^d_r}\,\parr^d_l$$

Figure 6 A cut-free derivation of Bierman's example in BiILL$_{dn}$.

3.3 The Equivalence of the Deep and Shallow Nested Sequent Calculi

From BiILL$_{dn}$ to BiILL$_{sn}$, it is enough to show that every deep inference rule is *cut-free derivable* in BiILL$_{sn}$. For the identity and the constant rules, this follows from the fact that hollow structures can be weakened away, as they add nothing to provability (see [10]). For the other logical rules, a key idea to their soundness is that the context splitting operation is derivable in BiILL$_{sn}$. This is a consequence of the following lemma (see [10]).

▶ **Lemma 14.** *The following rules are derivable in* BiILL$_{sn}$ *without cut:*

$$\dfrac{(\mathcal{X}_1 \Rightarrow \mathcal{Y}_1), (\mathcal{X}_2 \Rightarrow \mathcal{Y}_2), \mathcal{U} \Rightarrow \mathcal{V}}{(\mathcal{X}_1, \mathcal{X}_2 \Rightarrow \mathcal{Y}_1, \mathcal{Y}_2), \mathcal{U} \Rightarrow \mathcal{V}}\,dist_l \qquad \dfrac{\mathcal{U} \Rightarrow \mathcal{V}, (\mathcal{X}_1 \Rightarrow \mathcal{Y}_1), (\mathcal{X}_2 \Rightarrow \mathcal{Y}_2)}{\mathcal{U} \Rightarrow \mathcal{V}, (\mathcal{X}_1, \mathcal{X}_2 \Rightarrow \mathcal{Y}_1, \mathcal{Y}_2)}\,dist_r$$

Intuitively, these rules embody the weak distributivity formalised by the Grishin (b) rule.

▶ **Lemma 15.** *If* $\mathcal{X} \in \mathcal{X}_1 \bullet \mathcal{X}_2$ *then the rules below are cut-free derivable in* BiILL$_{sn}$*:*

$$\dfrac{\mathcal{X}_1, \mathcal{X}_2, \mathcal{U} \Rightarrow \mathcal{V}}{\mathcal{X}, \mathcal{U} \Rightarrow \mathcal{V}}\,m_l \qquad \dfrac{\mathcal{U} \Rightarrow \mathcal{V}, \mathcal{X}_1, \mathcal{X}_2}{\mathcal{U} \Rightarrow \mathcal{V}, \mathcal{X}}\,m_r$$

Proof. This follows straightforwardly from Lem. 14. ◀

▶ **Lemma 16.** *Suppose* $X[\ ] \in X_1[\ ] \bullet X_2[\ ]$ *and suppose there exists* $Y[\ ]$ *such that for any* $\mathcal{U}$ *and any* $\rho \in \{drp_1, drp_2, rp_1, rp_2\}$*, the figure below left is a valid inference rule in* BiILL$_{sn}$*:*

$$\dfrac{Y[\mathcal{U}]}{X[\mathcal{U}]}\,\rho \qquad \dfrac{Y_1[\mathcal{U}]}{X_1[\mathcal{U}]}\,\rho \qquad \dfrac{Y_2[\mathcal{U}]}{X_2[\mathcal{U}]}\,\rho$$

Then there exists $Y_1[\ ]$ *and* $Y_2[\ ]$ *such that* $Y[\ ] \in Y_1[\ ] \bullet Y_2[\ ]$ *and the second and the third figures above are also valid instances of* ρ *in* BiILL$_{sn}$*.*

Proof. This follows from the fact that $X[\ ]$, $X_1[\ ]$ and $X_2[\ ]$ have exactly the same nested structure, so whatever display rule applies to one also applies to the others. ◀

▶ **Theorem 17.** *If a sequent* X *is provable in* BiILL$_{dn}$ *then it is cut-free provable in* BiILL$_{sn}$*.*

Proof. We show that every rule of BiILL$_{dn}$ is cut-free derivable in BiILL$_{sn}$. We show here a derivation of the rule $\multimap^d_l$; the rest can be proved similarly. So suppose the conclusion of the rule is $X[\mathcal{S}, A \multimap B \Rightarrow \mathcal{T}]$, and the premises are $X_1[\mathcal{S}_1 \Rightarrow A, \mathcal{T}_1]$ and $X_2[\mathcal{S}_2, B \Rightarrow \mathcal{T}_2]$, where $X[\ ] \in X_1[\ ] \bullet X_2[\ ]$, $\mathcal{S} \in \mathcal{S}_1 \bullet \mathcal{S}_2$ and $\mathcal{T} \in \mathcal{T}_1 \bullet \mathcal{T}_2$. There are two cases to consider, depending on whether $X[\ ]$ is positive or negative. We show here the former case, as the latter case is similar. Prop. 13 entails that $X[\mathcal{S}, A \multimap B \Rightarrow \mathcal{T}]$ is display equivalent to $\mathcal{U} \Rightarrow (\mathcal{S}, A \multimap B \Rightarrow \mathcal{T})$ for some $\mathcal{U}$. By Lem. 16, we have $\mathcal{U}_1$ and $\mathcal{U}_2$ such that $\mathcal{U} \in \mathcal{U}_1 \bullet \mathcal{U}_2$, and $(\mathcal{U}_1 \Rightarrow \mathcal{V})$ and $(\mathcal{U}_2 \Rightarrow \mathcal{V})$ are display equivalent to, respectively, $X_1[\mathcal{V}]$ and $X_2[\mathcal{V}]$, for any $\mathcal{V}$. The derivation of $\multimap_l^d$ in BiILL$_{sn}$ is thus constructed as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{X_1[\mathcal{S}_1 \Rightarrow A, \mathcal{T}_1]}{\mathcal{U}_1 \Rightarrow (\mathcal{S}_1 \Rightarrow A, \mathcal{T}_1)}\text{Lem. 16}}{\mathcal{U}_1, \mathcal{S}_1 \Rightarrow A, \mathcal{T}_1}rp_2 \quad \dfrac{\dfrac{X_2[\mathcal{S}_2, B \Rightarrow \mathcal{T}_2]}{\mathcal{U}_2 \Rightarrow (\mathcal{S}_2, B \Rightarrow \mathcal{T}_2)}\text{Lem. 16}}{\mathcal{U}_2, \mathcal{S}_2, B \Rightarrow \mathcal{T}_2}rp_2}{\mathcal{U}_1, \mathcal{U}_2, \mathcal{S}_1, \mathcal{S}_2, A \multimap B \Rightarrow \mathcal{T}_1, \mathcal{T}_2}\multimap_l}{\mathcal{U}, \mathcal{S}, A \multimap B \Rightarrow \mathcal{T}}ml; ml; mr}{\mathcal{U} \Rightarrow (\mathcal{S}, A \multimap B \Rightarrow \mathcal{T})}rp_1}{X[\mathcal{S}, A \multimap B \Rightarrow \mathcal{T}]}\text{Prop. 13}$$

◀

The other direction of the equivalence is proved by a permutation argument: we first add the structural rules to BiILL$_{dn}$, then we show that these structural rules permute up over all (non-constant) logical rules of BiILL$_{dn}$. Then when the structural rules appear just below the id^d or the constant rules, they become redundant. There are quite a number of cases to consider, but they are not difficult once one observes the following property of BiILL$_{dn}$: in every rule, every context in the premise(s) has the same tree structure as the context in the conclusion of the rule. This observation takes care of permuting up structural rules that affect only the context. The non-trivial cases are those where the application of the structural rules changes the sequent where the logical rule is applied. We illustrate a case in the following lemma. The detailed proof can be found in [10].

▶ **Lemma 18.** *The rules drp_1, rp_1, drp_2, rp_2, gl, and gr permute up over all logical rules of* BiILL$_{dn}$.

Proof. (*Outline*) We illustrate here a non-trivial interaction between a structural rule and $\multimap_l$, where the conclusion sequent of $\multimap_l$ is changed by that structural rule. The other non-trivial cases follow the same pattern, i.e., propagation rules are used to move the principal formula to the required structural context.

$$\dfrac{\dfrac{\mathcal{S}_1, \mathcal{T}_1 \Rightarrow C, \mathcal{U}_1 \quad \mathcal{S}_2, \mathcal{T}_2, B \Rightarrow \mathcal{U}_2}{\mathcal{S}, C \multimap B, \mathcal{T} \Rightarrow \mathcal{U}}\multimap_l}{\mathcal{S}, C \multimap B \Rightarrow (\mathcal{T} \Rightarrow \mathcal{U})}rp_1 \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\mathcal{S}_1, \mathcal{T}_1 \Rightarrow C, \mathcal{U}_1}{\mathcal{S}_1 \Rightarrow (\mathcal{T}_1 \Rightarrow C, \mathcal{U}_1)}rp_1 \quad \dfrac{\mathcal{S}_2, \mathcal{T}_2, B \Rightarrow \mathcal{U}_2}{\mathcal{S}_2 \Rightarrow (\mathcal{T}_2, B \Rightarrow \mathcal{U}_2)}rp_1}{\mathcal{S} \Rightarrow (C \multimap B, \mathcal{T} \Rightarrow \mathcal{U})}\multimap_l}{\mathcal{S}, C \multimap B \Rightarrow (\mathcal{T} \Rightarrow \mathcal{U})}pl_1$$

◀

▶ **Theorem 19.** *If a sequent X is cut-free* BiILL$_{sn}$*-derivable then it is also* BiILL$_{dn}$*-derivable.*

▶ **Corollary 20.** *A formula is cut-free* BiILL$_{dc}$*-derivable iff it is* BiILL$_{dn}$*-derivable.*

4 Separation, Conservativity, and Decidability

In this section we return our attention to the relationship between our calculi and the categorical semantics (Defs. 5 and 6). Def. 10 gave a translation of nested sequents to formulae; we can hence define validity for nested sequents.

▶ **Definition 21.** A nested sequent S is BiILL-*valid* if there is an arrow $I \to \tau^s(S)$ in the free BiILL-category.

A nested sequent is a (nested) *FILL-sequent* if it has no nesting of sequents on the left of $\Rightarrow$, and no occurrences of $\prec$ at all. The formula translation of Def. 10 hence maps FILL-sequents to FILL-formulae. Such a sequent S is FILL-valid if there is an arrow $I \to \tau^s(S)$ in the free FILL-category.

The calculus BiILL$_{dn}$ enjoys a 'separation' property between the FILL fragment using only $\bot$, I, $\otimes$, $⅋$, and $\multimap$ and the dual fragment using only $\bot$, I, $\otimes$, $⅋$, $\prec$. Let us define FILL$_{dn}$ as the proof system obtained from BiILL$_{dn}$ by restricting to FILL-sequents and removing the rules pr_1, pl_2, $\prec_l^d$ and $\prec_r^d$.

▶ **Theorem 22** (Separation). *Nested FILL-sequents are* FILL$_{dn}$*-provable iff they are* BiILL$_{dn}$*-provable.*

Proof. One direction, from FILL$_{dn}$ to BiILL$_{dn}$, is easy. The other holds because every sequent in a BiILL$_{dn}$ derivation of a FILL-sequent is also a FILL-sequent. ◀

Thm. 22 tells us that every deep inference proof of a FILL-sequent is entirely constructed from FILL-sequents, each with a τ-translation to FILL-formulae. This contrasts with display calculus proofs, which must introduce the FILL-untranslatable $<$ even for simple theorems. By separation, and the equivalence of BiILL$_{dc}$ and BiILL$_{dn}$ (Cor. 20), the conservativity of BiILL over FILL reduces to checking the soundness of each rule of FILL$_{dn}$.

▶ **Lemma 23.** *An arrow $A \otimes B \to C$ exists in the free FILL-category iff an arrow $A \to B \multimap C$ exists. Further, arrows of the following types exist for all formulae A, B, C:*

(i) $A \multimap B \multimap C \;\to\; A \otimes B \multimap C$ *and* $A \otimes B \multimap C \;\to\; A \multimap B \multimap C$

(ii) $(A \multimap B) ⅋ C \;\to\; A \multimap B ⅋ C$.

In the proofs below we will abuse notation by omitting explicit reference to τ^a and τ^s, writing $\Gamma_1 \multimap \Delta_1$ for $\tau^a(\Gamma_1) \multimap \tau^s(\Delta_1)$ for example.

▶ **Lemma 24.** *Let $X[\ ]$ be a positive FILL-context. If there exists an arrow $f : \tau^s(S) \to \tau^s(T)$ in the free FILL-category then there also exists an arrow $\tau^s(X[S]) \to \tau^s(X[T])$. Hence if $X[S]$ is FILL-valid then so is $X[T]$.*

▶ **Lemma 25.** *Given a multiset $\mathcal{V}$ of* hollow *FILL-sequents, there exists an arrow $\bot \to \tau^s(\mathcal{V})$ in the free FILL-category.*

Proof. We will prove this for a single sequent first, by induction on its size. The base case is the sequent $\cdot \Rightarrow \cdot$, whose τ^s-translation is $I \multimap \bot$. The existence of an arrow $\bot \to I \multimap \bot$ is, by Lem. 23, equivalent to the existence of $\bot \otimes I \to \bot$; this is the unit arrow ρ. The induction case involves the sequent $\cdot \to T_1, \ldots, T_l$, with each T_i hollow; the required arrow exists by composing the arrows given by the induction hypothesis with $\bot \to \bot ⅋ \cdots ⅋ \bot$. The multiset case then follows easily by considering the cases where $\mathcal{V}$ is empty and non-empty. ◀

▶ **Lemma 26.** *Given a multiset $\mathcal{T} \in \mathcal{T}_1 \bullet \mathcal{T}_2$ of sequents and formulae, there is an arrow $\tau^s(\mathcal{T}_1) ⅋ \tau^s(\mathcal{T}_2) \to \tau^s(\mathcal{T})$ in the free FILL-category.*

Proof. We prove this for a single sequent first, by induction on its size. The base case requires an arrow $(\Gamma_1 \multimap \Delta_1) ⅋ (\Gamma_2 \multimap \Delta_2) \;\to\; \Gamma_1 \otimes \Gamma_2 \multimap \Delta_1 ⅋ \Delta_2$ (ref. Lem. 14), which exists by Lem. 23(ii) and (i). The induction case follows similarly. The multiset case then follows easily by considering the cases where $\mathcal{T}$ is empty and non-empty. ◀

▶ **Lemma 27.** *Take* $X[\ ] \in X_1[\ ] \bullet X_2[\ ]$ *and* $\mathcal{T} \in \mathcal{T}_1 \bullet \mathcal{T}_2$*. Then the following arrows exist in the free FILL-category for all* A, B, Γ_1 *and* Γ_2*:*
(i) $\tau^s(X_1[\Gamma_1 \Rightarrow A, \mathcal{T}_1]) \otimes \tau^s(X_2[\Gamma_2 \Rightarrow B, \mathcal{T}_2]) \rightarrow \tau^s(X[\Gamma_1, \Gamma_2 \Rightarrow A \otimes B, \mathcal{T}])$;
(ii) $\tau^s(X_1[\Gamma_1 \Rightarrow A, \mathcal{T}_1]) \otimes \tau^s(X_2[\Gamma_2, B \Rightarrow \mathcal{T}_2]) \rightarrow \tau^s(X[\Gamma_1, \Gamma_2, A \multimap B \Rightarrow \mathcal{T}])$;
(iii) $\tau^s(X_1[\Gamma_1, A \Rightarrow \mathcal{T}_1]) \otimes \tau^s(X_2[\Gamma_2, B \Rightarrow \mathcal{T}_2]) \rightarrow \tau^s(X[\Gamma_1, \Gamma_2, A \parr B \Rightarrow \mathcal{T}])$;

Proof. All three cases follow by induction on the size of $X[\ ]$. In all three cases the induction step is easy, and so we focus on the base cases. By Lem. 23 the base case for (i) requires an arrow:

$$(\Gamma_1 \multimap A \parr \mathcal{T}_1) \otimes (\Gamma_2 \multimap B \parr \mathcal{T}_2) \otimes \Gamma_1 \otimes \Gamma_2 \quad \rightarrow \quad (A \otimes B) \parr \mathcal{T}. \tag{4}$$

By the 'evaluation' arrows ε there is an arrow from the left hand side of (4) to $(A \parr \mathcal{T}_1) \otimes (B \parr \mathcal{T}_2)$. Composing this with weak distributivity takes us to $((A \parr \mathcal{T}_1) \otimes B) \parr \mathcal{T}_2$, and then to $(A \otimes B) \parr \mathcal{T}_1 \parr \mathcal{T}_2$. Lem. 26 completes the result. The base cases for (ii) and (iii) follow by similar arguments (App. B). ◀

▶ **Theorem 28.** *For every rule of* FILL$_{dn}$*, if the premises are FILL-valid then so is the conclusion.*

Proof. As FILL-sequents nest no sequents to the left of $\Rightarrow$, we can modify the rules of Fig. 5 to replace the multisets $\mathcal{S}, \mathcal{S}'$ of sequents and formulae with multisets Γ, Γ' of formulae only, and remove the hollow multisets of sequents $\mathcal{U}$ entirely (see App. B).

Therefore by Lem. 24 the soundness of pl_1 amounts to the existence in the free FILL-category of an arrow

$$\Gamma \multimap (A \otimes \Gamma' \multimap \mathcal{T}') \parr \mathcal{T} \quad \rightarrow \quad \Gamma \otimes A \multimap (\Gamma' \multimap \mathcal{T}') \parr \mathcal{T}.$$

This follows by two uses of Lem. 23(i). Similarly pr_2 requires an arrow

$$\Gamma \multimap \mathcal{T} \parr A \parr (\Gamma' \multimap \mathcal{T}') \quad \rightarrow \quad \Gamma \multimap \mathcal{T} \parr (\Gamma' \multimap \mathcal{T}' \parr A)$$

which exists by Lem. 23(ii).

id^d: by induction on the size of $X[\ ]$. The base case requires an arrow $I \rightarrow p \multimap p \parr \mathcal{V}$, which exists by Lems. 25 and 23. Induction involves a sequent $\cdot \Rightarrow X[p \Rightarrow p, \mathcal{V}], \mathcal{T}'$, with $\mathcal{T}'$ hollow, and hence requires an arrow $I \rightarrow I \multimap X[p \Rightarrow p, \mathcal{V}] \parr \mathcal{T}'$. By Lem. 23 and the arrow $I \otimes I \rightarrow I$ we need an arrow $I \rightarrow X[p \Rightarrow p, \mathcal{V}] \parr \mathcal{T}'$; by the induction hypothesis we have $I \rightarrow X[p \Rightarrow p, \mathcal{V}]$; this extends to $I \rightarrow X[p \Rightarrow p, \mathcal{V}] \parr \bot$; Lem. 25 completes the proof.

$\bot_l^d$: by another induction on $X[\ ]$. The base case $I \rightarrow \bot \multimap \mathcal{V}$ follows by Lems. 23 and 25; induction follows as with id^d.

$\bot_r^d$: By Lem. 24 and the unit property of $\bot$.

I_l^d: By Lem. 24 we need an arrow $(\Gamma \multimap \mathcal{T}) \otimes \Gamma \otimes I \rightarrow \mathcal{T}$; this exists by the unit property of I and the 'evaluation' arrow ε.

I_r^d: another induction on $X[\ ]$. The base case arrow $I \rightarrow I \multimap I \parr \mathcal{V}$ exists by Lems. 23 and 25; induction follows as with id^d.

$\otimes_l^d$, $\multimap_r^d$, and $\parr_r^d$ are trivial by the formula translation.

$\otimes_r^d$: compose the arrow $I \rightarrow I \otimes I$ with the arrows defined by the validity of the premises, then use Lem. 27(i). $\multimap_l^d$ and $\parr_r^d$ follow similarly via Lem. 27(ii) and (iii). ◀

▶ **Theorem 29.** *A FILL-formula is* FILL*-valid iff it is* FILL$_{dn}$*-provable, and BiILL is conservative over FILL.*

Proof. By Cors. 9 and 20 and Thms. 22 and 28. ◀

Note that it is also possible to prove soundness of FILL$_{dn}$ w.r.t. FILL syntactically, i.e., via a translation into Schellinx's sequent calculus for FILL [26]. See [10] for details.

Thm. 29 gives us a sound and complete calculus for FILL that enjoys a genuine subformula property. This in turn allows one to prove NP-completeness of the tautology problem for FILL (i.e., deciding whether a formula is provable or not), as we show next. The complexity does not in fact change even when one adds exclusion to FILL.

▶ **Theorem 30.** *The tautology problems for BiILL and FILL are NP-complete.*

Proof. (*Outline.*) Membership in NP is proved by showing that every cut-free proof of a formula A in BiILL$_{dn}$ can be checked in PTIME in the size of A. This is not difficult to prove given that each connective in A is introduced exactly once in the proof. NP-hardness is proved by encoding Constants-Only MLL (COMLL), which is NP-hard [23], in FILL$_{dn}$. ◀

5 Conclusion

We have given three cut-free sequent calculi for FILL without complex annotations, showing that, far from being a curiosity that demands new approaches to proof theory, FILL is in a broad family of linear and substructural logics captured by display calculi.

Various substructural logics can be defined by using a (possibly non-associative or non-commutative) multiplicative conjunction and its left and right residual(s) (implications). Many of these logics have cut-free sequent calculi with comma-separated structures in the antecedent and a single formula in the succedent. Each of these logics has a dual logic with disjunction and its residual(s) (exclusions); their proof theory requires sequents built out of comma-separated structures in the succedent and a single formula in the antecedent. These logics can then be combined using numerous "distribution principles" [19, 25], of which weak distributivity is but one example. However, obtaining an adequate sequent calculus for these combinations is often non-trivial. On the other hand, display calculi for these logics, their duals, and their combinations, are extremely easy to obtain using the known methodology for building display calculi [3, 16]. We followed this methodology to obtain BiILL in this paper, but needed a conservativity result to ensure the resulting calculus BiILL$_{dc}$ was sound for FILL. We finally note some specific variations on FILL deserving particular attention.

Grishin (a). Adding the converse of Grishin (b) to FILL recovers MLL. For example $(B \multimap \bot) \parr C \vdash B \multimap C$ is provable using Grn(b), but its converse requires Grn(a). Thus there is another 'full' non-classical extension of MILL with Grishin (a) as its interaction principle *instead* of (b). We do not know what significance this logic may have.

Mix rules. It is easy to give structural rules for the *mix* sequents $A, B \vdash A, B$ and $\Phi \vdash \Phi$ which have been studied in FILL [12, 1] and so it is natural to ask if the results of this paper can be extended to them. Intriguingly, our new structural connectives suggest a new mix rule with sequent form $A < B \vdash B > A$ which, given Grishin (b), is stronger than the mix rule for comma (given Grishin (a), it is weaker).

Exponentials. Adding exponentials [5] to our display calculus for FILL may be possible [2].

Additives. While it has been suggested that FILL could be extended with additives, the only attempt in the literature is erroneous [15]. It is not clear how easy this extension would be [8, Sec. 1]; it is certainly not straightforward with the display calculus. The problem is most easily seen through the categorical semantics: additive conjunction $\wedge$ and its unit $\top$ are limits, and $p \parr$ - is a right adjoint in BiILL but is not necessarily so in FILL. But right adjoints preserve limits. Then BiILL plus additives is not conservative over FILL plus additives, because the sequents $(p \parr q) \wedge (p \parr r) \vdash p, (q \wedge r)$ and $\top \vdash p, \top$ are valid in the former but not the latter, despite the absence of $\prec$ or $<$. We are currently investigating solutions.

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Propagation rules:

$$\frac{X[\Gamma \Rightarrow (A, \Gamma' \Rightarrow \mathcal{T}'), \mathcal{T}]}{X[\Gamma, A \Rightarrow (\Gamma' \Rightarrow \mathcal{T}'), \mathcal{T}]}\, pl_1 \qquad \frac{X[\Gamma \Rightarrow \mathcal{T}, A, (\Gamma' \Rightarrow \mathcal{T}')]}{X[\Gamma \Rightarrow \mathcal{T}, (\Gamma' \Rightarrow \mathcal{T}', A)]}\, pr_2$$

Identity and logical rules: In branching rules, $X[\ ] \in X_1[\ ] \bullet X_2[\ ]$ and $\mathcal{T} \in \mathcal{T}_1 \bullet \mathcal{T}_2$.

$$\frac{X[\ ] \text{ and } \mathcal{V} \text{ are hollow.}}{X[p \Rightarrow p, \mathcal{V}]}\, id^d \qquad \frac{X[\ ] \text{ and } \mathcal{V} \text{ are hollow.}}{X[\bot \Rightarrow \mathcal{V}]}\, \bot_l^d \qquad \frac{X[\Gamma \Rightarrow \mathcal{T}]}{X[\Gamma \Rightarrow \mathcal{T}, \bot]}\, \bot_r^d$$

$$\frac{X[\Gamma \Rightarrow \mathcal{T}]}{X[\Gamma, \mathrm{I} \Rightarrow \mathcal{T}]}\, \mathrm{I}_l^d \qquad \frac{X[\ ] \text{ and } \mathcal{V} \text{ are hollow.}}{X[\cdot \Rightarrow \mathrm{I}, \mathcal{V}]}\, \mathrm{I}_r^d$$

$$\frac{X[\Gamma, A, B \Rightarrow \mathcal{T}]}{X[\Gamma, A \otimes B \Rightarrow \mathcal{T}]}\, \otimes_l^d \qquad \frac{X_1[\Gamma_1 \Rightarrow A, \mathcal{T}_1] \quad X_2[\Gamma_2 \Rightarrow B, \mathcal{T}_2]}{X[\Gamma_1, \Gamma_2 \Rightarrow A \otimes B, \mathcal{T}]}\, \otimes_r^d$$

$$\frac{X_1[\Gamma_1 \Rightarrow A, \mathcal{T}_1] \quad X_2[\Gamma_2, B \Rightarrow \mathcal{T}_2]}{X[\Gamma_1, \Gamma_2, A \multimap B \Rightarrow \mathcal{T}]}\, \multimap_l^d \qquad \frac{X[\Gamma \Rightarrow \mathcal{T}, (A \Rightarrow B)]}{X[\Gamma \Rightarrow \mathcal{T}, A \multimap B]}\, \multimap_r^d$$

$$\frac{X_1[\Gamma_1, A \Rightarrow \mathcal{T}_1] \quad X_2[\Gamma_2, B \Rightarrow \mathcal{T}_2]}{X[\Gamma_1, \Gamma_2, A \parr B \Rightarrow \mathcal{T}]}\, \parr_l^d \qquad \frac{X[\Gamma \Rightarrow A, B, \mathcal{T}]}{X[\Gamma \Rightarrow A \parr B, \mathcal{T}]}\, \parr_r^d$$

Figure 7 The deep inference system FILL_{dn}.

A Display Calculus

We outline the conditions that are easily checked to confirm that display calculi enjoy cut-admissibility (Thm. 4):

▶ **Definition 31** (Belnap's Conditions C1-C8)**.** The set of display conditions appears in various guises in the literature. Here we follow the presentation given in Kracht [22].

(C1) Each formula variable occurring in some premise of a rule ρ is a subformula of some formula in the conclusion of ρ.

(C2) *Congruent parameters* is a relation between parameters of the identical structure variable occurring in the premise and conclusion sequents.

(C3) Each parameter is congruent to at most one structure variable in the conclusion. Equivalently, no two structure variables in the conclusion are congruent to each other.

(C4) Congruent parameters are either all antecedent or all succedent parts of their respective sequent.

(C5) A formula in the conclusion of a rule ρ is either the entire antecedent or the entire succedent. Such a formula is called a **principal formula** of ρ.

(C6/7) Each rule is closed under simultaneous substitution of arbitrary structures for congruent parameters.

(C8) If there are rules ρ and σ with respective conclusions $X \vdash A$ and $A \vdash Y$ with formula A principal in both inferences (in the sense of C5) and if *cut* is applied to yield $X \vdash Y$, then either $X \vdash Y$ is identical to either $X \vdash A$ or $A \vdash Y$; or it is possible to pass from the premises of ρ and σ to $X \vdash Y$ by means of inferences falling under *cut* where the cut-formula always is a proper subformula of A.

B Conservativity of BiILL over FILL

Fig. 7 explicitly gives the proof rules for FILL_{dn}, the nested sequent calculus with deep inference for FILL. These are easily derived from BiILL_{dn} (Fig. 5).

Proof of Lemma 23. This is basic category theory; we give one example to illustrate the techniques used. Given an arrow $f : A \otimes B \to C$, we get a new arrow $A \to B \multimap C$ by composing $B \multimap f$ with the 'co-evaluation' arrow $\eta : A \to B \multimap (A \otimes B)$. ◀

Proof of Lemma 24. By induction on the size of $X[\ ]$. The base case, where $X[\ ]$ is a hole, is trivial. The induction case involves a context $\Gamma \Rightarrow X[\ ], \mathcal{T}$ and hence requires an arrow

$$\Gamma \multimap X[S] \parr \mathcal{T} \quad \to \quad \Gamma \multimap X[T] \parr \mathcal{T}.$$

This exists by the induction hypothesis and the inductive definitions of Lem. 6. The validity of $X[S]$ then transfers to $X[T]$ via composition with the arrow $I \to X[S]$. ◀

Proof of Lemma 27(ii) and (iii). (ii): The base case requires an arrow

$$(\Gamma_1 \multimap A \parr \mathcal{T}_1) \otimes (\Gamma_2 \otimes B \multimap \mathcal{T}_2) \otimes \Gamma_1 \otimes \Gamma_2 \otimes (A \multimap B) \quad \to \quad \mathcal{T}. \tag{5}$$

Applying an evaluation to the left of (5) gives $(A \parr \mathcal{T}_1) \otimes (\Gamma_2 \otimes B \multimap \mathcal{T}_2) \otimes \Gamma_2 \otimes (A \multimap B)$; weak distributivity gives $\mathcal{T}_1 \parr (A \otimes (\Gamma_2 \otimes B \multimap \mathcal{T}_2) \otimes \Gamma_2 \otimes (A \multimap B))$; two more evaluations give $\mathcal{T}_1 \parr \mathcal{T}_2$ and Lem. 26 completes the result.

(iii): The base case requires an arrow

$$(\Gamma_1 \otimes A \multimap \mathcal{T}_1) \otimes (\Gamma_2 \otimes B \multimap \mathcal{T}_2) \otimes \Gamma_1 \otimes \Gamma_2 \otimes (A \parr B) \quad \to \quad \mathcal{T}. \tag{6}$$

Two applications of weak distributivity map the left of (6) to

$$((\Gamma_1 \otimes A \multimap \mathcal{T}_1) \otimes \Gamma_1 \otimes A) \parr ((\Gamma_2 \otimes B \multimap \mathcal{T}_2) \otimes \Gamma_2 \otimes B).$$

Two evaluations and Lem. 26 complete the result. ◀

C Annotated Sequent Calculi Proofs

On the next page we present cut-free proofs of the Bierman example (2) in the style of the three cut-free annotated sequent calculi in the literature: that due to Bierman [4]; that due to Bellin reported in [4], and that due to Bräuner and de Paiva [6]. Note that all three proofs contain the same sequence of proof rules; strip out the annotations and they are MLL proofs of the sequent. The difference between the calculi lies in the nature of their annotations, all of which come into play to verify that the final rule application, of $(\multimap R)$, is legal. The reader is invited to compare these proofs to those presented in the paper using display calculus (Fig. 2) and deep inference (Fig. 6).

Bierman-style proof; $(\multimap R)$ is legal because v and $(w \parr x \multimap y) \parr z$ share no free variables.

$$\dfrac{v : a \vdash v : a \qquad w : b \vdash w : b}{v \parr w : a \parr b \vdash v : a, w : b}$$

$$\dfrac{v \parr w : a \parr b \vdash v : a, w : b \qquad x : c \vdash x : c}{(v \parr w) \parr x : (a \parr b) \parr c \vdash v : a, w : b, x : c}$$

$$\dfrac{(v \parr w) \parr x : (a \parr b) \parr c \vdash v : a, w : b, x : c}{(v \parr w) \parr x : (a \parr b) \parr c \vdash v : a, w \parr x : b \parr c}$$

$$\dfrac{(v \parr w) \parr x : (a \parr b) \parr c \vdash v : a, w \parr x : b \parr c \qquad y : d \vdash y : d}{(v \parr w) \parr x : (a \parr b) \parr c, w \parr x \multimap y : b \parr c \multimap d \vdash v : a, y : d}$$

$$\dfrac{(v \parr w) \parr x : (a \parr b) \parr c, w \parr x \multimap y : b \parr c \multimap d \vdash v : a, y : d \qquad z : e \vdash z : e}{(v \parr w) \parr x : (a \parr b) \parr c, (w \parr x \multimap y) \parr z : (b \parr c \multimap d) \parr e \vdash v : a, y : d, z : e}$$

$$\dfrac{(v \parr w) \parr x : (a \parr b) \parr c, (w \parr x \multimap y) \parr z : (b \parr c \multimap d) \parr e \vdash v : a, y : d, z : e}{(v \parr w) \parr x : (a \parr b) \parr c, (w \parr x \multimap y) \parr z : (b \parr c \multimap d) \parr e \vdash v : a, y \parr z : d \parr e}$$

$$\dfrac{(v \parr w) \parr x : (a \parr b) \parr c, (w \parr x \multimap y) \parr z : (b \parr c \multimap d) \parr e \vdash v : a, y \parr z : d \parr e}{(v \parr w) \parr x : (a \parr b) \parr c \vdash v : a, \lambda (w \parr x \multimap y) \parr z^{(b \parr c \multimap d) \parr e}.(y \parr z) : (b \parr c \multimap d) \parr e \multimap d \parr e}$$

Bellin-style proof; $(\multimap R)$ is legal because r is not free in let t be $u \parr$ - in let u be $v \parr$ - in v. We apologise for the extremely small font size necessary to fit this proof on the page.

$$\dfrac{v : a \vdash v : a \qquad w : b \vdash w : b}{u : a \parr b \vdash \text{let } u \text{ be } v \parr \text{- in } v : a, \text{let } u \text{ be -} \parr w \text{ in } w : b}$$

$$\dfrac{u : a \parr b \vdash \text{let } u \text{ be } v \parr \text{- in } v : a, \text{let } u \text{ be -} \parr w \text{ in } w : b \qquad x : c \vdash x : c}{t : (a \parr b) \parr c \vdash \text{let } t \text{ be } u \parr \text{- in let } u \text{ be } v \parr \text{- in } v : a, \text{let } t \text{ be } u \parr \text{- in let } u \text{ be -} \parr w \text{ in } w : b, \text{let } t \text{ be -} \parr x \text{ in } x : c}$$

$$\dfrac{\cdots}{t : (a \parr b) \parr c \vdash \text{let } t \text{ be } u \parr \text{- in let } u \text{ be } v \parr \text{- in } v : a, (\text{let } t \text{ be } u \parr \text{- in let } u \text{ be -} \parr w \text{ in } w) \parr (\text{let } t \text{ be -} \parr x \text{ in } x) : b \parr c}$$

$$\dfrac{\cdots \qquad y : d \vdash y : d}{t : (a \parr b) \parr c, s : b \parr c \multimap d \vdash \text{let } t \text{ be } u \parr \text{- in let } u \text{ be } v \parr \text{- in } v : a, (s(\text{let } t \text{ be } u \parr \text{- in let } u \text{ be -} \parr w \text{ in } w) \parr (\text{let } t \text{ be -} \parr x \text{ in } x)) : d}$$

$$\dfrac{\cdots \qquad z : e \vdash z : e}{t : (a \parr b) \parr c, r : (b \parr c \multimap d) \parr e \vdash \text{let } t \text{ be } u \parr \text{- in let } u \text{ be } v \parr \text{- in } v : a, \text{let } r \text{ be } s \parr \text{- in } (s(\text{let } t \text{ be } u \parr \text{- in let } u \text{ be -} \parr w \text{ in } w) \parr (\text{let } t \text{ be -} \parr x \text{ in } x)) : d, \text{let } s \text{ be -} \parr z \text{ in } z : e}$$

$$\dfrac{\cdots}{t : (a \parr b) \parr c, r : (b \parr c \multimap d) \parr e \vdash \text{let } t \text{ be } u \parr \text{- in let } u \text{ be } v \parr \text{- in } v : a, (\text{let } r \text{ be } s \parr \text{- in } (s(\text{let } t \text{ be } u \parr \text{- in let } u \text{ be -} \parr w \text{ in } w) \parr (\text{let } t \text{ be -} \parr x \text{ in } x))) \parr (\text{let } s \text{ be -} \parr z \text{ in } z) : d \parr e}$$

$$\dfrac{\cdots}{t : (a \parr b) \parr c \vdash \text{let } t \text{ be } u \parr \text{- in let } u \text{ be } v \parr \text{- in } v : a, \lambda r^{(b \parr c \multimap d) \parr e}.(\text{let } r \text{ be } s \parr \text{- in } (s(\text{let } t \text{ be } u \parr \text{- in let } u \text{ be -} \parr w \text{ in } w) \parr (\text{let } t \text{ be -} \parr x \text{ in } x))) \parr (\text{let } s \text{ be -} \parr z \text{ in } z) : (b \parr c \multimap d) \parr e \multimap d \parr e}$$

Bräuner and de Paiva-style proof; $(\multimap R)$ is legal because $(b \parr c \multimap d) \parr e$ is not related to a.

$$(a, a) \quad \dfrac{}{a \vdash a} \qquad \dfrac{}{b \vdash b} \quad (b, b)$$

$$(a \parr b, a), (a \parr b, b) \quad \dfrac{a \vdash a \qquad b \vdash b}{a \parr b \vdash a, b}$$

$$\dfrac{}{c \vdash c} \quad (c, c)$$

$$((a \parr b) \parr c, a), ((a \parr b) \parr c, b), ((a \parr b) \parr c, c) \quad \dfrac{a \parr b \vdash a, b \qquad c \vdash c}{(a \parr b) \parr c \vdash a, b, c}$$

$$((a \parr b) \parr c, a), ((a \parr b) \parr c, b \parr c) \quad \dfrac{(a \parr b) \parr c \vdash a, b, c}{(a \parr b) \parr c \vdash a, b \parr c}$$

$$\dfrac{}{d \vdash d} \quad (d, d)$$

$$((a \parr b) \parr c, a), ((a \parr b) \parr c, d), (b \parr c \multimap d, d) \quad \dfrac{(a \parr b) \parr c \vdash a, b \parr c \qquad d \vdash d}{(a \parr b) \parr c, b \parr c \multimap d \vdash a, d}$$

$$\dfrac{}{e \vdash e} \quad (e, e)$$

$$((a \parr b) \parr c, a), ((a \parr b) \parr c, d), ((b \parr c \multimap d) \parr e, d), ((b \parr c \multimap d) \parr e, e) \quad \dfrac{(a \parr b) \parr c, b \parr c \multimap d \vdash a, d \qquad e \vdash e}{(a \parr b) \parr c, (b \parr c \multimap d) \parr e \vdash a, d, e}$$

$$((a \parr b) \parr c, a), ((a \parr b) \parr c, d \parr e), ((b \parr c \multimap d) \parr e, d \parr e) \quad \dfrac{(a \parr b) \parr c, (b \parr c \multimap d) \parr e \vdash a, d, e}{(a \parr b) \parr c, (b \parr c \multimap d) \parr e \vdash a, d \parr e}$$

$$((a \parr b) \parr c, a), ((a \parr b) \parr c, (b \parr c \multimap d) \parr e \multimap d \parr e) \quad \dfrac{(a \parr b) \parr c, (b \parr c \multimap d) \parr e \vdash a, d \parr e}{(a \parr b) \parr c \vdash a, (b \parr c \multimap d) \parr e \multimap d \parr e}$$