A Proof of Kamp's theorem

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Abstract

We provide a simple proof of Kamp's theorem.

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1 Introduction

Temporal Logic (*TL*) introduced to Computer Science by Pnueli in [8] is a convenient framework for reasoning about "reactive" systems. This has made temporal logics a popular subject in the Computer Science community, enjoying extensive research in the past 30 years. In *TL* we describe basic system properties by *atomic propositions* that hold at some points in time, but not at others. More complex properties are expressed by formulas built from the atoms using Boolean connectives and *Modalities* (temporal connectives): A k-place modality M transforms statements $\varphi_1, \ldots, \varphi_k$ possibly on 'past' or 'future' points to a statement $M(\varphi_1, \ldots, \varphi_k)$ on the 'present' point t_0. The rule to determine the truth of a statement $M(\varphi_1, \ldots, \varphi_k)$ at t_0 is called a *truth table* of M. The choice of particular modalities with their truth tables yields different temporal logics. A temporal logic with modalities $M_1, \ldots, M_k$ is denoted by $TL(M_1, \ldots, M_k)$.

The simplest example is the one place modality $\Diamond X$ saying: "X holds some time in the future." Its truth table is formalized by $\varphi_\Diamond(t_0, X) := (\exists t > t_0)X(t)$. This is a formula of the First-Order Monadic Logic of Order (*FOMLO*) - a fundamental formalism in Mathematical Logic where formulas are built using atomic propositions $P(t)$, atomic relations between elements $t_1 = t_2$, $t_1 < t_2$, Boolean connectives and first-order quantifiers $\exists t$ and $\forall t$. Two more natural modalities are the modalities Until ("*Until*") and Since ("*Since*"). $X\mathsf{Until}Y$ means that X will hold from now until a time in the future when Y will hold. $X\mathsf{Since}Y$ means that Y was true at some point of time in the past and since that point X was true until (not necessarily including) now. Both modalities have truth tables in *FOMLO*. Most modalities used in the literature are defined by such *FOMLO* truth tables, and as a result, every temporal formula translates directly into an equivalent *FOMLO* formula. Thus, the different temporal logics may be considered as a convenient way to use fragments of *FOMLO*. *FOMLO* can also serve as a yardstick by which one is able to check the strength of temporal logics: A temporal logic is *expressively complete* for a fragment L of *FOMLO* if every formula of L with a single free variable t_0 is equivalent to a temporal formula.

Actually, the notion of expressive completeness refers to a temporal logic and to a model (or a class of models), since the question whether two formulas are equivalent depends on the domain over which they are evaluated. Any (partially) ordered set with monadic predicates is a model for *TL* and *FOMLO*, but the main, *canonical*, linear time intended models are the non-negative integers $\langle \mathbb{N}, < \rangle$ for discrete time and the reals $\langle \mathbb{R}, < \rangle$ for continuous time.

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Kamp's theorem [7] states that the temporal logic with modalities Until and Since is expressively complete for *FOMLO* over the above two linear time canonical[1] models.

> This seminal theorem initiated the whole study of expressive completeness, and it remains one of the most interesting and distinctive results in temporal logic; very few, if any, similar 'modal' results exist. Several alternative proofs of it and stronger results have appeared; none of them are trivial (at least to most people) [6].

The objective of this paper is to provide a simple proof of Kamp's theorem.

The rest of the paper is organized as follows: In Section 2 we recall the definitions of the monadic logic, the temporal logics and state Kamp's theorem. Section 3 introduces formulas in a normal form and states their simple properties. In Section 4 we prove Kamp's theorem. The proof of one proposition is postponed to Section 5. Section 6 comments on the previous proofs of Kamp's theorem.

2 Preliminaries

In this section we recall the definitions of the first-order monadic logic of order, the temporal logics and state Kamp's theorem.

Fix a set Σ of *atoms*. We use $P, Q, R, S \dots$ to denote members of Σ. The syntax and semantics of both logics are defined below with respect to such Σ.

2.1 First-Order Monadic Logic of Order

Syntax. In the context of *FOMLO*, the atoms of Σ are referred to (and used) as *unary predicate symbols*. Formulas are built using these symbols, plus two binary relation symbols: $<$ and $=$, and a set of first-order variables (denoted: $x, y, z, \dots$). Formulas are defined by the grammar:

$$atomic ::= \quad x < y \ \mid \ x = y \ \mid \ P(x) \qquad \text{(where } \ P \in \Sigma)$$

$$\varphi ::= \quad atomic \ \mid \ \neg\varphi_1 \ \mid \ \varphi_1 \vee \varphi_2 \ \mid \ \varphi_1 \wedge \varphi_2 \ \mid \ \exists x \varphi_1 \ \mid \ \forall x \varphi_1$$

We will also use the standard abbreviated notation for ***bounded quantifiers***, e.g., $(\exists x)_{>z}(\dots)$ denotes $\exists x((x > z) \wedge (\dots))$, and $(\forall x)^{<z}(\dots)$ denotes $\forall x((x < z) \rightarrow (\dots))$, and $(\forall x)^{<z_2}_{>z_1}(\dots)$ denotes $\forall x((z_1 < x < z_2) \rightarrow (\dots))$, etc.

Semantics. Formulas are interpreted over *labeled linear orders* which are called *chains*. A Σ-***chain*** is a triplet $\mathcal{M} = (T, <, \mathcal{I})$ where T is a set - the *domain* of the chain, $<$ is a linear order relation on T, and $\mathcal{I} : \Sigma \rightarrow \mathcal{P}(T)$ is the *interpretation* of Σ (where $\mathcal{P}$ is the powerset notation). We use the standard notation $\mathcal{M}, t_1, t_2, \dots t_n \models \varphi(x_1, x_2, \dots x_n)$ to indicate that the formula φ with free variables among $x_1, \dots, x_n$ is satisfiable in $\mathcal{M}$ when x_i are interpreted as elements t_i of $\mathcal{M}$. For atomic $P(x)$ this is defined by: $\mathcal{M}, t \models P(x)$ iff $t \in \mathcal{I}(P)$; the semantics of $<, =, \neg, \wedge, \vee, \exists$ and $\forall$ is defined in a standard way.

2.2 $TL(\mathsf{Until}, \mathsf{Since})$ Temporal Logic

In this section we recall the syntax and semantics of a temporal logic with ***strict***-Until and ***strict***-Since modalities, denoted by $TL(\mathsf{Until}, \mathsf{Since})$.

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[1] the technical notion which unifies $\langle \mathbb{N}, <\rangle$ and $\langle \mathbb{R}, <\rangle$ is Dedekind completeness.

In the context of temporal logics, the atoms of Σ are used as *atomic propositions* (also called *propositional atoms*). Formulas of $TL(\mathsf{Until}, \mathsf{Since})$ are built using these atoms, Boolean connectives and ***strict***-Until and ***strict***-Since binary modalities. The formulas are defined by the grammar:

$$F ::= \quad \text{True} \;\mid\; P \;\mid\; \neg F_1 \;\mid\; F_1 \vee F_2 \;\mid\; F_1 \wedge F_2 \;\mid\; F_1 \mathsf{Until} F_2 \;\mid\; F_1 \mathsf{Since} F_2,$$

where $P \in \Sigma$.

Semantics. Formulas are interpreted at *time-points* (or *moments*) in chains (elements of the domain). The semantics of $TL(\mathsf{Until}, \mathsf{Since})$ formulas is defined inductively: Given a chain $\mathcal{M} = (T, <, \mathcal{I})$ and $t \in T$, define when a formula F *holds* in $\mathcal{M}$ at t - denoted $\mathcal{M}, t \models F$:

- $\mathcal{M}, t \models P$ iff $t \in \mathcal{I}(P)$, for any propositional atom P.
- $\mathcal{M}, t \models F \vee G$ iff $\mathcal{M}, t \models F$ or $\mathcal{M}, t \models G$; similarly for $\wedge$ and $\neg$.
- $\mathcal{M}, t \models F_1 \mathsf{Until} F_2$ iff there is $t' > t$ such that $\mathcal{M}, t' \models F_2$ and $\mathcal{M}, t_1 \models F_1$ for all $t_1 \in (t, t')$.
- $\mathcal{M}, t \models F_1 \mathsf{Since} F_2$ iff there is $t' < t$ such that $\mathcal{M}, t' \models F_2$ and $\mathcal{M}, t_1 \models F_1$ for all $t_1 \in (t', t)$.

We will use standard abbreviations. As usual $\Box F$ (respectively, $\overleftarrow{\Box} F$) is an abbreviation for $\neg(\text{True}\mathsf{Until}(\neg F))$ (respectively, $\neg(\text{True}\mathsf{Since}(\neg F))$), and $\mathsf{K}^+(F)$ (respectively, $\mathsf{K}^-(F)$) is an abbreviation for $\neg((\neg F)\mathsf{Until}\text{True})$ (respectively, $\neg((\neg F)\mathsf{Since}\text{True})$).

1. $\Box F$ (respectively, $\overleftarrow{\Box} F$) holds at t iff F holds everywhere after (respectively, before) t.
2. $\mathsf{K}^-(F)$ holds at a moment t iff $t = \sup(\{t' \mid t' < t \text{ and } F \text{ holds at } t'\})$.
3. $\mathsf{K}^+(F)$ holds at a moment t iff $t = \inf(\{t' \mid t' > t \text{ and } F \text{ holds at } t'\})$.

Note that $\mathsf{K}^+(\text{True})$ (respectively, $\mathsf{K}^-(\text{True})$) holds at t in $\mathcal{M}$ if t is a right limit (respectively, a left limit) point of the underlining order. In particular, both $\mathsf{K}^+(\text{True})$ and $\mathsf{K}^-(\text{True})$ are equivalent to False in the chains over $(\mathbb{N}, <)$,

2.3 Kamp's Theorem

Equivalence between temporal and monadic formulas is naturally defined: F is equivalent to $\varphi(x)$ over a class $\mathcal{C}$ of structures iff for any $\mathcal{M} \in \mathcal{C}$ and $t \in \mathcal{M}$: $\mathcal{M}, t \models F \Leftrightarrow \mathcal{M}, t \models \varphi(x)$. If $\mathcal{C}$ is the class of all chains, we will say that F is equivalent to φ.

A linear order $(T, <)$ is *Dedekind complete* if for every non-empty subset S of T, if S has a lower bound in T then it has a greatest lower bound, written $\inf(S)$, and if S has an upper bound in T then it has a least upper bound, written $\sup(S)$. The canonical linear time models $(\mathbb{N}, <)$ and $(\mathbb{R}, <)$ are Dedekind complete, while the order of the rationals is not Dedekind complete. A chain is Dedekind complete if its underlying linear order is Dedekind complete.

The fundamental theorem of Kamp's states that $TL(\mathsf{Until}, \mathsf{Since})$ is expressively equivalent to *FOMLO* over Dedekind complete chains.

▶ Theorem 2.1 (Kamp [7]). **1.** Given any $TL(\mathsf{Until}, \mathsf{Since})$ formula A there is a *FOMLO* formula $\varphi_A(x)$ which is equivalent to A over all chains.

2. Given any *FOMLO* formula $\varphi(x)$ with one free variable, there is a $TL(\mathsf{Until}, \mathsf{Since})$ formula which is equivalent to φ over Dedekind complete chains.

The meaning preserving translation from $TL(\mathsf{Until}, \mathsf{Since})$ to *FOMLO* is easily obtained by structural induction. The contribution of our paper is a proof of Theorem 2.1 (2). The proof is constructive. An algorithm which for every *FOMLO* formula $\varphi(x)$ constructs a $TL(\mathsf{Until}, \mathsf{Since})$ formula which is equivalent to φ over Dedekind complete chains is easily extracted from our proof. However, this algorithms is not efficient in the sense of complexity theory. This is unavoidable because there is a non-elementary succinctness gap between *FOMLO* and $TL(\mathsf{Until},\mathsf{Since})$ even over the class of finite chains, i.e., for every $m,n \in \mathbb{N}$ there is a *FOMLO* formula $\varphi(x_0)$ of size $|\varphi| > n$ which is not equivalent (even over finite chains) to any $TL(\mathsf{Until},\mathsf{Since})$ formula of size $\leq \exp(m,|\varphi|)$, where the m-iterated exponential function $\exp(m,n)$ is defined by induction on m so that $\exp(1,n) = 2^n$, and $\exp(m+1,n) = 2^{\exp(m,n)}$.

3 $\vec{\exists}\forall$ formulas

First, we introduce $\vec{\exists}\forall$ formulas which are instances of the Decomposition formulas of [3].

▶ Definition 3.1 ($\vec{\exists}\forall$-formulas). Let Σ be a set of monadic predicate names. An $\vec{\exists}\forall$-formula over Σ is a formula of the form:

$$\begin{aligned}
\psi(z_0,\dots,z_m) := \exists x_n \dots \exists x_1 \exists x_0 & \\
\left(\bigwedge_{k=0}^{m} z_k = x_{i_k}\right) \wedge (x_n > x_{n-1} > \dots > x_1 > x_0) & \quad \text{“ordering of } x_i \text{ and } z_j\text{”}\\
\wedge \bigwedge_{j=0}^{n} \alpha_j(x_j) & \quad \text{“Each } \alpha_j \text{ holds at } x_j\text{”}\\
\wedge \bigwedge_{j=1}^{n} [(\forall y)_{>x_{j-1}}^{<x_j} \beta_j(y)] & \quad \text{“Each } \beta_j \text{ holds along } (x_{j-1},x_j)\text{”}\\
\wedge (\forall y)_{>x_n} \beta_{n+1}(y) & \quad \text{“}\beta_{n+1} \text{ holds everywhere after } x_n\text{”}\\
\wedge (\forall y)^{<x_0} \beta_0(y) & \quad \text{“}\beta_0 \text{ holds everywhere before } x_0\text{”}
\end{aligned}$$

with a prefix of $n+1$ existential quantifiers and with all α_j, β_j quantifier free formulas with one variable over Σ. (ψ has $m+1$ free variables $z_0,\dots,z_m$ and $m+1 \leq n+1$ existential quantifiers are dummy and are introduced just in order to simplify notations.)

It is clear that

▶ Lemma 3.2. **1.** Conjunction of $\vec{\exists}\forall$-formulas is equivalent to a disjunction of $\vec{\exists}\forall$-formulas.

2. Every $\vec{\exists}\forall$-formula is equivalent to a conjunction of $\vec{\exists}\forall$-formulas with at most two free variables.

3. For every $\vec{\exists}\forall$-formula φ the formula $\exists x\varphi$ is equivalent to a $\vec{\exists}\forall$-formula.

▶ Definition 3.3 ($\vee\vec{\exists}\forall$-formulas). A formula is a $\vee\vec{\exists}\forall$ formula if it is equivalent to a disjunction of $\vec{\exists}\forall$-formulas.

▶ Lemma 3.4 (closure properties). The set of $\vee\vec{\exists}\forall$ formulas is closed under disjunction, conjunction, and existential quantification.

Proof. By (1) and (3) of Lemma 3.2, and distributivity of $\exists$ over $\vee$. ◀

The set of $\vee\vec{\exists}\forall$ formulas is not closed under negation[2]. However, we show later (see Proposition 4.3) that the negation of a $\vee\vec{\exists}\forall$ formula is equivalent to a $\vee\vec{\exists}\forall$ formula in the expansion of the chains by all $TL(\mathsf{Until},\mathsf{Since})$ definable predicates.

The $\vee\vec{\exists}\forall$ formulas with one free variable can be easily translated to $TL(\mathsf{Until},\mathsf{Since})$.

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[2] The truth table of $P\mathsf{Until}Q$ is an $\vec{\exists}\forall$ formula $(\exists x')_{>x}(Q(x') \wedge (\forall y)_{>x}^{<x'} P(y))$, yet we can prove that its negation is not equivalent to any $\vee\vec{\exists}\forall$ formula.

▶ Proposition 3.5 (From $\vee\overrightarrow{\exists}\forall$-formulas to $TL(\mathsf{Until}, \mathsf{Since})$ formulas). Every $\vee\overrightarrow{\exists}\forall$-formula with one free variable is equivalent to a $TL(\mathsf{Until}, \mathsf{Since})$ formula.

Proof. By a simple formalization we show that every $\overrightarrow{\exists}\forall$-formula with one free variable is equivalent to a $TL(\mathsf{Until}, \mathsf{Since})$ formula. This immediately implies the proposition.

Let $\psi(z_0)$ be an $\overrightarrow{\exists}\forall$-formula

$$\exists x_n \ldots \exists x_1 \exists x_0 z_0 = x_k \wedge \ (x_n > x_{n-1} > \cdots > x_1 > x_0) \wedge \ \bigwedge_{j=0}^{n} \alpha_j(x_j)$$

$$\wedge \ \bigwedge_{j=1}^{n} (\forall y)^{<x_j}_{>x_{j-1}} \beta_j(y) \wedge \ (\forall y)^{<x_0} \beta_0(y) \wedge \ (\forall y)_{>x_n} \beta_{n+1}(y)$$

Let A_i and B_i be temporal formulas equivalent to α_i and β_i (A_i and B_i do not even use Until and Since modalities). It is easy to see that ψ is equivalent to the conjunction of

$$A_k \wedge (B_{k+1}\mathsf{Until}(A_{k+1} \wedge (B_{k+2}\mathsf{Until} \cdots (A_{n-1} \wedge (B_n\mathsf{Until}(A_n \wedge \Box B_{n+1})) \cdots))$$

and

$$A_k \wedge (B_{k-1}\mathsf{Since}(A_{k-1} \wedge (B_{k-2}\mathsf{Since}(\cdots A_1 \wedge (B_1\mathsf{Since}(A_0 \wedge \overleftarrow{\Box} B_0)) \cdots))$$ ◀

4 Proof of Kamp's theorem

The next definition plays a major role in the proof of both Kamp's and Stavi's theorems [3].

▶ Definition 4.1. Let $\mathcal{M}$ be a Σ chain. We denote by $\mathcal{E}[\Sigma]$ the set of unary predicate names $\Sigma \cup \{A \mid A$ is an $TL(\mathsf{Until}, \mathsf{Since})$-formula over $\Sigma\}$. The canonical $TL(\mathsf{Until}, \mathsf{Since})$-expansion of $\mathcal{M}$ is an expansion of $\mathcal{M}$ to an $\mathcal{E}[\Sigma]$-chain, where each predicate name $A \in \mathcal{E}[\Sigma]$ is interpreted as $\{a \in \mathcal{M} \mid \mathcal{M}, a \models A\}$[3]. We say that first-order formulas in the signature $\mathcal{E}[\Sigma] \cup \{<\}$ are equivalent over $\mathcal{M}$ (respectively, over a class of Σ-chains $\mathcal{C}$) if they are equivalent in the canonical expansion of $\mathcal{M}$ (in the canonical expansion of every $\mathcal{M} \in \mathcal{C}$).

Note that if A is a $TL(\mathsf{Until}, \mathsf{Since})$ formula over $\mathcal{E}[\Sigma]$ predicates, then it is equivalent to a $TL(\mathsf{Until}, \mathsf{Since})$ formula over Σ, and hence to an atomic formula in the canonical $TL(\mathsf{Until}, \mathsf{Since})$-expansions.

From now on we say that "formulas are equivalent in a chain $\mathcal{M}$" instead of "formulas are equivalent in the canonical $TL(\mathsf{Until}, \mathsf{Since})$-expansion of $\mathcal{M}$." The $\overrightarrow{\exists}\forall$ and $\vee\overrightarrow{\exists}\forall$ formulas are defined as previously, but now they can use as atoms $TL(\mathsf{Until}, \mathsf{Since})$ definable predicates.

It is clear that all the results stated above hold for this modified notion of $\vee\overrightarrow{\exists}\forall$ formulas. In particular, every $\vee\overrightarrow{\exists}\forall$ formula with one free variable is equivalent to an $TL(\mathsf{Until}, \mathsf{Since})$ formula, and the set of $\vee\overrightarrow{\exists}\forall$ formulas is closed under conjunction, disjunction and existential quantification. However, now the set of $\vee\overrightarrow{\exists}\forall$ formulas is also closed under negation, due to the next proposition whose proof is postponed to Sect. 5.

▶ Proposition 4.2. (Closure under negation) The negation of $\overrightarrow{\exists}\forall$-formulas with at most two free variables is equivalent over Dedekind complete chains to a disjunction of $\overrightarrow{\exists}\forall$-formulas.

As a consequence we obtain

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[3] We often use "$a \in \mathcal{M}$" instead of "a is an element of the domain of $\mathcal{M}$"

▶ Proposition 4.3. Every first-order formula is equivalent over Dedekind complete chains to a disjunction of $\overrightarrow{\exists}\forall$-formulas.

Proof. We proceed by structural induction.

Atomic It is clear that every atomic formula is equivalent to a disjunction of (even quantifier free) $\overrightarrow{\exists}\forall$-formulas.

Disjunction - immediate.

Negation If φ is an $\overrightarrow{\exists}\forall$-formula, then by Lemma 3.2(2) it is equivalent to a conjunction of $\overrightarrow{\exists}\forall$ formulas with at most two free variables. Hence, $\neg\varphi$ is equivalent to a disjunction of $\neg\psi_i$ where ψ_i are $\overrightarrow{\exists}\forall$-formulas with at most two free variables. By Proposition 4.2, $\neg\psi_i$ is equivalent to a disjunction of $\overrightarrow{\exists}\forall$ formulas γ_i^j. Hence, $\neg\varphi$ is equivalent to a disjunction $\bigvee_i \bigvee_j \gamma_i^j$ of $\overrightarrow{\exists}\forall$ formulas.
If φ is a disjunction of $\overrightarrow{\exists}\forall$ formulas φ_i, then $\neg\varphi$ is equivalent to the conjunction of $\neg\varphi_i$. By the above, $\neg\varphi_i$ is equivalent to a $\vee\overrightarrow{\exists}\forall$ formula. Since, $\vee\overrightarrow{\exists}\forall$ formulas are closed under conjunction (Lemma 3.4), we obtain that $\neg\varphi$ is equivalent to a disjunction of $\overrightarrow{\exists}\forall$ formulas.

∃-quantifier For ∃-quantifier, the claim follows from Lemma 3.4. ◀

Now, we are ready to prove Kamp's Theorem:

▶ Theorem 4.4. For every *FOMLO* formula $\varphi(x)$ with one free variable, there is a $TL(\mathsf{Until}, \mathsf{Since})$ formula which is equivalent to φ over Dedekind complete chains.

Proof. By Proposition 4.3, $\varphi(x)$ is equivalent over Dedekind complete chains to a disjunction of $\overrightarrow{\exists}\forall$ formulas $\varphi_i(x)$. By Proposition 3.5, $\varphi_i(x)$ is equivalent to a $TL(\mathsf{Until}, \mathsf{Since})$ formula. Hence, $\varphi(x)$ is equivalent over Dedekind complete chains to a $TL(\mathsf{Until}, \mathsf{Since})$ formula. ◀

This completes our proof of Kamp's theorem except Proposition 4.2 which is proved in the next section.

5 Proof of Proposition 4.2

Let $\psi(z_0, z_1)$ be an $\overrightarrow{\exists}\forall$-formula

$$\exists x_n \dots \exists x_1 \exists x_0 [z_0 = x_m \wedge z_1 = x_k \wedge\ (x_0 < x_1 < \dots < x_{n-1} < x_n) \wedge \bigwedge_{j=0}^{n} \alpha_j(x_j)$$

$$\wedge \bigwedge_{j=1}^{n} (\forall y)_{>x_{j-1}}^{<x_j} \beta_j(y) \wedge\ (\forall y)^{<x_0} \beta_0(y) \wedge\ (\forall y)_{>x_n} \beta_{n+1}(y)]$$

We consider two cases. In the first case $k = m$, i.e., $z_0 = z_1$ and in the second $k \neq m$.

If $k = m$, then ψ is equivalent to $z_0 = z_1 \wedge \psi'(z_0)$, where ψ' is an $\overrightarrow{\exists}\forall$-formula. By Proposition 3.5, ψ' is equivalent to an $TL(\mathsf{Until}, \mathsf{Since})$ formula A'. Therefore, ψ is equivalent to an $\overrightarrow{\exists}\forall$-formula $\exists x_0[z_0 = x_0 \wedge z_1 = x_0 \wedge A'(x_0)]$.

If $k \neq m$, w.l.o.g. we assume that $m < k$. Hence, ψ is equivalent to a conjunction of

1. $\psi_0(z_0)$ defined as:
$\exists x_0 \dots \exists x_{m-1} \exists x_m [z_0 = x_m \wedge\ (x_0 < x_1 < \dots < x_m) \wedge\ \bigwedge_{j=0}^{m} \alpha_j(x_j)$
$$\wedge \bigwedge_{j=1}^{m} (\forall y)_{>x_{j-1}}^{<x_j} \beta_j(y) \wedge\ (\forall y)^{<x_0} \beta_0(y)$$

2. $\psi_1(z_1)$ defined as:
$\exists x_k \dots \exists x_{k+1} \exists x_n [z_1 = x_k \wedge \ (x_k < x_{k+1} < \dots < x_n) \wedge \ \bigwedge_{j=k}^{n} \alpha_j(x_j)$

$$\wedge \bigwedge_{j=k+1}^{n} (\forall y)^{<x_j}_{>x_{j-1}} \beta_j(y) \wedge \ (\forall y)_{>x_n} \beta_{n+1}(y)]$$

3. $\varphi(z_0, z_1)$ defined as:
$\exists x_m \dots \exists x_k [(z_0 = x_m < x_{m+1} < \dots < x_k = z_1) \wedge \ \bigwedge_{j=m}^{k} \alpha_j(x_j)$

$$\wedge \bigwedge_{j=m+1}^{k} (\forall y)^{<x_j}_{>x_{j-1}} \beta_j(y)$$

The first two formulas are $\overrightarrow{\exists}\forall$-formulas with one free variable. Therefore, (by Proposition 3.5) they are equivalent to a $TL(\mathsf{Until}, \mathsf{Since})$ formulas (in the signature $\mathcal{E}[\Sigma]$). Hence, their negations are equivalent (over the canonical expansions) to atomic (and hence to $\overrightarrow{\exists}\forall$) formulas.

Therefore, it is sufficient to show that the negation of the third formula is equivalent over Dedekind complete chains to a disjunction of $\overrightarrow{\exists}\forall$-formulas. This is stated in the following lemma:

▶ **Lemma 5.1.** The negation of any formula of the form

$$\exists x_0 \dots \exists x_n [(z_0 = x_0 < \dots < x_n = z_1) \wedge \bigwedge_{j=0}^{n} \alpha_j(x_j) \wedge \bigwedge_{j=1}^{n} (\forall y)^{<x_j}_{>x_{j-1}} \beta_j(y)] \qquad (1)$$

where α_i, β_i are quantifier free, is equivalent (over Dedekind complete chains) to a disjunction of $\overrightarrow{\exists}\forall$-formulas.

In the rest of this section we prove Lemma 5.1. Our proof is organized as follows. In Lemma 5.3 we prove an instance of Lemma 5.1 where α_0, α_n and all β_i are equivalent to True. Then we derive a more general instance (Corollary 5.4) where β_n is equivalent to true. Finally we prove the full version of Lemma 5.1.

First, we introduce some helpful notations.

▶ **Notations 5.2.** We use the abbreviated notation $[\alpha_0, \beta_1 \dots, \alpha_{n-1}, \beta_n \alpha_n](z_0, z_1)$ for the $\overrightarrow{\exists}\forall$-formula as in (1).

In this notation Lemma 5.1 can be rephrased as $\neg[\alpha_0, \beta_1 \dots, \alpha_{n-1}, \beta_n \alpha_n](z_0, z_1)$ is equivalent (over Dedekind complete chains) to a $\vee\overrightarrow{\exists}\forall$ formula.

We start with the instance of Lemma 5.1 where all β_i are True.

▶ **Lemma 5.3.** $\neg \exists x_1 \dots \exists x_n \, (z_0 < x_1 < \dots < x_n < z_1) \wedge \bigwedge_{i=1}^{n} P_i(x_i)$ is equivalent over Dedekind complete chains to a $\vee\overrightarrow{\exists}\forall$ formula $O_n(P_1, \dots, P_n, z_0, z_1)$.

Proof. We proceed by induction.

Basis: $\neg(\exists x_1)^{<z_1}_{>z_0} P_1(x_1)$ is equivalent to $(\forall y)^{<z_1}_{>z_0} \neg P_1(y)$.

Inductive step: $n \mapsto n+1$. We assume that a $\vee\overrightarrow{\exists}\forall$ formula O_n was defined and construct a $\vee\overrightarrow{\exists}\forall$ formula O_{n+1}.

Observe that if the interval (z_0, z_1) is non-empty, then one of the following cases holds:

Case 1 There is no occurrence of P_1 in (z_0, z_1), i.e. $(\forall y)^{<z_1}_{>z_0} \neg P_1(y)$.

In this case $O_{n+1}(P_1, \dots, P_{n+1}, z_0, z_1)$ should be equivalent to True.

Case 2 If case 1 does not hold then let $r_0 = \inf\{z \in (z_0, z_1) \mid P_1(z)\}$ (such r_0 exists by Dedekind completeness. Note that $r_0 = z_0$ iff $\mathbf{K}^+(P_1)(z_0)$. If $r_0 > z_0$ then $r_0 \in (z_0, z_1)$ and r_0 is definable by the following $\vee\overrightarrow{\exists}\forall$ formula:

$$INF(z_0, r_0, z_1, P_1) := z_0 < r_0 < z_1 \wedge (\forall y)^{<r_0}_{>z_0} \neg P_1(y) \wedge \\ \wedge (P_1(r_0) \vee \mathbf{K}^+(P_1)(r_0)) \tag{2}$$

Subcase $r_0 = z_0$
In this subcase $O_n(P_2, \dots, P_n, z_0, z_1)$ and $O_{n+1}(P_1, \dots, P_{n+1}, z_0, z_1)$ should be equivalent.

Subcase $r_0 \in (z_0, z_1)$
In this subcase $O_n(P_2, \dots, P_n, r_0, z_1)$ and $O_{n+1}(P_1, \dots, P_{n+1}, z_0, z_1)$ should be equivalent.

Hence, $O_{n+1}(P_1, \dots, P_{n+1}, z_0, z_1)$ can be defined as the disjunction of "(z_0, z_1) is empty" and the following formulas:

1. $(\forall y)^{<z_1}_{>z_0} \neg P_1((y)$
2. $\mathbf{K}^+(P_1)(z_0) \wedge O_n(P_2, \dots, P_n, z_0, z_1)$
3. $(\exists r_0)^{<z_1}_{>z_0} \big(INF(z_0, r_0, z_1, P_1) \wedge O_n(P_2, \dots, P_n, r_0, z_1)\big)$

The first formula is a $\vee\overrightarrow{\exists}\forall$ formula. By the inductive assumptions O_n is a $\vee\overrightarrow{\exists}\forall$ formula. $\mathbf{K}^+(P_1)(z_0)$ is an atomic (and hence a $\vee\overrightarrow{\exists}\forall$) formula in the canonical expansion, and $INF(z_0, r_0, z_1, P_1)$ is a $\vee\overrightarrow{\exists}\forall$ formula. Since $\vee\overrightarrow{\exists}\forall$ formulas are closed under conjunction, disjunction and the existential quantification, we conclude that O_{n+1} is a $\vee\overrightarrow{\exists}\forall$ formula. ◀

As a consequence we obtain

▶ Corollary 5.4. **1.** $\neg(\exists z)^{<z_1}_{>z_0}[\alpha_0, \beta_1, \alpha_1, \beta_2, \dots, \alpha_{n-1}, \beta_n, \alpha_n](z_0, z)$ is equivalent over Dedekind complete chains to a $\vee\overrightarrow{\exists}\forall$ formula.

2. $\neg(\exists z)^{<z_1}_{>z_0}[\alpha_0, \beta_1, \alpha_1, \beta_2, \dots, \alpha_{n-1}, \beta_n, \alpha_n](z, z_1)$ is equivalent over Dedekind complete chains to a $\vee\overrightarrow{\exists}\forall$ formula.

Proof. (1) Define

$$F_n := \alpha_n$$
$$F_{i-1} := \alpha_{i-1} \wedge \beta_i \mathsf{Until} F_i \qquad \text{for } i = 1, \dots, n$$

Observe that there is $z \in (z_0, z_1)$ such that $[\alpha_0, \beta_1, \alpha_1, \beta_2, \dots, \alpha_{n-1}, \beta_n, \alpha_n](z_0, z)$ iff $F_0(z_0)$ and there is an increasing sequence $x_1 < \dots < x_n$ in an open interval (z_0, z_1) such that $F_i(x_i)$ for $i = 1, \dots, n$. Indeed, the direction $\Rightarrow$ is trivial. The direction $\Leftarrow$ is easily proved by induction.

The *basis* is trivial.

Inductive step: $n \mapsto n+1$. Assume $F_0(z_0)$ holds and that (z_0, z_1) contains an increasing sequence $x_1 < \dots < x_{n+1}$ such that $F_i(x_i)$ for $i = 1, \dots, n+1$. By the inductive assumption there is $y_1 \in (z_0, x_{n+1})$ such that

$$[\alpha_0, \beta_1, \alpha_1, \beta_2, \dots, \beta_{n-1}\alpha_{n-1}, \beta_n, (\alpha_n \wedge \beta_{n+1} \mathsf{Until} \alpha_{n+1})](z_0, y_1).$$

In particular, y_1 satisfies $(\alpha_n \wedge \beta_{n+1} \mathsf{Until} \alpha_{n+1})$. Hence, there is $y_2 > y_1$ such that y_2 satisfies α_{n+1} and β_{n+1} holds along (y_1, y_2).

If $y_2 \leq x_{n+1}$ then the required $z \in (z_0, z_1)$ equals to y_2, and we are done. Otherwise, $x_{n+1} < y_2$. Therefore, $x_{n+1} \in (y_1, y_2)$ and β_{n+1} holds along (y_1, x_{n+1}). Hence, the required z equals to x_{n+1}.

From the above observation and Lemma 5.3, it follows that $\neg F_0(z_0) \vee O_n(F_1, \ldots, F_n, z_0, z_1)$ is a $\vee\overrightarrow{\exists}\forall$ formula that is equivalent to $\neg(\exists z)^{<z_1}_{>z_0}[\alpha_0, \beta_1, \alpha_1, \beta_2, \ldots, \alpha_{n-1}, \beta_n, \alpha_n](z_0, z)$.

(2) is the mirror image of (1) and is proved similarly. ◀

Now we are ready to prove Lemma 5.1, i.e.,

$$\neg[\alpha_0, \beta_1 \ldots, \beta_{n-1}, \alpha_{n-1}, \beta_n, \alpha_n](z_0, z_1) \text{ is equivalent over Dedekind complete chains to a } \vee\overrightarrow{\exists}\forall \text{ formula.}$$

If the interval (z_0, z_1) is empty then the assertion is immediate. We assume that (z_0, z_1) is non-empty. Hence, at least one of the following cases holds:

Case 1 $\neg\alpha_0(z_0)$ or $\neg\alpha_n(z_1)$ or $\neg(\beta_1 \mathsf{Until} \alpha_1)(z_0)$ or $\neg(\beta_n \mathsf{Since} \alpha_{n-1})(z_1)$.

Case 2 $\alpha_0(z_0)$, and β_1 holds along (z_0, z_1).

Case 3

1. $\alpha_0(z_0) \wedge (\beta_1 \mathsf{Until} \alpha_1)(z_0)$, and
2. there is $x \in (z_0, z_1)$ such that $\neg\beta_1(x)$.

For each of these cases we construct a $\vee\overrightarrow{\exists}\forall$ formula $Cond_i$ which describes it (i.e., Case i holds iff $Cond_i$ holds) and show that if $Cond_i$ holds, then $\neg[\alpha_0, \beta_1 \ldots, \beta_{n-1}, \alpha_{n-1}, \beta_n, \alpha_n](z_0, z_1)$ is equivalent to a $\vee\overrightarrow{\exists}\forall$ formula $Form_i$. Hence, $\neg[\alpha_0, \beta_1 \ldots, \beta_{n-1}, \alpha_{n-1}, \beta_n, \alpha_n](z_0, z_1)$ is equivalent to $\vee_i[Cond_i \wedge Form_i]$ which is a $\vee\overrightarrow{\exists}\forall$ formula.

Case 1 This case is already explicitly described by the $\vee\overrightarrow{\exists}\forall$ formula (in the canonical expansion). In this case $\neg[\alpha_0, \beta_1 \ldots, \beta_{n-1}, \alpha_{n-1}, \beta_n, \alpha_n](z_0, z_1)$ is equivalent to True.

Case 2 This case is described by a $\vee\overrightarrow{\exists}\forall$ formula $\alpha_0(z_0) \wedge (\forall z)^{<z_1}_{>z_0}\beta_1$. In this case $\neg[\alpha_0, \beta_1 \ldots, \beta_{n-1}, \alpha_{n-1}, \beta_n, \alpha_n](z_0, z_1)$ is equivalent to "there is no $z \in (z_0, z_1)$ such that $[\alpha_1, \beta_2 \ldots, \beta_n, \alpha_n](z, z_1)$." By Corollary 5.4 this is expressible by a $\vee\overrightarrow{\exists}\forall$ formula.

Case 3 The first condition of Case 3 is already explicitly described by a $\vee\overrightarrow{\exists}\forall$ formula. When the first condition holds, then the second condition is equivalent to "there is (a unique) $r_0 \in (z_0, z_1)$ such that $r_0 = \inf\{z \in (z_0, z_1) \mid \neg\beta_1(z)\}$" (If $\beta_1 \mathsf{Until} \alpha_1$ holds at z_0 and there is $x \in (z_0, z_1)$ such that $\neg\beta_1(x)$, then such r_0 exists because we deal with Dedekind complete chains.) This r_0 is definable by the following $\vee\overrightarrow{\exists}\forall$ formula, i.e., it is a unique z which satisfies it[4]:

$$INF^{\neg\beta_1}(z_0, z, z_1) := z_0 < z < z_1 \wedge (\forall y)^{<z}_{>z_0}\beta_1(y) \wedge (\neg\beta_1(z) \vee \mathbf{K}^+(\neg\beta_1)(z)) \quad (3)$$

Hence, Case 3 is described by $\alpha_0(z_0) \wedge (\beta_1 \mathsf{Until} \alpha_1)(z_0) \wedge (\exists z)^{<z_1}_{>z_0} INF^{\neg\beta_1}(z_0, z, z_1)$. Since the set of $\vee\overrightarrow{\exists}\forall$ formulas is closed under conjunction, disjunction and $\exists$, this case is described by a $\vee\overrightarrow{\exists}\forall$ formula.

It is sufficient to show that $(\exists z)^{<z_1}_{>z_0} INF^{\neg\beta_1}(z) \wedge \neg[\alpha_0, \beta_1, \alpha_1, \ldots, \beta_{n+1}, \alpha_{n+1}](z_0, z_1)$ is equivalent to a $\vee\overrightarrow{\exists}\forall$ formula.

We prove this by induction on n.

The *basis* is trivial.

Inductive step $n \mapsto n+1$.

[4] We will use only existence and will not use uniqueness.

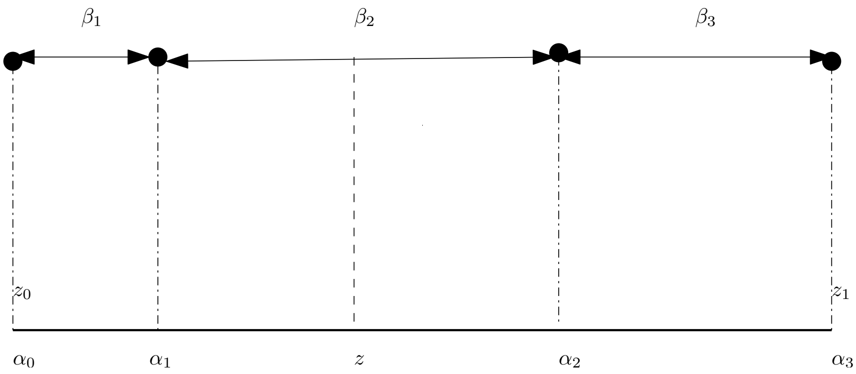

Figure 1 $B_2(z_0, z, z_1) := [\alpha_0, \beta_1, \alpha_1, \beta_2, \beta_2](z_0, z) \wedge [\beta_2, \beta_2, \alpha_2, \beta_3, \alpha_3](z, z_1)$.

Define:

$$\begin{aligned}
A_i^-(z_0, z) :=& [\alpha_0, \beta_1, \ldots, \beta_i, \alpha_i](z_0, z) && i = 1, \ldots, n\\
A_i^+(z, z_1) :=& [\alpha_i, \beta_{i+1}, \ldots \beta_{n+1}\alpha_{n+1}](z, z_1) && i = 1, \ldots, n\\
A_i(z_0, z, z_1) :=& A_i^-(z_0, z) \wedge A_i^+(z, z_1) && i = 1, \ldots, n\\
B_i^-(z_0, z) :=& [\alpha_0\beta_1, \ldots, \beta_{i-1}, \alpha_{i-1}, \beta_i, \beta_i](z_0, z) && i = 1, \ldots, n+1\\
B_i^+(z, z_1) :=& [\beta_i, \beta_i, \alpha_i\beta_{i+1}\alpha_{i+1}, \ldots, \beta_{n+1}, \alpha_{n+1}](z, z_1) && i = 1, \ldots, n+1\\
B_i(z_0, z, z_1) :=& B_i^-(z_0, z) \wedge B_i^+(z, z_1) && i = 1, \ldots, n+1
\end{aligned}$$

If the interval (z_0, z_1) is non-empty, these definitions imply

$$[\alpha_0, \beta_1, \alpha_1, \ldots, \beta_{n+1}, \alpha_{n+1}](z_0, z_1) \Leftrightarrow (\forall z)^{<z_1}_{>z_0}\Big(\bigvee_{i=1}^{n} A_i \vee \bigvee_{i=1}^{n+1} B_i\Big)$$

$$[\alpha_0, \beta_1, \alpha_1, \ldots, \beta_{n+1}, \alpha_{n+1}](z_0, z_1) \Leftrightarrow (\exists z)^{<z_1}_{>z_0}\Big(\bigvee_{i=1}^{n} A_i \vee \bigvee_{i=1}^{n+1} B_i\Big)$$

Hence, for every φ

$$(\exists z)^{<z_1}_{>z_0}\varphi(z) \wedge \neg[\alpha_0, \beta_1, \alpha_1, \ldots, \beta_{n+1}, \alpha_{n+1}](z_0, z_1)$$

is equivalent to

$$(\exists z)^{<z_1}_{>z_0}\Big(\varphi(z) \wedge \bigwedge_{i=1}^{n} \neg A_i \wedge \bigwedge_{i=1}^{n+1} \neg B_i\Big)$$

In particular,

$$(\exists z)^{<z_1}_{>z_0} INF^{\neg\beta_1}(z) \wedge \neg[\alpha_0, \beta_1, \alpha_1, \ldots, \beta_{n+1}, \alpha_{n+1}](z_0, z_1)$$

is equivalent to

$$(\exists z)^{<z_1}_{>z_0}\Big(INF^{\neg\beta_1}(z) \wedge \bigwedge_{i=1}^{n} \neg A_i \wedge \bigwedge_{i=1}^{n+1} \neg B_i\Big),$$

where $INF^{\neg\beta_1}(z)$ was defined in equation (3).

By the inductive assumption

(a) $\neg A_i$ is equivalent to a $\vee\overrightarrow{\exists}\forall$ formula for $i = 1, \ldots, n$.

(b) $\neg B_i$ is equivalent to a $\vee\overrightarrow{\exists}\forall$ formula for $i = 2, \ldots, n$.

Recall $B_1 := B_1^- \wedge B_1^+$ and $B_{n+1} := B_{n+1}^- \wedge B_{n+1}^+$.

(c) $\neg B_1^-$ and $\neg B_{n+1}^+$ are equivalent to $\vee\overrightarrow{\exists}\forall$ formulas, by the induction basis.

(d) $INF^{\neg\beta_1}(z) \wedge \neg B_1^+(z, z_1)$ is equivalent to $INF^{\neg\beta_1}(z)$, because if $INF^{\neg\beta_1}(z)$, then for no $x > z$, β_1 holds along $[z, x)$.

(e) $INF^{\neg\beta_1}(z) \wedge \neg B_{n+1}^-(z_0, z)$ is equivalent to $INF^{\neg\beta_1}(z) \wedge$ ("β_1 holds on (z_0, z)" $\wedge$ $\neg B_{n+1}^-(z_0, z)$). Since, by case 2, "β_1 holds on (z_0, z)" $\wedge \neg B_{n+1}^-(z_0, z)$ is equivalent to a $\vee\overrightarrow{\exists}\forall$ formula, and $INF^{\neg\beta_1}(z)$ is a $\vee\overrightarrow{\exists}\forall$ formula, we conclude that $INF^{\neg\beta_1}(z) \wedge \neg B_{n+1}^-(z_0, z)$ is equivalent to a $\vee\overrightarrow{\exists}\forall$ formula.

Since the set of $\vee\overrightarrow{\exists}\forall$ formulas is closed under conjunction, disjunction and $\exists$, by (a)-(e) we obtain that $(\exists z)_{>z_0}^{<z_1}\big(INF^{\neg\beta_1}(z) \wedge \bigwedge_{i=1}^{n} \neg A_i \wedge \bigwedge_{i=1}^{n+1} \neg B_i\big)$ is equivalent to a $\vee\overrightarrow{\exists}\forall$ formula. Therefore, $(\exists z)_{>z_0}^{<z_1} INF^{\neg\beta_1}(z) \wedge \neg[\alpha_0, \beta_1, \alpha_1, \ldots, \beta_{n+1}, \alpha_{n+1}](z_0, z_1)$ is also a $\vee\overrightarrow{\exists}\forall$ formula.

This completes our proof of Lemma 5.1 and of Proposition 4.2.

6 Related Works

Kamp's theorem was proved in

1. Kamp's thesis [7] (proof > 100pages).
2. Outlined by Gabbay, Pnueli, Stavi and Shelah [3] (Sect. 2) for $\mathbb{N}$ and stated that it can be extended to Dedekind complete orders using game arguments.
3. Was proved by Gabbay [1] by separation arguments for $\mathbb{N}$, and extended to Dedekind complete order in [2].
4. Was proved by Hodkinson [4] by game arguments and simplified in [5] (unpublished).

A temporal logic has the *separation* property if its formulas can be equivalently rewritten as a boolean combination of formulas, each of which depends only on the past, present or future. The separation property was introduced by Gabbay [1], and surprisingly, a temporal logic which can express $\Box$ and $\overleftarrow{\Box}$ has the separation property (over a class $\mathcal{C}$ of structures) iff it is expressively complete for *FOMLO* over $\mathcal{C}$.

The separation proof for $TL(\mathsf{Until}, \mathsf{Since})$ over $\mathbb{N}$ is manageable; however, over the real (and over Dedekind complete) chains it contains many rules and transformations and is not easy to follow. Hodkinson and Reynolds [6] write:

> The proofs of theorems 18 and 19 [Kamp's theorem over naturals and over reals, respectively] are direct, showing that each formula can be separated. They are tough and tougher, respectively. Nonetheless, they are effective, and so, whilst not quite providing an algorithm to determine if a set of connectives is expressively complete, they do suggest a potential way of telling in practice whether a given set of connectives is expressively complete – in Gabbay's words, *try to separate and see where you get stuck!*

The game arguments are easier to grasp, but they use complicated inductive assertions. The proof in [5] proceeds roughly as follows. Let $\mathcal{L}_r$ be the set of $TL(\mathsf{Until}, \mathsf{Since})$ formulas of nesting depth at most r. A formula of the form: $\exists\bar{x}\forall y\chi(\bar{x}, y, \bar{z})$ where $\bar{x}$ is an n-tuple of variables and χ is a quantifier free formula over $\{<, =\}$ and $\mathcal{L}_r$-definable monadic predicates is called $\langle n, r\rangle$-decomposition formula. The main inductive assertion is proved by "unusual back-and-forth games" and can be rephrased in logical terms as there is a function f :

$\mathbb{N} \to \mathbb{N}$ such that for every $n, r \in \mathbb{N}$, the negation of positive Boolean combinations $\langle n, r\rangle$-decomposition formula is equivalent to a positive Boolean combination of $\langle f(n), (n+r)\rangle$-decomposition formulas.

Our proof is inspired by [3] and [5]; however, it avoids games, and it separates general logical equivalences and temporal arguments.

Many temporal formalisms studied in computer science concern only future formulas - whose truth value at any moment is determined by what happens from that moment on. A formula (temporal, or monadic with a single free first-order variable) F is (*semantically*) *future* if for every chain $\mathcal{M}$ and moment $t_0 \in \mathcal{M}$:

$$\mathcal{M}, t_0 \models F \text{ iff } \mathcal{M}|_{\geq t_0}, t_0 \models F,$$

where $\mathcal{M}|_{\geq t_0}$ is the subchain of $\mathcal{M}$ over the interval $[t_0, \infty)$. For example, $P\mathsf{Until}Q$ and $\mathsf{K}^+(P)$ are future formulas, while $P\mathsf{Since}Q$ and $\mathsf{K}^-(P)$ are not future ones.

It was shown in [3] that over the *discrete* chains Kamp's theorem holds also for *future formulas* of *FOMLO*:

▶ Theorem 6.1. Every future *FOMLO* formula is equivalent over discrete orders (Natural, Integer, finite) to a *TL*(Until) formula.

Theorems 6.1 can be easily obtained from our proof of Kamp's theorem.

The temporal logic with the modalities Until and Since is not expressively complete for *FOMLO* over the rationals. Stavi introduced two additional modalities Until^s and Since^s and proved that $TL(\mathsf{Until}, \mathsf{Since}, \mathsf{Until}^s, \mathsf{Since}^s)$ is expressively complete for *FOMLO* over all linear orders [2]. In the full version of this paper we prove Stavi's theorem. The proof is similar to our proof of Kamp's theorem; however, it treats some additional cases related to gaps in orders, and replaces $\overrightarrow{\exists}\forall$-formulas by slightly more general formulas.

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