

Decidability Issues for Two-Variable Logics with Several Linear Orders *

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Abstract

We show that the satisfiability and the finite satisfiability problems for two-variable logic, FO^2 , over the class of structures with three linear orders, are undecidable. This sharpens an earlier result that FO^2 with eight linear orders is undecidable. The theorem holds for a restricted case in which linear orders are the only non-unary relations. Recently, a contrasting result has been shown, that the finite satisfiability problem for FO^2 with two linear orders and with no additional non-unary relations is decidable. We observe that our proof can be adapted to some interesting fragments of FO^2 , in particular it works for the two-variable guarded fragment, GF^2 , even if the order relations are used only as guards. Finally, we show that GF^2 with an arbitrary number of linear orders which can be used only as guards becomes decidable if except linear orders only unary relations are allowed.

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1 Introduction

In the field of logic in computer science the two-variable fragment of first order logic, FO^2 , plays a prominent role. With respect to the number of variables it appears to be the maximal fragment whose satisfiability problem is decidable. The importance of FO^2 can be justified by the fact that it, or its natural extensions and variants, embeds many formalisms used in computer science, such as modal, temporal or description logics.

The decidability of FO^2 was shown in [18] by establishing a finite model property, namely, that every satisfiable formula has a finite model of size at most doubly exponential with respect to its length. This bound on the size of models was later improved in [7] to singly exponential, which implied **NExpTime**-upper bound on the complexity of the satisfiability problem. A corresponding lower bound follows from [15, 5], so the satisfiability problem for FO^2 is **NExpTime**-complete.

One particular drawback of FO^2 is that it cannot express transitivity of a binary relation. Similarly, it is not possible to say that a relation is, e.g., an equivalence relation or a linear order. Such properties of relations are very natural and desirable in practical applications. Thus researchers started to investigate FO^2 over restricted classes of structures, in which some distinguished binary symbols have to be interpreted as transitive relations, equivalences, or linear orders. The idea for such a kind of research comes from the world of modal logics, where, e.g., in Kripke structures for multimodal logic **K4** accessibility relations are transitive and for multimodal logic **S5** they are equivalences. Linear orders are very natural

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when we consider temporal logics, where they model time flow, but can be also applicable in different scenarios, like in databases or description logics, to compare objects with respect to some parameters.

Unfortunately, the results are generally negative. It appeared that both the satisfiability and the finite satisfiability problems for FO^2 are undecidable in the presence of several equivalence or several transitive relations [8, 9]. These results were later strengthened: FO^2 is undecidable in the presence of two transitive relations [11, 10], three equivalence relations [12], one transitive and one equivalence relation [14], or eight linear orders [19]. On the positive side it is known that FO^2 with one or two equivalence relations [12, 14], or with one linear order [19] are decidable.

A related line of work, motivated by XML, concerns the so called *data words*. A data word is a word over a finite alphabet. Positions of a word are naturally ordered by the linear order and may be related by an equivalence relation (such an equivalence relation models equality of data values). It was shown in [3] that FO^2 is decidable over data words, even in the case when except the linear order we are allowed to use the associated successor relation. Some other interesting results related to data words have been recently obtained in [4], [17] and [20]. In particular it is shown in [17] that FO^2 is decidable over words whose positions are ordered by two linear orders, with the assumption that the orders are only accessible by the successor relations.

In this paper we perform a next step towards completing the classification of FO^2 with linear orders. We show that the satisfiability and the finite satisfiability problems for FO^2 are undecidable in the presence of three linear orders. The proof works for a restricted language, in which, besides three linear orders, only unary predicates are used. This theorem improves the above mentioned result from [19], where eight linear orders were used. It also sharpens a theorem from [21] that FO^2 is undecidable in the presence of two linear orders and one total preorder. Our result seems to be optimal with respect to the number of linear orders, since it contrasts with the main theorem from [21], that the finite satisfiability problem for FO^2 with a linear order and a total preorder (and thus also for FO^2 with two linear orders) is decidable. The proof of the last result works only in the case in which the order relations are the only non-unary symbols; it is very likely however that it can be extended to the general case.

It is an interesting question if there exists a natural decidable fragment of FO^2 in which elements could be compared by an unbounded number of linear orders (or, at least, by more than two orders). When looking for analogous fragments with transitive or equivalence relations the attention is often turned to the two-variable *guarded fragment*, GF^2 . In the guarded fragment each occurrence of a quantifier has to be relativised by an atomic formula containing all the variables that are free in the scope of this quantifier, e.g. $\forall xy(x < y \rightarrow (Px \wedge Qy \wedge Bxy))$. The guarded fragment was introduced in [1] to simulate the way accessibility relations in modal logics or roles in description logics are used. The satisfiability problem for the guarded fragment is 2ExpTime -complete and for its two-variable version – ExpTime -complete [6].

It appeared that GF^2 is decidable with an arbitrary number of transitive or equivalence relations, if the usage of transitive or equivalence symbols is restricted only to guards [22, 11, 13]. This last restriction is natural, since to the obtained fragment we may still translate multimodal logics K5 , S4 or some description logics with transitive roles.

In the case of linear orders the situation appears to be different. Our undecidability proof for FO^2 can be easily adapted to the case of GF^2 , even if linear orders are allowed to appear only as guards. This can be done by enforcing that some additional binary relations

are identical to the linear orders. On the other hand, if we assume that linear orders are the only non-unary symbols and are used only as guards then GF^2 becomes decidable. The obtained variant allows only for a very limited interaction among different linear orders (in fact, because of the syntactic restrictions, such interaction can be obtained only in an indirect way), however it seems that not much more can be done: we explain that, e.g., extending the fragment by allowing guards built from conjunctions of atoms instead of just atoms, e.g. guards like $x \leq_1 y \wedge x \leq_2 y \wedge y \leq_3 x$, leads to undecidability.

The organisation of the paper is as follows. In Section 2 we present our main undecidability result for FO^2 with three linear orders and discuss some of its refinements. In Section 3 we show that GF^2 with an arbitrary number of linear orders is decidable if linear orders are used only as guards and if there are no additional non-unary symbols.

2 Undecidability

2.1 Tilings and grids

The reduction of the tiling problem to satisfiability of some variants of two-variable logic was presented in [8, 9]. Some ramifications, particularly suited for the case of linear orders, were given in [19]. For convenience we present (adaptations of) some basic definitions and lemmas (without proofs) from [19].

Let $\mathfrak{G}_{\mathbb{Z}}$ be the canonical grid structure on $\mathbb{Z} \times \mathbb{Z}$: $\mathfrak{G}_{\mathbb{Z}} = (\mathbb{Z}^2, H, V)$, $H = \{((p, q), (p + 1, q)) : p, q \in \mathbb{Z}\}$, $V = \{((p, q), (p, q + 1)) : p, q \in \mathbb{Z}\}$. Similarly, let $\mathfrak{G}_{\mathbb{N}}$ be the canonical grid on $\mathbb{N} \times \mathbb{N}$ and let \mathfrak{G}_m denote the standard grid on a finite $m \times m$ torus: $\mathfrak{G}_m = (\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}, H, V)$, $H = \{((p, q), (p', q)) : p' - p \equiv 1 \pmod{m}\}$, $V = \{((p, q), (p, q')) : q' - q \equiv 1 \pmod{m}\}$.

Let $\mathfrak{G}_i = (G_i, H_i, V_i)$, $i = 1, 2$. \mathfrak{G}_1 is *homomorphically embeddable* into \mathfrak{G}_2 if there is a homomorphism $\pi : \mathfrak{G}_1 \rightarrow \mathfrak{G}_2$, i.e. a mapping π such that for all $v, v' \in G_1$: $(v, v') \in H_1 \Rightarrow (\pi(v), \pi(v')) \in H_2$ and $(v, v') \in V_1 \Rightarrow (\pi(v), \pi(v')) \in V_2$.

We are interested in structures which are grid-like in the following sense.

► **Definition 1.** An infinite structure $\mathfrak{G} = (G, H, V)$ is called *grid-like* if $\mathfrak{G}_{\mathbb{N}}$ is homomorphically embeddable into \mathfrak{G} ; a finite \mathfrak{G} is grid-like if some \mathfrak{G}_m is homomorphically embeddable into \mathfrak{G} .

Grid-likeness is implied by a simple local criterion. We say that H is *complete over V* in $\mathfrak{G} = (G, H, V)$ if \mathfrak{G} satisfies $\forall xyx'y'((Hxy \wedge Vxx' \wedge Vyy') \rightarrow Hx'y')$.

► **Lemma 2.** Assume that $\mathfrak{G} = (G, H, V)$ satisfies the FO^2 -axiom $\forall x(\exists yHxy \wedge \exists yVxy)$. If H is complete over V , then \mathfrak{G} is grid-like.

► **Lemma 3.** Let \mathcal{C} be a class of structures. If there exists an FO^2 sentence η such that:

- (a) $\mathfrak{G}_{\mathbb{Z}}$ can be expanded to a structure in \mathcal{C} satisfying η ,
- (b) for every $n \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that \mathfrak{G}_{kn} can be expanded to a structure in \mathcal{C} satisfying η ,
- (c) every model of η from \mathcal{C} is grid-like,

then both satisfiability and finite satisfiability of FO^2 over \mathcal{C} are undecidable. In fact FO^2 forms even a conservative reduction class over \mathcal{C} . If at least (a) and (c) hold then the (general) satisfiability problem is undecidable.

For a more detailed exposition of the technique see [19]. For some background on conservative reduction classes see [2].

The general idea of our proof of the undecidability of FO^2 with three linear orders is similar to the idea from the proof for the case of eight linear orders from [19], however details are much trickier.

To postpone some technical problems and present the main ideas of the proof clearly, in the first instance we consider only the (general) satisfiability case. First, we describe the expansion $\bar{\mathfrak{G}}_{\mathbb{Z}}$ of the standard infinite $\mathbb{Z} \times \mathbb{Z}$ grid by three linear orders \leq_1, \leq_2, \leq_3 and some unary predicates. Then we construct a formula η capturing some important properties of $\bar{\mathfrak{G}}_{\mathbb{Z}}$. We argue that every model of η , interpreting symbols \leq_i as linear orders, is grid-like. By Lemma 3 this implies the undecidability of the satisfiability problem for FO^2 over the class of structures with three linear orders.

Further, we describe expansions of the finite $12k \times 12k$ grids, $\bar{\mathfrak{G}}'_{12k}$ in a signature containing some additional unary symbols. We modify slightly the formula η , obtaining η' which will be satisfied in $\bar{\mathfrak{G}}'_{12k}$ for all $k \in \mathbb{N}$. It will also appear that η' satisfies all assumptions of Lemma 3, which shows that FO^2 forms a conservative reduction class (in particular the finite satisfiability problem is undecidable) over the class of structures with three linear orders.

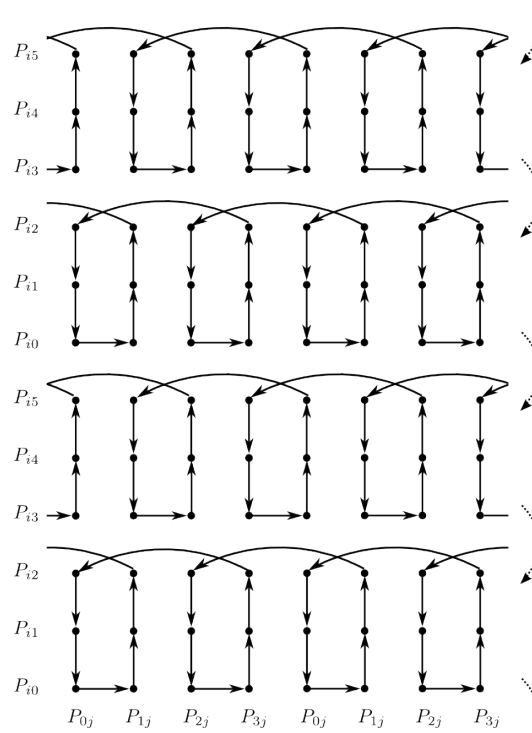
2.2 Intended infinite model

We describe the expansion $\bar{\mathfrak{G}}_{\mathbb{Z}}$ of the standard $\mathbb{Z} \times \mathbb{Z}$ grid. The basic repeating pattern of the grid expansion consists of 24 elements, forming a 4×6 rectangle. To distinguish types of elements inside such rectangles we use unary predicates P_{ij} , $0 \leq i \leq 3$, $0 \leq j \leq 5$. Namely, if $a = (k, l)$ then $\bar{\mathfrak{G}}_{\mathbb{Z}} \models P_{ij}a$ if and only if $i = k \bmod 4$ and $j = l \bmod 6$.

In Fig. 1 we illustrate the order \leq_1 . The set of elements $\mathbb{Z} \times \mathbb{Z}$ is divided into horizontal \leq_1 -zones, each of them consisting of three rows of elements. Formally, the \leq_1 -zones are defined as $Z_k^{\leq_1} = \{(i, 3k), (i, 3k + 1), (i, 3k + 2) : i \in \mathbb{Z}\}$ for $k \in \mathbb{Z}$. If $a \in Z_k^{\leq_1}$, $b \in Z_l^{\leq_1}$, and $k < l$ then $\bar{\mathfrak{G}}_{\mathbb{Z}} \models b \leq_1 a$. The points in a zone are organised in U-shaped six-element blocks, called \leq_1 -blocks. If for elements $a, b \in Z_k^{\leq_1}$, a belongs to a \leq_1 -block located to the left from the \leq_1 -block of b then $\bar{\mathfrak{G}}_{\mathbb{Z}} \models b \leq_1 a$. Look at Fig. 1 to see the \leq_1 -ordering inside the \leq_1 -blocks. Note that the \leq_1 -blocks in the odd zones are shifted by 1 with respect to the even zones.

The orders \leq_2 and \leq_3 follow the same pattern, but are shifted with respect to the order \leq_1 . To obtain the picture for \leq_2 we shift the picture for \leq_1 by the vector $(1, 1)$. Similarly, the picture for \leq_3 is obtained by shifting the picture for \leq_1 by $(0, 2)$. This implies that the zones determined by different order relations do not coincide. See Fig. 2 to see how \leq_i -blocks of all three orders are located in the grid. For clarity \leq_i -relations are shown only inside \leq_i -blocks. Recall that \leq_i -arrows among \leq_i -zones go from up to down and among the \leq_i -blocks inside a zone – from right to left.

Fig. 3 shows relations \leq_1 and \leq_3 between the neighbouring points from two consecutive rows of the grid. Note that elements connected by H or by V are related by \leq_1, \leq_3 incompatibly. This observation extends to the crucial property of $\bar{\mathfrak{G}}_{\mathbb{Z}}$, which will be used to axiomatise the grid relations H, V : all three orders coincide on points whose y -coordinates differ by at least 3, or which belong to the same row and their x -coordinates differ by at least 2; and, on the other hand, points connected by the grid relation H or by the grid relation V are related incompatibly by some pair of the orders. We state it precisely in the following observation. We also formalise another important property of our grid expansion (part (iv)) which will be captured by the formula η ; this will allow to show that in all models of η , H is complete over V .



■ **Figure 1** The order \leq_1 . Solid arrows represent the successor relation, dotted arrows illustrate relations among \leq_1 -zones. The lower-left element is the point $(0,0)$.

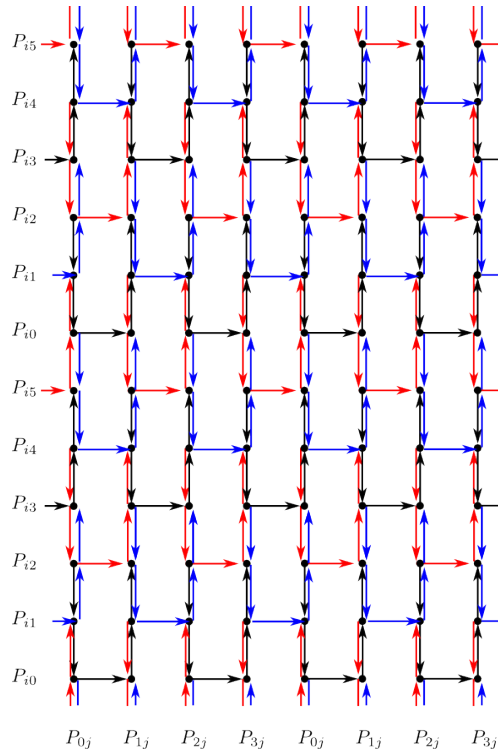
- **Observation 4.** (i) Let $(k, l), (k', l') \in \mathbb{Z} \times \mathbb{Z}$ be two points in $\bar{\mathfrak{G}}_{\mathbb{Z}}$, such that $l' - l \geq 3$. Then for all i we have $(k', l') \leq_i (k, l)$.
- (ii) Let $(k, l), (k', l') \in \mathbb{Z} \times \mathbb{Z}$ be two points in $\bar{\mathfrak{G}}_{\mathbb{Z}}$, such that $l = l'$ and $k' - k \geq 2$. Then for all i we have $(k', l') \leq_i (k, l)$.
- (iii) If (k, l) and (k', l') are connected by H or by V , i.e, if $k = k'$ and $l' - l = 1$ or $l = l'$ and $k' - k = 1$, then there exist i, j such that $(k, l) \leq_i (k', l')$ and $(k', l') \leq_j (k, l)$ or $(k, l) \leq_j (k', l')$ and $(k', l') \leq_i (k, l)$. Namely, if $l \bmod 3 = 0$ then $i = 1, j = 3$, if $l \bmod 3 = 1$ then $i = 1, j = 2$, and if $l \bmod 3 = 2$ then $i = 2, j = 3$.
- (iv) For all points $a, b, c, d \in \mathbb{Z} \times \mathbb{Z}$, if $\bar{\mathfrak{G}}_{\mathbb{Z}} \models Vba \wedge Hbc \wedge Vcd$, then there exist i, j such that $a \leq_i b \leq_i c \leq_i d$ and $d \leq_j c \leq_j b \leq_j a$. Namely, if $b = (k, l)$ then i, j can be chosen as in point (iii).

Proof. Claim (i) follows from the fact that for all i the point (k', l') belongs to a \leq_i -zone located above the zone of (k, l) . Claim (ii) follows from the fact that for all i both points belong to the same \leq_i -zone, and that for all orders the point (k, l) belongs to a \leq_i -block located to the left from the \leq_i -block of (k', l') . Claims (iii),(iv) follow from an inspection of Fig. 2. ◀

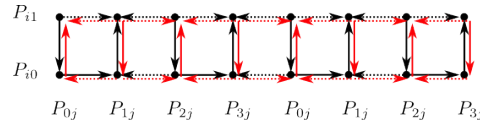
2.3 The formula η

The formula η consists of four conjuncts $\eta = \eta_G \wedge \eta_H \wedge \eta_V \wedge \eta_C$. The first conjunct explicitly enforces horizontal and vertical successors in the grid:

$$\eta_G = \forall x(\exists yHxy \wedge \exists yVxy).$$



■ **Figure 2** U-shaped blocks in the orders: \leq_1 (black arrows), \leq_2 (blue arrows), \leq_3 (red arrows). For clarity only successor \leq_i -connections inside \leq_i -blocks are shown.



■ **Figure 3** Relations \leq_1 (black arrows) and \leq_3 (red arrows) between the neighbouring points from two consecutive rows. Solid arrows represent successor relations, dotted arrows represent non-successor \leq_i -relations.

The next conjunct η_H axiomatises H :

$$\eta_H = \forall xy (Hxy \leftrightarrow \bigvee_{\substack{0 \leq i \leq 3 \\ 0 \leq j \leq 5}} (P_{ij}x \wedge P_{i+1,j}y \wedge \lambda_{ij}^H(x, y))),$$

where $i + 1$ is calculated modulo 4, and $\lambda_{ij}^H(x, y)$ says how points x, y are related by two of the three orders; namely λ_{i0}^H and λ_{i3}^H speak about \leq_1 and \leq_3 , λ_{i1}^H and λ_{i4}^H speak about \leq_1 and \leq_2 , λ_{i2}^H and λ_{i5}^H speak about \leq_2 and \leq_3 , e.g.:

$$\lambda_{00}^H = x \leq_1 y \wedge y \leq_3 x,$$

$$\lambda_{01}^H = x \leq_1 y \wedge y \leq_2 x,$$

$$\lambda_{31}^H = y \leq_1 x \wedge x \leq_2 y.$$

The next conjunct η_V speaks about V -connections. It is similar to η_H , however this time we impose only the implication from left to right:

$$\eta_V = \forall xy (Vxy \rightarrow \bigvee_{\substack{0 \leq i \leq 3 \\ 0 \leq j \leq 5}} (P_{ij}x \wedge P_{i,j+1}y \wedge \lambda_{ij}^V(x, y)),$$

where $j + 1$ is calculated modulo 6; again λ_{i0}^V and λ_{i3}^V speak about \leq_1 and \leq_3 , λ_{i1}^V and λ_{i4}^V speak about \leq_1 and \leq_2 , λ_{i2}^V and λ_{i5}^V speak about \leq_2 and \leq_3 , e.g.:

$$\lambda_{00}^V = y \leq_1 x \wedge x \leq_3 y,$$

$$\lambda_{10}^V = x \leq_1 y \wedge y \leq_3 x,$$

$$\lambda_{21}^V = y \leq_1 x \wedge x \leq_2 y.$$

Finally, η_C says that some points, related incompatibly by two of the three orders are connected by the third one in a specific way:

$$\eta_C = \forall xy \bigwedge_{\substack{0 \leq i \leq 3 \\ 0 \leq j \leq 5}} ((P_{ij}x \wedge P_{i+1,j}y) \rightarrow \kappa_{ij}(x, y)),$$

where $i + 1$ is calculated modulo 4. Formulae κ_{i0} and κ_{i3} enforce \leq_1 relations, κ_{i1} and κ_{i4} – \leq_2 relations, κ_{i2} and κ_{i5} – \leq_3 relations, e.g.:

$$\kappa_{01} = (x \leq_1 y \wedge y \leq_3 x) \rightarrow y \leq_2 x$$

$$\kappa_{15} = (x \leq_1 y \wedge y \leq_2 x) \rightarrow x \leq_3 y$$

It can be readily checked that $\bar{\mathfrak{G}}_{\mathbb{Z}}$ is a model of η . In particular, the implication from right to left in η_H does not impose any unwanted H -connections. Indeed, by Observation 4 (i), (ii) the formula $\bigvee_{i,j} (P_{ij}a \wedge P_{i+1,j}b \wedge \lambda_{ij}^H(a, b))$ is not satisfied by non-neighbouring points a, b of the grid, since $\lambda_{ij}^H(a, b)$ says that a, b are related incompatibly by some two orders.

2.4 Grid-likeness

Now let us see that every model $\mathfrak{M} \models \eta$ interpreting \leq_1, \leq_2, \leq_3 as linear orders¹ is grid like. Since $\mathfrak{M} \models \eta_G$, by Lemma 2, it suffices to check that H is complete over V . Assume, that $a, b, c, d \in M$ are such that $\mathfrak{M} \models Hbc \wedge Vba \wedge Vcd$. We want to see that $\mathfrak{M} \models Had$. We need to consider several cases, depending on the P_{ij} -type of b . Let us go through one of them. Assume that $\mathfrak{M} \models P_{00}b$. Then, the implications from left to right in η_H and η_V imply

$$\mathfrak{M} \models P_{10}c \wedge P_{01}a \wedge P_{11}d \wedge a \leq_1 b \wedge b \leq_1 c \wedge c \leq_1 d \wedge d \leq_3 c \wedge c \leq_3 b \wedge b \leq_3 a.$$

Since \leq_1 and \leq_3 are linear orders, and thus transitive, it follows that $\mathfrak{M} \models a \leq_1 d \wedge d \leq_3 a$. Now, consider η_C . It follows that $\mathfrak{M} \models \kappa_{01}(a, d)$. The implication in κ_{01} guarantees that $\mathfrak{M} \models d \leq_2 a$. Thus $\mathfrak{M} \models P_{01}a \wedge P_{11}d \wedge \lambda_{01}^H(a, d)$, so the implication from right to left in η_H finally enforces $\mathfrak{M} \models Had$.

The remaining cases can be treated in a similar way.

This finishes the proof of the undecidability of the general satisfiability problem. To obtain the undecidability of finite satisfiability we need to work further on some details.

¹ Actually, for grid-likeness it is enough to assume that they are interpreted as transitive relations.

2.5 Finite models

We describe first how to construct our intended expansions $\bar{\mathfrak{G}}'_{12k}$ of the standard grids on $12k \times 12k$ tori, for $k \geq 1$. Then we explain how to modify η to a formula η' which is satisfied in such expansions, without losing the property that all models of η' are grid-like.

To simplify the presentation we describe the grid expansion $\bar{\mathfrak{G}}'_{12}$. The larger grid expansions are constructed analogously. Let $\bar{\mathfrak{G}}_{12}$ be the restriction of $\bar{\mathfrak{G}}_{\mathbb{Z}}$ to the set $\{0, \dots, 11\} \times \{0, \dots, 11\}$ in which additionally, for all l , the element $(11, l)$ is connected by H to $(0, l)$, and the element $(l, 11)$ is connected by V to $(l, 0)$ (i.e. the sides of the square are appropriately glued). $\bar{\mathfrak{G}}_{12}$ is not a model of η for some trivial reasons; among other things the implication from left to right in η_H is violated, e.g. $(11, 1)$ is inappropriately related to $(0, 1)$ by \leq_1 .

Thus we slightly modify $\bar{\mathfrak{G}}_{12}$ to obtain $\bar{\mathfrak{G}}'_{12}$. In Fig. 4 the order \leq_1 is shown. The \leq_1 -zones are defined analogously to the infinite case. In all odd zones, i.e. zones built from elements of types P_{i3}, P_{i4}, P_{i5} , the \leq_1 -connections remain as they are in $\bar{\mathfrak{G}}_{12}$. The \leq_1 -connections are modified in even zones. We describe the zone built from the rows 0, 1, 2. The element $(10, 2)$ is made the minimal element in this zone. The next elements in the order are $(10, 1)$, $(10, 0)$, as in $\bar{\mathfrak{G}}_{12}$, the next one however is not $(11, 0)$ but $(8, 2)$. Then the order coincides with the order in $\bar{\mathfrak{G}}_{12}$ until the element $(1, 2)$ is reached. Its successor is $(11, 0)$, the next element is $(11, 1)$, and the maximal element in this zone is $(11, 2)$. More intuitively, we may think that the rightmost U in $\bar{\mathfrak{G}}_{12}$ is cut into two parts: the left one is made minimal in the zone and the right one – maximal, with respect to \leq_1 .

Analogously to the case of infinite models, to obtain the pictures for \leq_2 and \leq_3 we shift the picture for \leq_1 by the vectors $(1, 1)$ and $(0, 2)$, respectively, taking into account that this time shifts are made on a torus, so, e.g., the minimal element with respect to \leq_2 will be the element $(0, 0)$.

We also introduce new unary symbols $S_0 - S_3$, intended to mark four consecutive columns of the grid (columns 10, 11, 0, 1 in our example) and $Z_0 - Z_3$, intended to mark four consecutive rows of the grid (rows 2, 1, 0, 11 in our example). Their relevance will become clear in a moment.

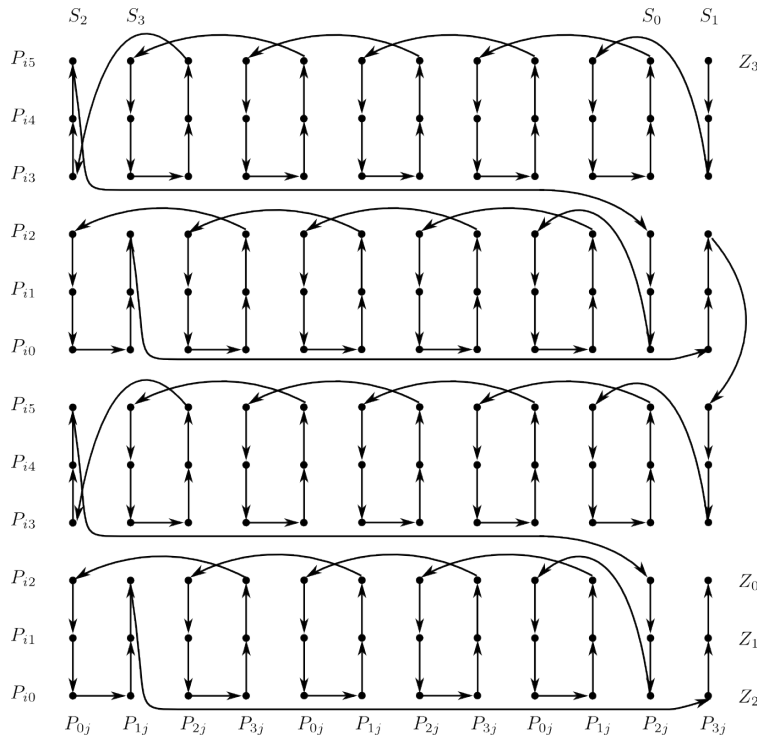
The described structure $\bar{\mathfrak{G}}'_{12}$ satisfies η_G, η_V and the implication from left to right in η_H . Unfortunately, parts (i) and (ii) of Observation 4 are not true this time, which makes the implication from right to left in η_H not satisfied. Let us explain why.

Note first that η_H enforces H -connections between distant elements from the same row. Consider e.g. the element $(11, 1)$. In its row this element is maximal with respect to \leq_1 and minimal with respect to \leq_2 . Thus η_H enforces a H -connection e.g. from $(2, 1)$ to $(11, 1)$.

Similarly, some unwanted H -connections are enforced also between elements from different zones. Each of the orders divides the set of elements into four zones. In Fig. 5 it is shown how \leq_1 -, \leq_2 - and \leq_3 -zones are related by \leq_1, \leq_2 and \leq_3 , respectively. Note that the elements in the row marked Z_1 belong to the \leq_3 -zone which is minimal with respect to \leq_3 , and to the \leq_2 - and \leq_3 -zones which are maximal with respect to, resp., \leq_1 and \leq_3 . Similarly, the elements in the row marked Z_2 belong to the \leq_2 - and \leq_3 -zones which are minimal with respect to, resp., \leq_2 and \leq_3 , and to the \leq_1 -zone which is maximal with respect to \leq_1 . This means that η_H enforces some unwanted H -connections to (or from) Z_1 and Z_2 , from (or to) some distant elements in the grid, e.g. the element $(1, 0)$ should be connected by H to $(2, 6)$.

To fix the problems we use the mentioned unary relations S_0, S_1, S_2, S_3 and Z_0, Z_1, Z_2, Z_3 . Let

$$\alpha^S(x, y) = (S_0x \wedge S_1y) \vee (S_1x \wedge S_2y) \vee (S_2x \wedge S_3y) \vee (\neg S_1x \wedge \neg S_1y \wedge \neg S_2x \wedge \neg S_2y),$$



■ **Figure 4** The order \leq_1 in the finite grid $\bar{\mathfrak{G}}'_{12}$

$$\alpha^Z(x, y) = \bigwedge_{0 \leq i \leq 3} (Z_i x \leftrightarrow Z_i y).$$

For $\bar{\mathfrak{G}}'_{12k}$ we have now a slightly weaker observation than Observation 4 part (i) and (ii).

► **Observation 5.** Let $a = (k, l), b = (k', l') \in \mathbb{Z} \times \mathbb{Z}$ be two distinct points in $\bar{\mathfrak{G}}'_{12k}$, such that $\bar{\mathfrak{G}}'_{12k} \models \alpha^S(a, b) \wedge \alpha^Z(a, b)$. Then:

- (i) If the distance in the torus between the row l and the row l' is at least 3 then for all i we have $(k', l') \leq_i (k, l)$ or for all i we have $(k, l) \leq_i (k', l')$.
- (ii) If $l = l'$ and the distance in the torus between columns k' and k is at least 2 then for all i we have $(k', l') \leq_i (k, l)$ or for all i we have $(k, l) \leq_i (k', l')$.

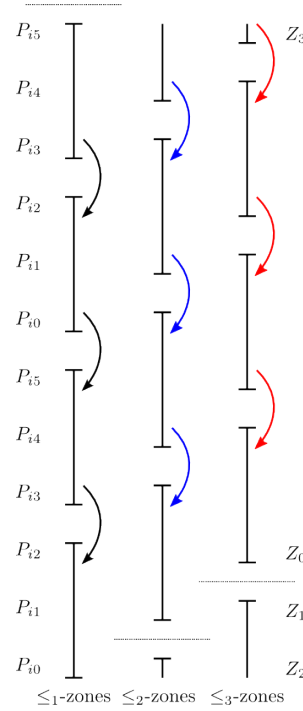
Proof. To see claim (i) note that elements a, b cannot belong to the rows marked by Z_i (since they satisfy $\alpha^Z(a, b)$). For the remaining rows an argument similar to the argument from the proof of Observation 4 (i) works. To see claim (ii) note that the elements a, b cannot belong to the columns marked S_1 or S_2 (since they satisfy $\alpha^S(a, b)$). For the remaining columns an argument similar to the argument from the proof of Observation 4 (ii) works. ◀

Observe that points (iii) and (iv) from Observation 4 remain true.

We modify η to allow and impose H -connection to S_1 only from S_0 , to S_2 only from S_1 , from S_1 only to S_2 , from S_2 only to S_3 ; and to Z_i only from Z_i , for $0 \leq i \leq 3$.

We change η_H to:

$$\eta'_H = \forall xy (Hxy \leftrightarrow (\alpha^S(x, y) \wedge \alpha^Z(x, y) \wedge \bigvee_{\substack{0 \leq i \leq 3 \\ 0 \leq j \leq 5}} (P_{ij}x \wedge P_{i+1,j}y \wedge \lambda_{ij}^H(x, y))).$$



■ **Figure 5** The \leq_1 - \leq_2 - and \leq_3 -zones in $\bar{\mathfrak{G}}'_{12}$

We define additional conjuncts which say that V -connected elements are given consistent S_i - and Z_i -values:

$$\eta_S = \bigwedge_{0 \leq i \leq 3} \forall xy (Vxy \rightarrow (S_i x \leftrightarrow S_i y)),$$

$$\eta_Z = \forall xy (Vxy \rightarrow ((\delta(x) \wedge \delta(y)) \vee (\delta(x) \wedge Z_3 y) \vee (Z_0 x \wedge \delta(y)) \vee \bigvee_{1 \leq i \leq 3} (Z_i x \wedge Z_{i-1} y))),$$

where $\delta(x) = \bigwedge_{0 \leq i \leq 3} \neg Z_i x$.

We modify also η_C , since it may also generate some unwanted relations:

$$\eta'_C = \forall xy \bigwedge_{\substack{0 \leq i \leq 3 \\ 0 \leq j \leq 5}} (P_{ij} x \wedge P_{i+1,j} y \wedge \alpha^S(x, y) \wedge \alpha^Z(x, y) \rightarrow \kappa_{ij}(x, y)).$$

Finally, let the conjunct η_U says that every element satisfies at most one of the S_i -predicates and at most one of the Z_i -predicates, and the S_i and Z_i -values imply proper P_{ij} values, e.g. $\forall x (Z_0 x \rightarrow (P_{02} x \vee P_{12} x \vee P_{22} x \vee P_{32} x))$.

Now let $\eta' = \eta_G \wedge \eta'_H \wedge \eta_V \wedge \eta_S \wedge \eta_Z \wedge \eta'_C \wedge \eta_U$. Every grid \mathfrak{G}_{12k} can now be expanded to a model $\bar{\mathfrak{G}}'_{12k}$ of η' analogously to the described expansion of \mathfrak{G}_{12} . Also the infinite grid \mathfrak{G}_Z has an expansion to a model of η' . It is enough to take $\bar{\mathfrak{G}}_Z$ and mark columns $-2, -1, 0, 1$ with, resp., S_0, S_1, S_2, S_3 , and rows $2, 1, 0, -1$ with, resp., Z_0, Z_1, Z_2, Z_3 .

Let us finally sketch a fragment of the argument that every model \mathfrak{M} of η' interpreting \leq_i as linear orders is grid-like. Assume, that $a, b, c, d \in M$ are such that $\mathfrak{M} \models Hbc \wedge Vba \wedge Vcd$. We want to see that $\mathfrak{M} \models Had$. This time the cases we have to consider are distinguished not only by the values of P_{ij} but also by the values of the additional relations S_i, Z_i . Let us

consider one of the cases, namely, $\mathfrak{M} \models P_{00}b \wedge S_2b \wedge Z_2b$. Then, the implication from left to right in η'_H , and the formulae η_V , η_S , and η_Z imply

$$\mathfrak{M} \models P_{10}c \wedge P_{01}a \wedge P_{11}d \wedge S_2a \wedge Z_1a \wedge S_3c \wedge Z_2c \wedge S_3d \wedge Z_1d$$

and

$$\mathfrak{M} \models a \leq_1 b \wedge b \leq_1 c \wedge c \leq_1 d \wedge d \leq_3 c \wedge c \leq_3 b \wedge b \leq_3 a.$$

Since \leq_1 and \leq_3 are linear orders, and thus transitive, it follows that $\mathfrak{M} \models a \leq_1 d \wedge d \leq_3 a$. Now, consider η'_C . Note that $\alpha^S(a, d)$ and $\alpha^Z(a, d)$ are true. It follows that $\mathfrak{M} \models \kappa_{01}(a, d)$. The implication in κ_{01} guarantees that $\mathfrak{M} \models d \leq_2 a$. Thus $\mathfrak{M} \models P_{01}a \wedge P_{11}d \wedge \lambda_{01}^H(a, d) \wedge \alpha^S(a, d) \wedge \alpha^Z(a, d)$. Finally, the implication from right to left in η_H enforces $\mathfrak{M} \models Had$.

We left the remaining cases to the reader.

We have proved that FO^2 forms a conservative reduction class over the class of structures with three linear orders.

2.6 Remarks on the proof and discussion

In our proof we use the binary symbols H and V . They are convenient to present the construction but do not play a crucial role. In a reduction from the tiling problem they can be simulated by combinations of unary predicates and the order relations. Namely, in η the conjunct η_G can be substituted by the conjunction of formulae enforcing for every x the existence of two elements related to x by the linear orders in a specific way: $\bigwedge_{i,j} \forall x (P_{ij}x \rightarrow \exists y (\eta_{ij}^H(x, y) \wedge P_{i+1,j}y) \wedge \exists y (\eta_{ij}^V(x, y) \wedge P_{i,j+1}y))$. The formulae η_H and η_V can then be omitted. Thus we obtain the following, strong version of our main undecidability result.

► **Theorem 6.** *FO^2 forms a conservative reduction class over the structures with three linear orders and no additional non-unary symbols.*

A question arises whether there exists an elegant and useful fragment of FO^2 which is decidable in the presence of an arbitrary number of linear orders (or at least in the presence of three linear orders). A natural candidate is the two-variable guarded fragment, GF^2 . Let us recall the definition of the guarded fragment. The guarded fragment, GF , of first-order logic is defined as the least set of formulae such that: (i) every atomic formula belongs to GF ; (ii) GF is closed under logical connectives $\neg, \vee, \wedge, \rightarrow$; and (iii) quantifiers are relativised by atoms, i.e. if $\varphi(\mathbf{x}, \mathbf{y})$ is a formula of GF and $\gamma(\mathbf{x}, \mathbf{y})$ is an atomic formula containing all the free variables of φ , then the formulae $\forall \mathbf{y}(\gamma(\mathbf{x}, \mathbf{y}) \rightarrow \varphi(\mathbf{x}, \mathbf{y}))$ and $\exists \mathbf{y}(\gamma(\mathbf{x}, \mathbf{y}) \wedge \varphi(\mathbf{x}, \mathbf{y}))$ belong to GF . The atoms $\gamma(\mathbf{x}, \mathbf{y})$ are called *guards*.

Syntactically, not all of the formulae we use in our undecidability proof are guarded. However there is no problem to make them guarded, since linear orders are total and thus can be used as guards if necessary, e.g. $\forall xy\psi(x, y)$ can be rewritten as $\forall xy(x \leq_1 y \rightarrow \psi(x, y)) \wedge \forall xy(y \leq_1 x \rightarrow \psi(x, y))$. Thus GF^2 is undecidable in the presence of three linear orders. This situation is similar to the case of GF^2 with equivalence or transitive relations, which are also undecidable (with three equivalences [12], and with two transitive relations [11, 10]). However if we restrict the usage of special relations (i.e. equivalence or transitive relations) to guards only, then GF^2 becomes decidable, with an arbitrary number of special relations [22, 11, 13]. Unfortunately, a similar restriction does not help in the case of linear orders. A simple formula $\forall xy(x \leq_i y \rightarrow (x \neq y \rightarrow (R_i xy \wedge \neg R_i yx))) \wedge \forall x R_i xx$, in which R_i is a fresh binary symbol, enforces R_i to behave exactly as \leq_i . Thus R_i can replace all occurrences of \leq_i outside guards.

► **Corollary 7.** *The satisfiability and the finite satisfiability problems for GF^2 , in the class of structures with three linear orders, are undecidable even if linear orders are used only in guards.*

We emphasise that to obtain Corollary 7 some binary symbols except linear orders are required. It appears that if we allow only unary symbols except linear orders and allow to use linear orders only as guards then, as it is argued in the next section, GF^2 becomes decidable with an arbitrary number of linear orders. The obtained decidable variant, which will be called *monadic* GF^2 , allows only for a very restricted interaction among different linear orders (in fact, because of the syntactic restrictions such interaction can be obtained only in an indirect way). However, it seems that the situation cannot be improved too much. For example, if instead of just linear orders we allow conjunctions of linear orders as guards then the logic becomes undecidable. This fact can be inferred using the exponential translation of FO^2 to a variant of Boolean modal logic from [16], but can be also proved directly, by observing that the formulae we construct in Section 2 can be rewritten to the desired variant. Indeed, consider the place which looks most problematically, i.e. the formula η_C . It says e.g. that elements x, y satisfying $P_{01}x$ and $P_{11}y$ which are related by \leq_1 and \leq_3 in the following way: $x \leq_1 y \wedge y \leq_3 x$ should satisfy also $y \leq_2 x$. This can be enforced by saying: $\forall xy((x \leq_1 y \wedge y \leq_3 x \wedge x \leq_2 y) \rightarrow (\neg P_{01}x \vee \neg P_{11}y))$. Again we use the fact that x, y has to be connected by \leq_2 and we only forbid the connection in the unwanted direction.

► **Corollary 8.** *The satisfiability and the finite satisfiability problems for the extension of monadic GF^2 with three linear orders, which allows conjunctions of atoms of the form $x \leq_i y$ and $y \leq_i x$ (for $i = 1, 2, 3$) as guards, are undecidable.*

3 Decidability

In this section we work with signatures of the form $(\sigma, \leq_1, \dots, \leq_k)$, where σ is a set of unary symbols and \leq_i are binary symbols. We assume that the equality is also allowed. Formally, monadic GF^2 is the fragment of GF^2 containing formulae over such signatures in which symbols \leq_1, \dots, \leq_k are used only as guards. We consider satisfiability of monadic GF^2 in the class of structures in which \leq_1, \dots, \leq_k are interpreted as linear orders, which we denote as $\mathcal{LIN}(\leq_1, \dots, \leq_k)$. We will simply say that a monadic GF^2 sentence φ has a model (is satisfiable, finitely satisfiable) if it has a model (is satisfiable, finitely satisfiable) in $\mathcal{LIN}(\leq_1, \dots, \leq_k)$.

A 1-type (over σ) is a subset of σ . If α is a 1-type then we denote by $\alpha(x)$ the conjunction of the atoms Px , for all $P \in \alpha$, and the atoms $\neg Qx$, for all $Q \notin \alpha$. For a given structure \mathfrak{A} we say that an element a realises a type α if $\mathfrak{A} \models \alpha(a)$.

► **Definition 9.** A monadic GF^2 sentence φ is in *normal form* if it is a conjunction of formulae of the following form:

- $\exists x(\gamma(x) \wedge \psi(x))$,
- $\forall x(\gamma(x) \rightarrow \exists y(x \leq_i y \wedge x \neq y \wedge \psi(x, y)))$,
- $\forall x(\gamma(x) \rightarrow \exists y(y \leq_i x \wedge x \neq y \wedge \psi(x, y)))$,
- $\forall x(\gamma(x) \rightarrow \psi(x))$,
- $\forall xy(x \leq_i y \rightarrow (x \neq y \rightarrow \psi(x, y)))$.

where all $\gamma(x)$ are atomic formulae (possibly of the form $x = x$), and $\psi(x)$, $\psi(x, y)$ are quantifier-free formulae over monadic vocabulary σ .

The (finite) satisfiability problem for monadic GF^2 can be reduced to the (finite) satisfiability problem for disjunctions of exponential number of linearly bounded monadic GF^2

sentences in normal form. See [22] for the proof of a similar result. Since we are going to show that (finite) satisfiability is in NExpTime it is enough to consider formulae in normal form.

The decidability proof for monadic GF^2 is based on the proof for FO^2 with one linear order from [19]. Roughly speaking, after fixing the universe, the (slightly simplified) construction from [19] is applied here to the particular orders. Below we present a sketch of the proof.

3.1 General satisfiability

► **Definition 10.** Let a tuple $(\mathcal{T}, \mathcal{K}, \mathcal{S}^1, \dots, \mathcal{S}^k)$ be such that:

- \mathcal{T} is a set of 1-types over σ ,
- \mathcal{K} is a subset of \mathcal{T} , called the set of *royal* 1-types,
- for every $1 \leq i \leq k$, $\mathcal{S}^i = (S_1^i, \dots, S_{k_i}^i)$ is a sequence of subsets of \mathcal{T} , such that $\bigcup_{j=1}^{k_i} S_j^i = \mathcal{T}$, each type from \mathcal{K} belongs to exactly one set from \mathcal{S}^i , and the types from \mathcal{K} appear only in singletons.

We say that such a tuple is a *certificate of satisfiability* for a normal form monadic GF^2 sentence φ if the following conditions hold:

- (a) For every conjunct of φ of the form $\exists x(\gamma(x) \wedge \psi(x))$ there exists a type $\alpha \in \mathcal{T}$ such that $\alpha(x) \models \gamma(x) \wedge \psi(x)$.
- (b) For every i, j , for every type $\alpha \in S_j^i$ and for every conjunct of φ of the form $\forall x(\gamma(x) \rightarrow \exists y(x \leq_i y \wedge x \neq y \wedge \psi(x, y)))$, if $\alpha(x) \models \gamma(x)$ then there exists α' in $S_{j'}^i$, such that $\alpha(x), \alpha'(y) \models \psi(x, y)$, where $j' \geq j$, and if $\alpha \in \mathcal{K}$ then $j' > j$.
- (c) For every i, j , for every type $\alpha \in S_j^i$ and for every conjunct of φ of the form $\forall x(\gamma(x) \rightarrow \exists y(y \leq_i x \wedge x \neq y \wedge \psi(x, y)))$, if $\alpha(x) \models \gamma(x)$ then there exists α' in $S_{j'}^i$, such that $\alpha(x), \alpha'(y) \models \psi(x, y)$, where $j' \leq j$, and if $\alpha \in \mathcal{K}$ then $j' < j$.
- (d) For every type $\alpha \in \mathcal{T}$ and for every conjunct of φ of the form $\forall x(\gamma(x) \rightarrow \psi(x))$, we have $\alpha(x) \models \gamma(x) \rightarrow \psi(x)$.
- (e) For every $i, j \leq j'$, for every pair of types $\alpha \in S_j^i, \alpha' \in S_{j'}^i$, such that it is not the case that $\alpha = \alpha'$ and $\alpha \in \mathcal{K}$, then for every conjunct of the form $\forall xy(x \leq_i y \rightarrow (x \neq y \rightarrow \psi(x, y)))$, we have $\alpha(x), \alpha'(y) \models \psi(x, y)$.

► **Lemma 11.** *Let φ be a monadic GF^2 sentence in the normal form. Then φ is satisfiable if and only if it has a certificate of satisfiability.*

Proof. \Leftarrow Assume that $(\mathcal{T}, \mathcal{K}, \mathcal{S}^1, \dots, \mathcal{S}^k)$ is a certificate of satisfiability for φ . We build a model \mathfrak{A} whose universe A consists of exactly one realisation of each type from \mathcal{K} and infinitely many realisations of each type from $\mathcal{T} \setminus \mathcal{K}$. For every i we define the order \leq_i . We split A into sets $A_1^i, \dots, A_{k_i}^i$ in such a way that A_j^i contains infinitely many realisations of $\alpha \in S_j^i$ if $\alpha \notin \mathcal{K}$ and exactly one realisation of $\alpha \in S_j^i$ if $\alpha \in \mathcal{K}$. Now if $a \in A_j^i, a' \in A_{j'}^i$, and $j < j'$ then we set $\mathfrak{A} \models a \leq_i a'$. If S_j^i consists of non-royal types $\alpha^0, \dots, \alpha^{l-1}$ then we make the order \leq_i on A_j^i isomorphic to the natural order on \mathbb{Z} , in such a way that the element corresponding to the number m has 1-type $\alpha^{m \bmod l}$. It is readily checked that the conditions on the certificate imply that $\mathfrak{A} \models \varphi$.

\Rightarrow Let $\mathfrak{A} \models \varphi$. We show how to extract a certificate of satisfiability for φ from \mathfrak{A} . We define \mathcal{T} to be the set of 1-types realised in \mathfrak{A} and \mathcal{K} to be the set of 1-types realised exactly once in \mathfrak{A} .

For a given i and a type $\alpha \in \mathcal{T}$ we define B_α^i to be the minimal set containing all the realisations of α such that for all $a \leq_i b \leq_i c$, if $a, c \in B_\alpha^i$ then $b \in B_\alpha^i$. Let us denote by

$A_{\alpha-}^i$ the subset of A consisting of the elements smaller than all the elements from B_{α}^i , and by $A_{\alpha+}^i$ the union of B_{α}^i and $A_{\alpha-}^i$. Observe that the sets $A_{\alpha+}^i, A_{\alpha-}^i$ are closed downwards. Let $F_0^i, \dots, F_{k_i}^i$ be an ordered list of all the sets from $\{A_{\alpha+}^i, A_{\alpha-}^i : \alpha \in \mathcal{T}\}$ such that if $k < l$ then $F_k^i \subseteq F_l^i$. Note that k_i is linear in the number of 1-types and that $F_0^i = \emptyset$ and $F_{k_i}^i = A$. Let us define $D_j^i = F_j^i \setminus F_{j-1}^i$ for $1 \leq j \leq k_i$. The sets D_j^i are called *i-regions*.

Let S_j^i be the set of 1-types realised in D_j^i . This finishes the definition of $(\mathcal{T}, \mathcal{K}, \mathcal{S}^1, \dots, \mathcal{S}^k)$.

The properties (a)-(d) of the certificate are satisfied in an obvious way. Let us prove (e). If $j < j'$ then the conclusion is straightforward. Similarly, if $j = j'$ and $\alpha = \alpha'$ is a royal type. Assume that $j = j'$ and $\alpha, \alpha' \in S_j^i$ are two non-royal types. It is enough to show that in \mathfrak{A} there are two distinct elements a_{α}, b_{α} of type α and two distinct elements $a'_{\alpha'}, b'_{\alpha'}$ of type α' such that $\mathfrak{A} \models a_{\alpha} \leq_i a'_{\alpha'}$ and $\mathfrak{A} \models b'_{\alpha'} \leq_i b_{\alpha}$.

If $\alpha = \alpha'$ then we have at least two realisations a, b of α in \mathfrak{A} . Assume that $a \leq_i b$. Then we can take $a_{\alpha} = b'_{\alpha} = a$ and $a'_{\alpha} = b_{\alpha} = b$. If $\alpha \neq \alpha'$ consider elements a, a' from \mathfrak{A} , of types α, α' , respectively, in the i -region D_j^i . Assume that $a \leq_i a'$ (the opposite case is symmetric). Assume that there is no realisation of α , which is greater, with respect to \leq_i than a' . It means that $a' \notin B_{\alpha}^i$ (since otherwise B_{α}^i would not be minimal). Since $a \in B_{\alpha}^i$ we have a contradiction with the assumption that a, a' are member of the same i -region. So there exists a realisation b of α such that $\mathfrak{A} \models a' \leq_i b$. We can take $a_{\alpha} = a, b_{\alpha} = b, a'_{\alpha'} = b_{\alpha'} = a'$. ◀

The construction in the proof of the lemma shows that every satisfiable monadic GF^2 sentence φ in the normal form has a certificate of size polynomial in the number of 1-types. Since we may assume that σ contains only symbols appearing in φ it implies that the size of a certificate can be bounded exponentially in $|\varphi|$. Checking that a given tuple is indeed a certificate of satisfiability can easily be done in polynomial time. Thus:

► **Corollary 12.** *The satisfiability problem for monadic GF^2 is decidable in NExpTime .*

3.2 Finite satisfiability

The case of finite satisfiability is even simpler than the case of general satisfiability.

► **Lemma 13.** *Let φ be a GF^2 sentence in normal form. If φ is finitely satisfiable then φ has a model with at most $2k \cdot 2^{|\sigma|}$ elements, where k is the number of linear orders in the signature.*

Proof. Let \mathfrak{A} be a model of φ . Mark in \mathfrak{A} all the elements whose 1-types are realised only once. For every 1-type α , such that there are at least two realisations of α in \mathfrak{A} , and for every $0 \leq i \leq k$, mark the \leq_i -minimal and the \leq_i -maximal realisations of α . Let \mathfrak{A}' be the substructure of \mathfrak{A} induced by the marked elements. It is easy to verify that $\mathfrak{A}' \models \varphi$. ◀

► **Corollary 14.** *The finite satisfiability problem for monadic GF^2 is in NExpTime .*

3.3 Lower bound

The satisfiability problem for GF^2 in the class of all structures is in ExpTime [6]. On the other hand FO^2 is NExpTime -hard even if only unary symbols are allowed [15, 5]. Clearly such monadic FO^2 can be reduced to monadic GF^2 with just one linear order \leq , since this order can always be used as a guard, e.g. a formula $\forall xy\psi(x, y)$ can be translated to $\forall xy(x \leq y \rightarrow \psi(x, y)) \wedge \forall xy(y \leq x \rightarrow \psi(x, y))$. Thus:

► **Theorem 15.** *The satisfiability and the finite satisfiability problems for monadic GF^2 are NExpTime -complete.*

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