Labelings for Decreasing Diagrams*

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Abstract

This paper is concerned with automating the decreasing diagrams technique of van Oostrom for establishing confluence of term rewrite systems. We study abstract criteria that allow to lexicographically combine labelings to show local diagrams decreasing. This approach has two immediate benefits. First, it allows to use labelings for linear rewrite systems also for left-linear ones, provided some mild conditions are satisfied. Second, it admits an incremental method for proving confluence which subsumes recent developments in automating decreasing diagrams. The techniques proposed in the paper have been implemented and experimental results demonstrate how, e.g., the rule labeling benefits from our contributions.

1998 ACM Subject Classification F.3.1 Specifying and Verifying and Reasoning about Programs, F.4.2 Grammars and Other Rewriting Systems

Keywords and phrases term rewriting, confluence, decreasing diagrams, labeling

Digital Object Identifier 10.4230/LIPIcs.RTA.2011.377

Category Regular Research Paper

1 Introduction

The decreasing diagrams technique of van Oostrom [10] is a powerful method for showing confluence of abstract rewrite systems, i.e., it is complete for countable systems. The main idea of the approach is to show confluence by establishing local confluence under the side condition that rewrite steps of the joining sequences must *decrease* with respect to some well-founded order. For term rewrite systems however, the main problem for automation of decreasing diagrams is that in general infinitely many local peaks must be considered. To reduce this problem to a finite set of local peaks one can label rewrite steps with functions that satisfy special properties. In [12] van Oostrom presented the rule labeling that allows to conclude confluence of *linear* rewrite systems by checking decreasingness of the critical peaks (those emerging from critical overlaps). The rule labeling has recently been implemented by Aoto [1] and Hirokawa and Middeldorp [8]. Already in [12] van Oostrom presented constraints that allow to apply the rule labeling to *left-linear* systems. This approach has recently been implemented and extended by Aoto [1]. Our framework subsumes the above ideas.

The contributions of this paper comprise the extraction of abstract constraints on a labeling such that for a (left-)linear rewrite system decreasingness of the critical peaks ensures confluence. We show that the rule labeling adheres to our constraints and present additional labeling functions. Furthermore such labeling functions can be combined lexicographically to obtain new labeling functions satisfying our constraints. This approach allows the formulation of an abstract criterion that makes virtually every labeling function for linear rewrite systems also applicable to left-linear systems. Consequently, confluence of the TRS in Example 1.1

* This research is supported by FWF (Austrian Science Fund) project P22467.

22nd International Conference on Rewriting Techniques and Applications (RTA'11).
Editor: M. Schmidt-Schauß; pp. 377–392
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

can be established automatically, e.g., by the rule labeling, while current approaches based on the decreasing diagrams technique [1,8] as well as standard confluence criteria fail.

▶ **Example 1.1.** Consider the TRS $\mathcal{R}$ (from [14]) consisting of the rules

$1\colon x + (y + z) \to (x + y) + z$	$5\colon x + y \to y + x$	$7\colon \mathsf{s}(x) + y \to x + \mathsf{s}(y)$
$2\colon (x + y) + z \to x + (y + z)$	$6\colon x \times y \to y \times x$	$8\colon x + \mathsf{s}(y) \to \mathsf{s}(x) + y$
$3\colon \mathsf{sq}(x) \to x \times x$		$9\colon x \times \mathsf{s}(y) \to x + (x \times y)$
$4\colon \mathsf{sq}(\mathsf{s}(x)) \to (x \times x) + \mathsf{s}(x + x)$		$10\colon \mathsf{s}(x) \times y \to (x \times y) + y$

This system is locally confluent since all its 34 critical pairs are joinable.

The remainder of this paper is organized as follows. After recalling preliminaries in Section 2 we present constraints (on a labeling) such that decreasingness of the critical peaks ensures confluence for (left-)linear rewrite systems in Section 3. The merits of this approach are assessed in Section 4. Implementation issues are addressed in Section 5 before Section 6 gives an empirical evaluation of our results. Section 7 concludes.

2 Preliminaries

We assume familiarity with term rewriting [4,15].

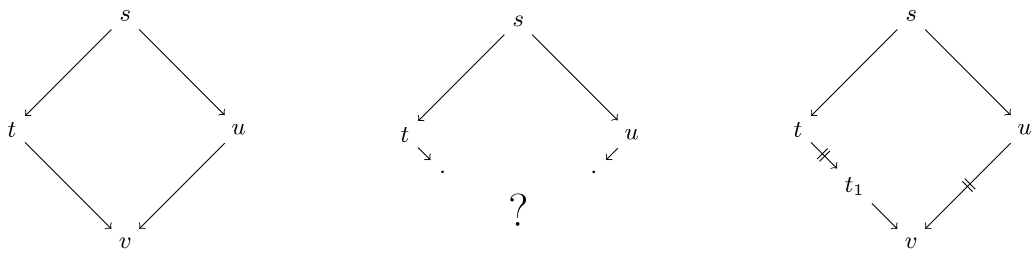
Let $\mathcal{F}$ be a signature and let $\mathcal{V}$ be a set of variables disjoint from $\mathcal{F}$. By $\mathcal{T}(\mathcal{F},\mathcal{V})$ we denote the set of terms over $\mathcal{F}$ and $\mathcal{V}$. The expression $|t|_x$ indicates how often variable x occurs in term t. The *set of positions* of a term t is defined as $\mathcal{P}\mathsf{os}(t) = \{\epsilon\}$ if t is a variable and as $\mathcal{P}\mathsf{os}(t) = \{\epsilon\} \cup \{iq \mid q \in \mathcal{P}\mathsf{os}(t_i)\}$ if $t = f(t_1,\ldots,t_n)$. We write $p \leqslant q$ if $q = pp'$ for some position p', in which case $q \backslash p$ is defined to be p'. Furthermore $p < q$ if $p \leqslant q$ and $p \neq q$. Finally, $p \parallel q$ if neither $p \leqslant q$ nor $q < p$. Positions are used to address occurrences of subterms. The subterm of t at position $p \in \mathcal{P}\mathsf{os}(t)$ is defined as $t|_p = t$ if $p = \epsilon$ and as $t|_p = t_i|_q$ if $p = iq$. We write $s[t]_p$ for the result of replacing $s|_p$ with t in s. The set of function symbol positions $\mathcal{P}\mathsf{os}_{\mathcal{F}}(t)$ is $\{p \in \mathcal{P}\mathsf{os}(t) \mid t|_p \notin \mathcal{V}\}$ and $\mathcal{P}\mathsf{os}_{\mathcal{V}}(t) = \mathcal{P}\mathsf{os}(t) \setminus \mathcal{P}\mathsf{os}_{\mathcal{F}}(t)$.

A rewrite rule is a pair of terms (l, r), written $l \to r$ such that l is not a variable and all variables in r are contained in l. A rewrite rule $l \to r$ is duplicating if $|l|_x < |r|_x$ for some $x \in \mathcal{V}$. A term rewrite system (TRS) is a signature together with a finite set of rewrite rules over this signature. In the sequel signatures are implicit. By $\mathcal{R}_{\mathsf{d}}$ and $\mathcal{R}_{\mathsf{nd}}$ we denote the duplicating and non-duplicating rules of a TRS $\mathcal{R}$, respectively. A rewrite relation is a binary relation on terms that is closed under contexts and substitutions. For a TRS $\mathcal{R}$ we define $\to_{\mathcal{R}}$ to be the smallest rewrite relation that contains $\mathcal{R}$. As usual $\to^{=}$ ($\to^*$) denotes the reflexive (reflexive and transitive) closure of $\to$ and $\mathrel{+\!\!\!\!\to}$ denotes rewriting at parallel positions.

A relative TRS $\mathcal{R}/\mathcal{S}$ is a pair of TRSs $\mathcal{R}$ and $\mathcal{S}$ with the induced rewrite relation $\to_{\mathcal{R}/\mathcal{S}} = \to_{\mathcal{S}}^* \cdot \to_{\mathcal{R}} \cdot \to_{\mathcal{S}}^*$. Sometimes we identify a TRS $\mathcal{R}$ with the relative TRS $\mathcal{R}/\varnothing$ and vice versa. A TRS $\mathcal{R}$ (relative TRS $\mathcal{R}/\mathcal{S}$) is terminating if $\to_{\mathcal{R}}$ ($\to_{\mathcal{R}/\mathcal{S}}$) is well-founded. Two relations $\geqslant$ and $>$ are called compatible if $\geqslant \cdot > \cdot \geqslant \subseteq >$. A monotone reduction pair $(\geqslant, >)$ consists of a quasi-order $\geqslant$ and a well-founded order $>$ such that $\geqslant$ and $>$ are compatible and closed under contexts and substitutions. We recall how to prove relative termination incrementally according to Geser [6]:

▶ **Theorem 2.1.** *A relative TRS $\mathcal{R}/\mathcal{S}$ is terminating if $\mathcal{R} = \varnothing$ or there exists a monotone reduction pair $(\geqslant, >)$ such that $\mathcal{R} \cup \mathcal{S} \subseteq \geqslant$ and $(\mathcal{R} \setminus >)/(\mathcal{S} \setminus >)$ is terminating.* ◀

An overlap $(l_1 \to r_1, p, l_2 \to r_2)_\mu$ of a TRS $\mathcal{R}$ consists of variants $l_1 \to r_1$ and $l_2 \to r_2$ of rewrite rules of $\mathcal{R}$ without common variables, a position $p \in \mathcal{P}\mathsf{os}_{\mathcal{F}}(l_2)$, and a most general



(a) (parallel) **(b)** (critical overlap) **(c)** (variable overlap)

Figure 1 Three kinds of local peaks.

unifier μ of l_1 and $l_2|_p$. If $p = \epsilon$ then we require that $l_1 \to r_1$ and $l_2 \to r_2$ are not variants. From an overlap $(l_1 \to r_1, p, l_2 \to r_2)_\mu$ we obtain a critical peak $l_2\mu[r_1\mu]_p \leftarrow l_2\mu \rightarrow r_2\mu$ and a critical pair $l_2\mu[r_1\mu]_p \leftarrow\!\rtimes\!\rightarrow r_2\mu$.

We write $\langle A, \{\to_\alpha\}_{\alpha \in I}\rangle$ to denote the ARS $\langle A, \to\rangle$ where $\to$ is the union of $\to_\alpha$ for all $\alpha \in I$. Let $\langle A, \{\to_\alpha\}_{\alpha \in I}\rangle$ be an ARS and let $>$ be a relation on I. We write $\overset{\vee}{\to}_{\alpha_1 \ldots \alpha_n}$ for the union of $\to_\beta$ where $\beta < \alpha_i$ for some $1 \leqslant i \leqslant n$. We say $\to_\alpha$ and $\to_\beta$ are *extended locally decreasing* (with respect to $\geqslant$ and $>$) if ${}_\alpha\!\leftarrow \cdot \to_\beta \subseteq \overset{\vee}{\to}{}^*_\alpha \cdot \overset{\curlyeqsucc}{\to}{}^=_\beta \cdot \overset{\vee}{\to}{}^*_{\alpha\beta} \cdot {}^*_{\alpha\beta}\!\overset{\vee}{\leftarrow} \cdot {}^=_\alpha\!\overset{\curlyeqsucc}{\leftarrow} \cdot {}^*_\beta\!\overset{\vee}{\leftarrow}$. An ARS $\langle A, \{\to_\alpha\}_{\alpha \in I}\rangle$ is *extended locally decreasing* if there exists a quasi-order $\geqslant$ and a well-founded order $>$ such that $\geqslant$ and $>$ are compatible and $\to_\alpha$ and $\to_\beta$ are extended locally decreasing for all $\alpha, \beta \in I$ with respect to $\geqslant$ and $>$.

The following theorem is from [8], reformulating a result obtained by van Oostrom [10].

▶ **Theorem 2.2.** *Every extended locally decreasing ARS is confluent.* ◀

3 Confluence by Labeling

In this section we present constraints (on a labeling) such that extended local decreasingness of the critical peaks ensures confluence of linear (Section 3.1) and left-linear (Section 3.2) TRSs. Furthermore, we show that if two labelings satisfy these conditions then also their lexicographic combination satisfies them.

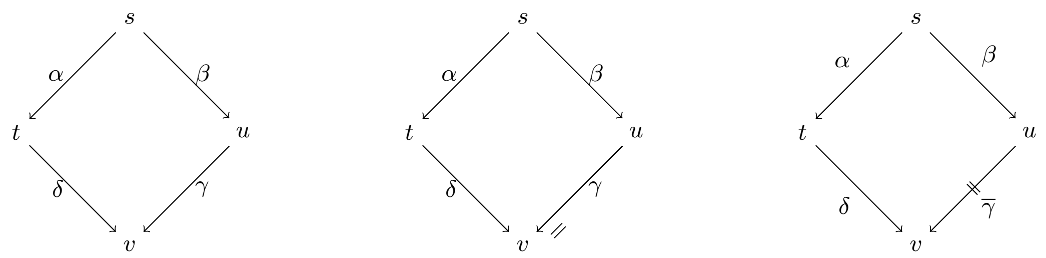
There are three possibilities for a local peak (modulo symmetry):

$$t = s[r_1\sigma]_p \leftarrow s[l_1\sigma]_p = s = s[l_2\sigma]_q \to s[r_2\sigma]_q = u \tag{1}$$

- $p \parallel q$ (parallel)
- $q \leqslant p$ and $p \in \mathcal{P}\mathsf{os}_\mathcal{F}(s[l_2]_q)$ (critical overlap)
- $q < p$ and $p \notin \mathcal{P}\mathsf{os}_\mathcal{F}(s[l_2]_q)$ (variable overlap)

These cases are visualized in Figure 1. Figure 1(a) shows the shape of a local peak where the reductions take place at parallel positions. Here we have $s \to_{p, l_1 \to r_1} t$ and $u \to_{p, l_1 \to r_1} v$, i.e., the reductions drawn at opposing sides in the diagram are due to the same rules. The question mark in Figure 1(b) conveys that joinability of critical overlaps may depend on auxiliary rules. Variable overlaps (Figure 1(c)) can again be joined by the rules involved in the diverging step. More precisely, if q' is the unique position in $\mathcal{P}\mathsf{os}_\mathcal{V}(l_2)$ such that $qq' \leqslant p$, $x = l_2|_{q'}$, $|l_2|_x = m$, and $|r_2|_x = n$ then we have $t \to^{m-1}_{l_1 \to r_1} t_1$, $t_1 \to_{l_2 \to r_2} v$, and $u \to^n_{l_1 \to r_1} v$.

Labelings are used to compare rewrite steps. In the sequel we denote the set of all rewrite steps for a TRS $\mathcal{R}$ by $\Lambda_\mathcal{R}$ and elements from this set by capital Greek letters Γ and Δ. Furthermore if $\Gamma = s \to_{p, l \to r} t$ then $C(\Gamma\sigma)$ denotes the rewrite step $C[s\sigma] \to_{p'p, l \to r} C[t\sigma]$ for any substitution σ and context C with $C|_{p'} = \Box$.



(a) (parallel) **(b)** (variable-linear) **(c)** (variable-left-linear)

Figure 2 Labeled peaks.

▶ **Definition 3.1.** Let $\mathcal{R}$ be a TRS. A *labeling function* $\ell\colon \Lambda_{\mathcal{R}} \to W$ is a mapping from rewrite steps into some set W. A *labeling* $(\ell, \geqslant, >)$ for $\mathcal{R}$ consists of a labeling function ℓ, a quasi-order $\geqslant$, and a well-founded order $>$ such that $\geqslant$ and $>$ are compatible and for all rewrite steps $\Gamma, \Delta \in \Lambda_{\mathcal{R}}$, contexts C and substitutions σ:

1. $\ell(\Gamma) \geqslant \ell(\Delta)$ implies $\ell(C[\Gamma\sigma]) \geqslant \ell(C[\Delta\sigma])$ and
2. $\ell(\Gamma) > \ell(\Delta)$ implies $\ell(C[\Gamma\sigma]) > \ell(C[\Delta\sigma])$

All labelings we discuss satisfy $> \subseteq \geqslant$ which allows to avoid tedious case distinctions. In the sequel W, $\geqslant$, and $>$ are left implicit when clear from the context and a labeling is identified with the labeling function ℓ. We use the terminology that a labeling ℓ is *monotone* and *stable* if properties 1 and 2 of Definition 3.1 hold. Abstract labels, i.e., labels that are unknown, are represented by lowercase Greek letters α, β, γ, etc. We write $s \xrightarrow{\alpha}_{p,l\to r} t$ (or simply $s \xrightarrow{\alpha} t$ or $s \to_\alpha t$) if $\ell(s \to_{p,l\to r} t) = \alpha$. Often we leave the labeling ℓ implicit and just attach labels to arrows. A local peak $t \leftarrow s \to u$ is called *decreasing for* ℓ if $t \;{}_{\alpha}\!\leftarrow s \to_\beta u$, and $\to_\alpha$ and $\to_\beta$ are extended locally decreasing with respect to $\geqslant$ and $>$. To employ Theorem 2.2 for TRSs, extended local decreasingness of the ARS $\langle \mathcal{T}(\mathcal{F},\mathcal{V}), \{\to_w\}_{w\in W}\rangle$ must be shown.

In the sequel we investigate conditions on a labeling such that local peaks according to (parallel) and (variable overlap) are decreasing automatically. This is desirable since in general there are infinitely many local peaks corresponding to these cases (even if the underlying TRS has finitely many rules). There are also infinitely many local peaks according to (critical overlap) in general, but for a finite TRS they are captured by the finitely many overlaps. Still, it is undecidable if they are decreasingly joinable [8].

For later reference, Figure 2 shows labeled peaks for the case (parallel) (Figure 2(a)) and (variable overlap) if the rule $l_2 \to r_2$ in (1) is linear (Figure 2(b)) and left-linear (Figure 2(c)), respectively. In Figure 2(c) the expression $\overline{\gamma}$ means a sequence of labels $\gamma_1, \ldots, \gamma_n$. Since the step from u to v is parallel we can choose any permutation of $\overline{\gamma}$.

3.1 Linear TRSs

The next definition presents sufficient abstract conditions on a labeling such that local peaks according to the cases (parallel) and (variable-linear) in Figure 2 are decreasing. We use the observation that the former can be seen as an instance of the latter to shorten proofs.

▶ **Definition 3.2.** Let ℓ be a labeling for a TRS $\mathcal{R}$. We call ℓ an *L-labeling (for $\mathcal{R}$)* if for local peaks according to (parallel) and (variable-linear) we have $\alpha \geqslant \gamma$ and $\beta \geqslant \delta$ in Figures 2(a) and 2(b), respectively.

The local diagram in Figure 3(a) visualizes the conditions on an L-labeling more succinctly. We call the critical peaks of a TRS $\mathcal{R}$ Φ-*decreasing* if there exists a Φ-labeling ℓ for $\mathcal{R}$ such that the critical peaks of $\mathcal{R}$ are decreasing for ℓ.

The next theorem states that L-labelings may be used to show confluence of linear TRSs.

▶ **Theorem 3.3.** *Let $\mathcal{R}$ be a linear TRS. If the critical peaks of $\mathcal{R}$ are L-decreasing then $\mathcal{R}$ is confluent.*

Proof. By assumption the critical peaks of $\mathcal{R}$ are decreasing for some L-labeling ℓ. We establish confluence of $\mathcal{R}$ by Theorem 2.2, i.e., show extended local decreasingness of the ARS $\langle \mathcal{T}(\mathcal{F},\mathcal{V}), \rightarrow_\mathcal{R}\rangle$ where rewrite steps are labeled according to ℓ. Since $\mathcal{R}$ is linear, local peaks have the shape (parallel), (variable-linear), or (critical overlap). By definition of an L-labeling the former two are extended locally decreasing. Now consider a local peak according to (critical overlap), i.e., for the peak in (1) we have $q \leqslant p$ and $p \in \mathcal{P}\mathsf{os}_\mathcal{F}(s[l_2]_q)$. Let $p' = p \backslash q$. Then $(s|_q)[r_1\sigma]_{p'} = (s[r_1\sigma]_p)|_q \; {}_{p'}\!\leftarrow s|_q \rightarrow_\epsilon r_2\sigma$ must be an instance of a critical peak which is decreasing by assumption. By monotonicity and stability of ℓ we obtain extended local decreasingness of the local peak (1). ◀

We recall the rule labeling of van Oostrom [12], parametrized by a mapping $i\colon \mathcal{R} \to \mathbb{N}$.

▶ **Definition 3.4.** Let $\mathcal{R}$ be a TRS. Then $\ell^i_{\mathrm{rl}}(s \rightarrow_{p,l\to r} t) = i(l \to r)$.

Often i is left implicit. The rule labeling satisfies the constraints of an L-labeling.

▶ **Lemma 3.5.** *Let $\mathcal{R}$ be a TRS. Then $(\ell^i_{\mathrm{rl}}, \geqslant_\mathbb{N}, >_\mathbb{N})$ is an L-labeling for $\mathcal{R}$.*

Proof. First we show that $(\ell^i_{\mathrm{rl}}, \geqslant_\mathbb{N}, >_\mathbb{N})$ is a labeling. The quasi-order $\geqslant_\mathbb{N}$ and the well-founded order $>_\mathbb{N}$ are compatible. Furthermore $\ell^i_{\mathrm{rl}}(s \rightarrow_{p,l\to r} t) = i(l \to r)$ which ensures monotonicity and stability of ℓ^i_{rl}. Hence $(\ell^i_{\mathrm{rl}}, \geqslant_\mathbb{N}, >_\mathbb{N})$ is a labeling. Next we show the properties demanded in Definition 3.2. For local peaks according to cases (parallel) and (variable-linear) we recall that the steps drawn at opposite sides in the diagram, e.g., the steps labeled with α and γ (β and δ) in Figures 2(a) and 2(b), are due to applications of the same rule. Hence $\alpha = \gamma$ ($\beta = \delta$) in Figures 2(a) and 2(b), which shows the result. ◀

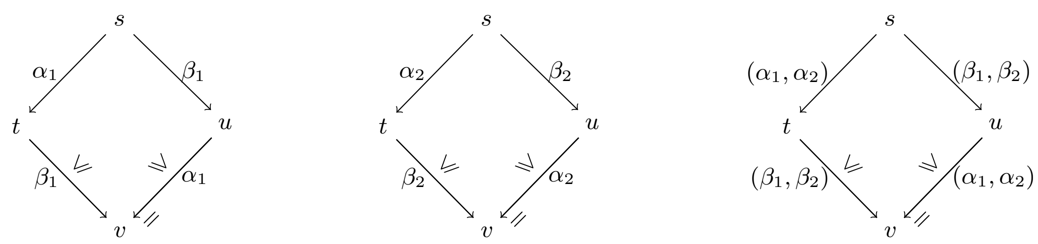
Inspired by [8] we propose a labeling based on relative termination.

▶ **Definition 3.6.** Let $\mathcal{R}$ be a TRS. Then $\ell_{\mathrm{sn}}(s \to t) = s$.

▶ **Lemma 3.7.** *Let $\mathcal{R}$ be a TRS. Then $\ell^\mathcal{S}_{\mathrm{sn}} := (\ell_{\mathrm{sn}}, \rightarrow^*_\mathcal{R}, \rightarrow^+_{\mathcal{S}/\mathcal{R}})$ is an L-labeling for $\mathcal{R}$, provided $\rightarrow_\mathcal{S} \subseteq \rightarrow_\mathcal{R}$ and $\mathcal{S}/\mathcal{R}$ is terminating.*

Proof. Let $\geqslant := \rightarrow^*_\mathcal{R}$ and $> := \rightarrow^+_{\mathcal{S}/\mathcal{R}}$. First we show that $(\ell_{\mathrm{sn}}, \geqslant, >)$ is a labeling. By definition of relative rewriting, $\rightarrow^*_\mathcal{R}$ and $\rightarrow^+_{\mathcal{S}/\mathcal{R}}$ are compatible and $\rightarrow^+_{\mathcal{S}/\mathcal{R}}$ is well-founded by the termination assumption of $\mathcal{S}/\mathcal{R}$. Since rewriting is closed under contexts and substitutions, $\ell^\mathcal{S}_{\mathrm{sn}}$ is monotone and stable and hence a labeling. Next we show the properties demanded in Definition 3.2. The assumption $\rightarrow_\mathcal{S} \subseteq \rightarrow_\mathcal{R}$ yields $\rightarrow^+_{\mathcal{S}/\mathcal{R}} \subseteq \rightarrow^*_\mathcal{R}$ which ensures $> \subseteq \geqslant$. Combining $\alpha = s = \beta$, $\gamma = u$, and $\delta = t$ with $s \rightarrow_\mathcal{R} t$ and $s \rightarrow_\mathcal{R} u$ yields $\alpha = \beta \geqslant \gamma, \delta$ for local peaks according to (parallel) and (variable-linear) in Figures 2(a) and 2(b). ◀

The L-labeling from the previous lemma allows to establish a decrease with respect to some steps of $\mathcal{R}$. The next lemma allows to combine L-labelings. Let $\ell_1\colon \Lambda_\mathcal{R} \to W_1$ and $\ell_2\colon \Lambda_\mathcal{R} \to W_2$. Then $(\ell_1, \geqslant_1, >_1) \times (\ell_2, \geqslant_2, >_2)$ is defined as $(\ell_1 \times \ell_2, \geqslant_{12}, >_{12})$ where $\ell_1 \times \ell_2\colon \Lambda_\mathcal{R} \to W_1 \times W_2$ with $(\ell_1 \times \ell_2)(\Gamma) = (\ell_1(\Gamma), \ell_2(\Gamma))$. Furthermore $(x_1, x_2) \geqslant_{12} (y_1, y_2)$ if and only if $x_1 > y_1$ or $x_1 \geqslant_1 y_1$ and $x_2 \geqslant_2 y_2$ and $(x_1, x_2) >_{12} (y_1, y_2)$ if and only if $x_1 >_1 y_1$ or $x_1 \geqslant_1 y_1$ and $x_2 >_2 y_2$.



(a) Labeling ℓ_1 **(b)** Labeling ℓ_2 **(c)** Labeling $\ell_1 \times \ell_2$

Figure 3 Lexicographic combination of L-labelings.

▶ **Lemma 3.8.** *Let ℓ_1 and ℓ_2 be L-labelings. Then $\ell_1 \times \ell_2$ is an L-labeling.*

Proof. First we remark that $\ell_1 \times \ell_2$ is a labeling whenever ℓ_1 and ℓ_2 are labelings. Next we show that $\ell_1 \times \ell_2$ satisfies the properties from Definition 3.2. If ℓ_1 and ℓ_2 are L-labelings then the diagram of Figure 2(b) has the shape as in Figure 3(a) and 3(b), respectively. It is easy to see that the lexicographic combination is again an L-labeling (cf. Figure 3(c)). ◀

3.2 Left-linear TRSs

For left-linear TRSs the notion of an LL-labeling is introduced.

▶ **Definition 3.9.** Let ℓ be a labeling for a TRS $\mathcal{R}$. We call ℓ an LL-labeling (for $\mathcal{R}$) if $\alpha \geqslant \gamma$ and $\beta \geqslant \delta$ in Figure 2(a) and $\alpha > \overline{\gamma}$ and $\beta \geqslant \delta$ in Figure 2(c). Here $\alpha > \overline{\gamma}$ means $\alpha \geqslant \gamma_1$ and $\alpha > \gamma_i$ for $2 \leqslant i \leqslant n$.

The next theorem states that LL-labelings allow to show confluence of left-linear TRSs.

▶ **Theorem 3.10.** *Let $\mathcal{R}$ be a left-linear TRS. If the critical peaks of $\mathcal{R}$ are LL-decreasing then $\mathcal{R}$ is confluent.*

Proof. By assumption the critical peaks of $\mathcal{R}$ are decreasing for some LL-labeling ℓ. We establish confluence of $\mathcal{R}$ by Theorem 2.2, i.e., show extended local decreasingness of the ARS $\langle \mathcal{T}(\mathcal{F}, \mathcal{V}), \to_{\mathcal{R}} \rangle$ by labeling rewrite steps according to ℓ. By definition of an LL-labeling local peaks according to (parallel) and (variable-left-linear) are extended locally decreasing. The reasoning for local peaks according to (critical overlap) is the same as in the proof of Theorem 3.3. ◀

The rule labeling from Definition 3.4 is not an LL-labeling since in Figure 2(c) we have $\alpha = \gamma_i$ for $1 \leqslant i \leqslant n$ which does not satisfy $\alpha > \overline{\gamma}$ if $n > 1$. (See also [8, Example 5].) In contrast, the L-labeling from Lemma 3.7 can be adapted to an LL-labeling.

▶ **Lemma 3.11.** *Let $\mathcal{R}$ be a left-linear TRS. Then $\ell_{\mathrm{sn}}^{\mathcal{R}_{\mathsf{d}}}$ is an LL-labeling, provided $\mathcal{R}_{\mathsf{d}}/\mathcal{R}_{\mathsf{nd}}$ is terminating.*

Proof. By Theorem 2.1 the relative TRS $\mathcal{R}_{\mathsf{d}}/\mathcal{R}_{\mathsf{nd}}$ is terminating if and only if $\mathcal{R}_{\mathsf{d}}/\mathcal{R}$ is. Hence $(\ell_{\mathrm{sn}}^{\mathcal{R}_{\mathsf{d}}}, \geqslant, >)$ is a labeling by Lemma 3.7. Here $\geqslant \,:=\, \to_{\mathcal{R}}^{*}$ and $> \,:=\, \to_{\mathcal{R}_{\mathsf{d}}/\mathcal{R}}^{+}$. Since $\ell_{\mathrm{sn}}(s \to t) = s$, we have $\alpha = \beta$ in Figures 2(a) and 2(c). We have $> \subseteq \geqslant$. Hence $\alpha \geqslant \gamma$ and $\alpha \geqslant \delta$ in Figure 2(a) and if $l_2 \to r_2$ in (1) is linear also in Figure 2(c). If $l_2 \to r_2$ is not linear then it must be duplicating and hence $\alpha > \gamma_i$ for $1 \leqslant i \leqslant n$. Combining this with $\alpha \geqslant \delta$ from above we obtain that $\ell_{\mathrm{sn}}^{\mathcal{R}_{\mathsf{d}}}$ is an LL-labeling for $\mathcal{R}$. ◀

To combine the previous lemma with the rule labeling we study how different labelings can be combined and introduce the following notion.

▶ **Definition 3.12.** Let ℓ be an L-labeling. We call ℓ a *weak* LL-labeling if $\alpha \geqslant \overline{\gamma}$ and $\beta \geqslant \delta$. for peaks according to Figure 2(c). Here $\alpha \geqslant \overline{\gamma}$ means $\alpha \geqslant \gamma_i$ for $1 \leqslant i \leqslant n$.

▶ **Remark 3.13.** The L-labelings presented so far (cf. Lemmata 3.5 and 3.7) are weak LL-labelings. Furthermore if ℓ_1 and ℓ_2 are weak LL-labelings then so are $\ell_1 \times \ell_2$ and $\ell_2 \times \ell_1$.

▶ **Lemma 3.14.** *Let ℓ_1 be an LL-labeling and let ℓ_2 be a weak LL-labeling. Then $\ell_1 \times \ell_2$*[1] *and $\ell_2 \times \ell_1$ are LL-labelings.*

Proof. By the proof of Lemma 3.8 $\ell_1 \times \ell_2$ and $\ell_2 \times \ell_1$ are labelings. The only interesting case of (variable-left-linear) is when $l_2 \to r_2$ in (1) is non-linear, i.e., $\overline{\gamma}$ contains more than one element. First we show that $\ell_1 \times \ell_2$ is an LL-labeling. Here labels according to ℓ_1 are suffixed with $_1$ and similarly for ℓ_2. Recall Figure 2(c). By assumption we have $\alpha_1 > \overline{\gamma_1}$, $\beta_1 \geqslant \delta_1$ and $\alpha_2 \geqslant \overline{\gamma_2}$, $\beta_2 \geqslant \delta_2$, which yields the desired $(\alpha_1, \alpha_2) \geqslant (\gamma_{11}, \gamma_{21})$, $(\alpha_1, \alpha_2) > (\gamma_{1i}, \gamma_{2i})$ for $2 \leqslant i \leqslant n$, and $(\beta_1, \beta_2) \geqslant (\delta_1, \delta_2)$. In the proof for $\ell_2 \times \ell_1$ the assumptions yield $(\alpha_2, \alpha_1) \geqslant (\gamma_{21}, \gamma_{11})$, $(\alpha_2, \alpha_1) > (\gamma_{2i}, \gamma_{1i})$ for $2 \leqslant i \leqslant n$, and $(\beta_2, \beta_1) \geqslant (\delta_2, \delta_1)$. ◀

In particular LL-labelings can be composed lexicographically.

▶ **Lemma 3.15.** *Every LL-labeling is a weak LL-labeling.*

Proof. By the global assumption that $> \subseteq \geqslant$. ◀

From Theorem 3.10 and Lemmata 3.11 and 3.14 we obtain the following result.

▶ **Corollary 3.16.** *Let $\mathcal{R}$ be a left-linear TRS. If $\mathcal{R}_{\mathsf{d}}/\mathcal{R}_{\mathsf{nd}}$ is terminating and all critical peaks of $\mathcal{R}$ are weakly LL-decreasing then $\mathcal{R}$ is confluent.*

Proof. By Lemma 3.11 $\ell_{\text{sn}}^{\mathcal{R}_{\mathsf{d}}}$ is an LL-labeling. By assumption the critical peaks of $\mathcal{R}$ are decreasing for some weak LL-labeling ℓ. By Lemma 3.14 also $\ell_{\text{sn}}^{\mathcal{R}_{\mathsf{d}}} \times \ell$ is an LL-labeling. It remains to show decreasingness of the critical peaks for $\ell_{\text{sn}}^{\mathcal{R}_{\mathsf{d}}} \times \ell$. This is obvious since for terms s, t, u with $s \to_{\mathcal{R}} t \to_{\mathcal{R}} u$ we have $\ell_{\text{sn}}^{\mathcal{R}_{\mathsf{d}}}(s \to t) \geqslant \ell_{\text{sn}}^{\mathcal{R}_{\mathsf{d}}}(t \to u)$. Hence decreasingness for ℓ implies decreasingness for $\ell_{\text{sn}}^{\mathcal{R}_{\mathsf{d}}} \times \ell$. Confluence of $\mathcal{R}$ follows from Theorem 3.10. ◀

We revisit the example from the introduction.

▶ **Example 3.17.** Recall the TRS $\mathcal{R}$ from Example 1.1. The polynomial interpretation

$$+_{\mathbb{N}}(x, y) = x + y \qquad \mathsf{s}_{\mathbb{N}}(x) = x + 1 \qquad \times_{\mathbb{N}}(x, y) = x^2 + xy + y^2 \qquad \mathsf{sq}_{\mathbb{N}}(x) = 3x^2 + 1$$

shows termination of $\mathcal{R}_{\mathsf{d}}/\mathcal{R}_{\mathsf{nd}}$. It is easy to check that ℓ_{rl}^{i} with $i(3) = i(6) = 2$, $i(4) = i(10) = 1$, and all other rules labeled 0, shows the 34 critical peaks decreasing.

The next example is more suitable to familiarize the reader with Corollary 3.16. Note that also here no standard criterion for confluence applies.

[1] Here the condition that ℓ_2 is a weak LL-labeling can be weakened to $\alpha \geqslant \gamma_1$ and $\beta \geqslant \delta$ in Figure 2(c).

▶ **Example 3.18.** Consider the TRS $\mathcal{R}$ consisting of the three rules

$$1\colon \mathsf{b} \to \mathsf{a} \qquad 2\colon \mathsf{a} \to \mathsf{b} \qquad 3\colon \mathsf{f}(\mathsf{g}(x,\mathsf{a})) \to \mathsf{g}(\mathsf{f}(x),\mathsf{f}(x))$$

We have $\mathcal{R}_\mathsf{d} = \{3\}$ and $\mathcal{R}_\mathsf{nd} = \{1,2\}$. Termination of $\mathcal{R}_\mathsf{d}/\mathcal{R}_\mathsf{nd}$ can be established by LPO with precedence $\mathsf{a} \sim \mathsf{b}$ and $\mathsf{f} > \mathsf{g}$. The rule labeling that takes the rule numbers as labels shows the only critical peak decreasing, i.e., $\mathsf{f}(\mathsf{g}(x,\mathsf{b}))\ {}_2\!\leftarrow \mathsf{f}(\mathsf{g}(x,\mathsf{a})) \to_3 \mathsf{g}(\mathsf{f}(x),\mathsf{f}(x))$ and $\mathsf{f}(\mathsf{g}(x,\mathsf{b})) \to_1 \mathsf{f}(\mathsf{g}(x,\mathsf{a})) \to_3 \mathsf{g}(\mathsf{f}(x),\mathsf{f}(x))$ which allows to establish confluence of $\mathcal{R}$ by Corollary 3.16.

▶ **Remark 3.19.** Using $\ell^i_{\mathrm{rl}}(\cdot) = 0$ as weak LL-labeling, Corollary 3.16 gives a condition (termination of $\mathcal{R}_\mathsf{d}/\mathcal{R}_\mathsf{nd}$) such that $s \to^= t$ or $t \to^= s$ for all critical pairs $s \leftarrow\!\rtimes\!\to t$ implies confluence of a left-linear TRS $\mathcal{R}$. This partially answers one question in RTA LooP #13.[2]

Next we prepare for a different LL-labeling. In [12, Example 20] van Oostrom suggests to count function symbols above the contracted redex, demands that this measurement decreases for variables that are duplicated, and combines this with the rule labeling. Consequently local peaks according to Figure 2(c) are decreasing. Below we exploit this idea but incorporate the following beneficial generalizations. First, we do not restrict to counting function symbols (which has been adopted and extended by Aoto in [1]) but represent the constraints as a relative termination problem. This abstract formulation allows to strictly subsume the criteria from [12, 1] (see Section 4) because more advanced techniques than counting symbols can be applied for proving termination. Additionally, our setting also allows to weaken these constraints significantly (see Lemma 3.27).

The next example motivates an LL-labeling that does not require termination of $\mathcal{R}_\mathsf{d}/\mathcal{R}_\mathsf{nd}$.

▶ **Example 3.20.** Consider the TRS $\mathcal{R}$ consisting of the six rules

$$\begin{array}{lll} \mathsf{f}(\mathsf{h}(x)) \to \mathsf{h}(\mathsf{g}(\mathsf{f}(x),x,\mathsf{f}(\mathsf{h}(\mathsf{a})))) & \mathsf{f}(x) \to \mathsf{a} & \mathsf{a} \to \mathsf{b} \\ \mathsf{h}(x) \to \mathsf{c} & \mathsf{b} \to \bot & \mathsf{c} \to \bot \end{array}$$

Since the duplicating rule admits an infinite sequence Corollary 3.16 cannot succeed.

In the sequel we let $\mathcal{G}$ be the signature consisting of unary function symbols $f_1, \ldots, f_n$ for every n-ary function symbol $f \in \mathcal{F}$.

▶ **Definition 3.21.** Let $x \in \mathcal{V}$. We define a partial mapping $\star$ from $\mathcal{T}(\mathcal{F},\mathcal{V}) \times \mathcal{P}\mathsf{os}(\mathcal{T}(\mathcal{F},\mathcal{V}))$ to terms in $\mathcal{T}(\mathcal{G},\mathcal{V})$ as follows:

$$\star(f(t_1,\ldots,t_n),p) = \begin{cases} f_i(\star(t_i,q)) & \text{if } p = iq \\ x & \text{if } p = \epsilon \end{cases}$$

For a left-linear TRS $\mathcal{R}$ we abbreviate $\mathcal{R}^\star_>/\mathcal{R}^\star_=$ by $\star(\mathcal{R})$. Here, for $\rtimes \in \{>,=\}$,

$$\mathcal{R}^\star_\rtimes = \{\star(l,p) \to \star(r,q) \mid l \to r \in \mathcal{R},\ l|_p = r|_q = y,\ y \in \mathcal{V}, \text{ and } |r|_y \rtimes 1\}$$

The next example illustrates the transformation $\star(\cdot)$.

▶ **Example 3.22.** Consider the TRS $\mathcal{R}$ from Example 3.20. The relative TRS $\star(\mathcal{R}) = \mathcal{R}^\star_>/\mathcal{R}^\star_=$ consists of the TRS $\mathcal{R}^\star_>$ with rules

$$\mathsf{f}_1(\mathsf{h}_1(x)) \to \mathsf{h}_1(\mathsf{g}_1(\mathsf{f}_1(x))) \qquad \mathsf{f}_1(\mathsf{h}_1(x)) \to \mathsf{h}_1(\mathsf{g}_2(x))$$

and the TRS $\mathcal{R}^\star_=$ which is empty.

---

[2] `http://rtaloop.mancoosi.univ-paris-diderot.fr/problems/13.html`

▶ **Definition 3.23.** Let $\mathcal{R}$ be a TRS. Then $\ell_\star(s \to_{p,l\to r} t) = \star(s,p)$.

Due to the next lemma a termination proof of $\star(\mathcal{R})$ yields an LL-labeling.

▶ **Lemma 3.24.** *Let $\mathcal{R}$ be a TRS. Then $(\ell_\star, \geqslant, >)$ is an LL-labeling, provided $(\geqslant, >)$ is a monotone reduction pair, $\mathcal{R}^\star_> \subseteq >$, and $\mathcal{R}^\star_> \cup \mathcal{R}^\star_= \subseteq \geqslant$.*

Proof. That $(\ell_\star, \geqslant, >)$ is a labeling for $\mathcal{R}$ follows from the assumption that $(\geqslant, >)$ is a monotone reduction pair. To see that the constraints of Definition 3.9 are satisfied we argue as follows. For Figure 2(a) we have $\alpha = \gamma$ and $\beta = \delta$ because the steps drawn at opposing sides in the diagram take place at the same positions and the function symbols above these positions stay the same. For Figure 2(c) we have $\beta = \delta = x$ since the corresponding reductions take place at the root position and hence $\beta \geqslant \delta$. To see $\alpha > \overline{\gamma}$ recall the peak (1). Let q' be the unique position in $\mathcal{P}\mathsf{os}_\mathcal{V}(l_2)$ such that $qq' \leqslant p$ with $x = l_2|_{q'}$ and $Q = \{q'_1, \dots, q'_n\}$ with $r_2|_{q'_i} = x$. By construction $\mathcal{R}^\star_>$ contains all rules $\star(s,q') \to \star(u,q'_i)$ for $1 \leqslant i \leqslant n$. Since $u \mathrel{+\!\!\!\!\!\Vdash}_Q v$ we obtain $\alpha > \gamma_i$ for $1 \leqslant i \leqslant n$ (from $\mathcal{R}^\star_> \subseteq >$) and hence the desired $\alpha > \overline{\gamma}$. ◀

From Lemma 3.24 we obtain the following corollary.

▶ **Corollary 3.25.** *Let $\mathcal{R}$ be a left-linear TRS and let ℓ be a weak LL-labeling. Let $\ell_\star\ell$ denote $\ell \times \ell_\star$ or $\ell_\star \times \ell$. If $\star(\mathcal{R})$ is terminating and the critical peaks of $\mathcal{R}$ are decreasing for $\ell_\star\ell$ then $\mathcal{R}$ is confluent.*

Proof. If $\star(\mathcal{R})$ is terminating then $\ell_\star$ is an LL-labeling by Lemma 3.24. Lemma 3.14 yields that $\ell_\star\ell$ is an LL-labeling. By assumption the critical peaks are decreasing for $\ell_\star\ell$ and hence Theorem 3.10 yields the confluence of $\mathcal{R}$. ◀

The next example illustrates the use of Corollary 3.25.

▶ **Example 3.26.** We show confluence of the TRS $\mathcal{R}$ from Example 3.20. Termination of $\star(\mathcal{R})$ (cf. Example 3.22) is easily shown, e.g., the polynomial interpretation

$$\mathsf{f_1}_\mathbb{N}(x) = 2x \qquad \mathsf{g_1}_\mathbb{N}(x) = \mathsf{g_2}_\mathbb{N}(x) = x \qquad \mathsf{h_1}_\mathbb{N}(x) = x + 1$$

orients both rules in $\mathcal{R}^\star_>$ strictly. To show decreasingness of the three critical peaks (two of which are symmetric) we use $\ell_\star \times \ell^i_{\mathrm{rl}}$ with $i(\mathsf{f}(\mathsf{h}(x)) \to \mathsf{h}(\mathsf{g}(\mathsf{f}(x), x, \mathsf{f}(\mathsf{h}(\mathsf{a}))))) = 1$ and all other rules receive label 0. Since it is impractical to label with $\ell_\star$ and compare labels with respect to the monotone reduction pair obtained from the above termination proof we label a step $s \to_{p,l\to r} t$ with the constant part of the interpretation of $\star(s,p)$ (cf. Lemma 5.2 below) and compare labels with $\geqslant_\mathbb{N}$ and $>_\mathbb{N}$. E.g., a step $\mathsf{f}(\mathsf{h}(\mathsf{b})) \to \mathsf{f}(\mathsf{h}(\bot))$ is labeled 2 since $\star(\mathsf{f}(\mathsf{h}(\mathsf{b})), 11) = \mathsf{f_1}(\mathsf{h_1}(x))$ and $[\mathsf{f_1}(\mathsf{h_1}(x))]_\mathbb{N} = 2x + 2$. Hence the critical peak $\mathsf{h}(\mathsf{g}(\mathsf{f}(x), x, \mathsf{f}(\mathsf{h}(\mathsf{a})))) \leftarrow_{0,1} \mathsf{f}(\mathsf{h}(x)) \to_{0,0} \mathsf{a}$ is closed decreasingly by

$$\mathsf{h}(\mathsf{g}(\mathsf{f}(x), x, \mathsf{f}(\mathsf{h}(\mathsf{a})))) \to_{0,0} \mathsf{c} \to_{0,0} \bot \ {}_{0,0}\!\leftarrow \mathsf{b} \ {}_{0,0}\!\leftarrow \mathsf{a}$$

and the critical peak $\mathsf{h}(\mathsf{g}(\mathsf{f}(x), x, \mathsf{f}(\mathsf{h}(\mathsf{a})))) \ {}_{0,1}\!\leftarrow \mathsf{f}(\mathsf{h}(x)) \to_{0,0} \mathsf{f}(\mathsf{c})$ is closed decreasingly by

$$\mathsf{h}(\mathsf{g}(\mathsf{f}(x), x, \mathsf{f}(\mathsf{h}(\mathsf{a})))) \to_{0,0} \mathsf{c} \to_{0,0} \bot \ {}_{0,0}\!\leftarrow \mathsf{b} \ {}_{0,0}\!\leftarrow \mathsf{a} \ {}_{0,0}\!\leftarrow \mathsf{f}(\mathsf{c})$$

which allows to prove confluence of $\mathcal{R}$ by Corollary 3.25.

By definition of $\alpha > \overline{\gamma}$ (cf. Definition 3.9) we observe that the definition of $\star(\mathcal{R})$ can be relaxed. If $l_2 \to r_2$ with $l_2|_{q'} = x \in \mathcal{V}$ and $\{q'_1, \dots, q'_n\}$ are the positions of the variable x in r_2 then it suffices if $n - 1$ instances of $\star(l_2, q') \to \star(r_2, q'_i)$ are put in $\mathcal{R}^\star_>$ while one $\star(l_2, q') \to \star(r_2, q'_j)$ can be put in $\mathcal{R}^\star_=$ (since the steps labeled $\overline{\gamma}$ in Figure 2(c) are at parallel positions we can choose the first closing step such that $\alpha \geqslant \gamma_1$). This improved version of $\star(\mathcal{R})$ is denoted by $\star\!\!\star(\mathcal{R}) = \mathcal{R}^{\star\star}_> / \mathcal{R}^{\star\star}_=$. We obtain the following variant of Lemma 3.24.

▶ **Lemma 3.27.** *Let $\mathcal{R}$ be a TRS. Then $(\ell_\star, \geqslant, >)$ is an LL-labeling, provided $(\geqslant, >)$ is a monotone reduction pair, $\mathcal{R}^{\star\star}_{>} \subseteq >$, and $\mathcal{R}^{\star\star}_{>} \cup \mathcal{R}^{\star\star}_{=} \subseteq \geqslant$.* ◀

Obviously any $\overset{\star}{\star}(\mathcal{R})$ is terminating whenever $\star(\mathcal{R})$ is. The next example shows that the reverse statement does not hold. In Section 5 we show how the intrinsic indeterminism of $\overset{\star}{\star}(\mathcal{R})$ is eliminated in the implementation.

▶ **Example 3.28.** Consider the TRS $\mathcal{R}$ from Example 1.1. Then $\star(\mathcal{R})$ consists of the rules

$$\begin{array}{rcc}
\mathcal{R}^\star_> & \multicolumn{2}{c}{\mathcal{R}^\star_=} \\
\mathsf{sq}_1(\mathsf{s}_1(x)) \to +_1(\times_1(x)) & \times_1(x) \to \times_2(x) & +_1(x) \to +_2(x) \\
\mathsf{sq}_1(\mathsf{s}_1(x)) \to +_1(\times_2(x)) & \times_2(y) \to \times_1(y) & +_1(\mathsf{s}_1(x)) \to +_1(x) \\
\mathsf{sq}_1(\mathsf{s}_1(x)) \to +_2(\mathsf{s}_1(+_1(x))) & \times_1(\mathsf{s}_1(x)) \to +_1(\times_1(x)) & +_2(y) \to +_1(y) \\
\mathsf{sq}_1(\mathsf{s}_1(x)) \to +_2(\mathsf{s}_1(+_2(x))) & \times_2(\mathsf{s}_1(y)) \to +_2(\times_2(y)) & +_1(+_1(x)) \to +_1(x) \\
\mathsf{sq}_1(x) \to \times_1(x) & +_1(x) \to +_1(\mathsf{s}_1(x)) & +_1(+_2(y)) \to +_2(+_1(y)) \\
\mathsf{sq}_1(x) \to \times_2(x) & +_2(\mathsf{s}_1(y)) \to +_2(y) & +_2(z) \to +_2(+_2(z)) \\
\dagger\colon \times_2(y) \to +_1(\times_2(y)) & +_2(+_2(z)) \to +_2(z) & +_1(x) \to +_1(+_1(x)) \\
\times_2(y) \to +_2(y) & +_2(y) \to +_2(\mathsf{s}_1(y)) & +_2(+_1(y)) \to +_1(+_2(y)) \\
\times_1(x) \to +_1(x) & & \\
\dagger\colon \times_1(x) \to +_2(\times_1(x)) & &
\end{array}$$

Let $\mathcal{R}^\star_\dagger$ denote the rules in $\mathcal{R}^\star_>$ marked with †. Termination of $\star(\mathcal{R})$ cannot be established (because $\mathcal{R}^\star_\dagger$ is non-terminating) but we stress that moving these rules into $\mathcal{R}^\star_=$ yields a valid $\overset{\star}{\star}(\mathcal{R})$ which can be proved terminating by the polynomial interpretation with

$$\mathsf{sq}_{1\mathbb{N}}(x) = x + 2 \qquad\qquad \times_{1\mathbb{N}}(x) = \times_{2\mathbb{N}}(x) = x + 1$$

that interprets the remaining function symbols by the identity function. We remark that Corollary 3.25 with the labeling from Lemma 3.27 establishes confluence of $\mathcal{R}$. Since all reductions in the 34 joining sequences have only + above the redex and $+_{1\mathbb{N}}(x) = +_{2\mathbb{N}}(x) = x$, the $\ell_\star$ labeling attaches zero to any of these steps. The rule labeling that assigns $i(3) = i(6) = 2$, $i(4) = i(10) = 1$, and zero to all other rules shows the 34 critical peaks decreasing.

4 Assessment

In this section we relate the results from this paper with each other and contributions from [1, 8]. First we observe that Corollaries 3.16 and 3.25 subsume Theorem 3.3 since the preconditions of the corollaries evaporate for linear systems. Note that both results extend Knuth and Bendix' criterion [9] (joinability of critical pairs for terminating systems) for left-linear systems. Next we compare the power of Corollaries 3.16 and 3.25. Example 3.20 and the TRS from the following example show that Corollaries 3.16 and 3.25 are incomparable.

▶ **Example 4.1.** It is easy to adapt the TRS from Example 3.18 such that $\star(\mathcal{R})$ becomes non-terminating. Consider the TRS $\mathcal{R}$

$$1\colon \mathsf{b} \to \mathsf{a} \qquad 2\colon \mathsf{a} \to \mathsf{b} \qquad 3\colon \mathsf{f}(\mathsf{g}(x,\mathsf{a})) \to \mathsf{g}(\mathsf{f}(x), \mathsf{f}(\mathsf{g}(x,\mathsf{c})))$$

for which termination of $\mathcal{R}_{\mathsf{d}}/\mathcal{R}_{\mathsf{nd}}$ and decreasingness of the critical peaks is proved similar to Example 3.18. Note that $\mathsf{f}_1(\mathsf{g}_1(x)) \to \mathsf{g}_2(\mathsf{f}_1(\mathsf{g}_1(x))) \in \mathcal{R}^{\star}_{>}$ is non-terminating.[3]

Neither of Corollaries 3.16 and 3.25 gives a necessary confluence criterion for left-linear systems. The TRSs from Example 3.20 and 4.1 are confluent. Hence (by renaming function symbols) so is their direct sum according to Toyama's result [17]. But the combined TRS does not satisfy either precondition of our corollaries.

Next we show that our setting subsumes one of the results from [8]. To this end we define the *critical pair steps* $\mathsf{CPS}(\mathcal{R}) = \{s \to t, s \to u \mid t \leftarrow s \to u \text{ is a critical peak of } \mathcal{R}\}$.

▶ **Theorem 4.2** ([8, Theorem 6]). *Let $\mathcal{R}$ be a left-linear TRS. The TRS $\mathcal{R}$ is confluent if* $\leftarrow\!\rtimes\!\to\ \subseteq\ \to^* \cdot {}^*\!\leftarrow$ *and* $(\mathsf{CPS}(\mathcal{R}) \cup \mathcal{R}_{\mathsf{d}})/\mathcal{R}_{\mathsf{nd}}$ *is terminating.*

By Theorem 2.2 termination of $(\mathsf{CPS}(\mathcal{R}) \cup \mathcal{R}_{\mathsf{d}})/\mathcal{R}_{\mathsf{nd}}$ implies termination of $\mathsf{CPS}(\mathcal{R})/\mathcal{R}$ and $\mathcal{R}_{\mathsf{d}}/\mathcal{R}_{\mathsf{nd}}$. Hence the above result corresponds to Corollary 3.16 using $\ell_{\mathrm{sn}}^{\mathsf{CPS}(\mathcal{R})}$ as weak LL-labeling. Note that our setting is strictly more liberal because of two reasons. First we do not demand a decrease already in the peak, i.e., we can cope with non-terminating $\mathsf{CPS}(\mathcal{R})$. Second, our approach allows to combine ℓ_{sn} lexicographically with further labelings. Examples 3.18 and 3.20 show that the inclusion is strict (for the first reason).

Next we show that Corollary 3.25 generalizes the results from [1, Sections 5 and 6]. It is not difficult to see that the encoding presented in [1, Theorem 5.4] can be mimicked by Corollary 3.25 where linear polynomial interpretations over $\mathbb{N}$ of the shape as in (1)

(1) $\quad f_{i\mathbb{N}}(x) = x + c_f \qquad\qquad$ (2) $\quad f_{i\mathbb{N}}(x) = x + c_{f_i}$

are used to prove termination of $\star(\mathcal{R})$ and $\ell_\star \times \ell_{\mathrm{rl}}$ is employed to show LL-decreasingness of the critical peaks. In contrast to [1, Theorem 5.4], which explicitly encodes these constraints in a single formula of linear arithmetic, our abstract formulation admits the following gains. First, we do not restrict to weight functions but allow powerful machinery for proving relative termination and second our approach allows to combine arbitrarily many labelings lexicographically (cf. Lemma 3.14). Furthermore we stress that our abstract treatment of $\star(\mathcal{R})$ allows to implement Corollary 3.25 based on $\overset{\star}{\star}(\mathcal{R})$ (cf. Section 5) which admits further gains in power (cf. Example 1.1 as well as Section 6).

The idea of the extension presented in [1, Example 6.1] amounts to using $\ell_{\mathrm{rl}} \times \ell_\star$ instead of $\ell_\star \times \ell_{\mathrm{rl}}$, which is an application of Lemma 3.14 in our setting. Finally, the extension discussed in [1, Example 6.3] suggests to use linear polynomial interpretations over $\mathbb{N}$ of the shape as in (2) to prove termination of $\star(\mathcal{R})$. Note that these interpretations are still weight functions. This explains why the approach from [1] fails to establish confluence of the TRSs in Examples 3.18 and 3.20 since a weight function cannot show termination of the rules $\mathsf{f}_1(\mathsf{g}_1(x)) \to \mathsf{g}_1(\mathsf{f}_1(x))$ and $\mathsf{f}_1(\mathsf{h}_1(x)) \to \mathsf{h}_1(\mathsf{g}_1(\mathsf{f}_1(x)))$, respectively.

Note that both recent approaches [1, 8] based on decreasing diagrams fail to prove the TRS $\mathcal{R}$ from Example 1.1 confluent. The former can, e.g., not cope with the non-terminating rule $\times_1(x) \to +_0(\times_1(x))$ in $\mathcal{R}^{\star}_{>}$ (cf. Example 3.28) while overlaps with the non-terminating rule $x + y \to y + x \in \mathcal{R}$ prevent the latter approach from succeeding. On the contrary Examples 3.17 and 3.28 give two confluence proofs based on our setting.

Finally we present a situation when the decreasing diagrams technique typically fails. (In a slightly different setting similar ideas are proposed in [13]. We remark that the recent

[3] We remark that it is easy to extend this example such that also $\overset{\star}{\star}(\mathcal{R})$ is non-terminating. Just consider the rule $\mathsf{f}(\mathsf{g}(x, \mathsf{a})) \to \mathsf{g}(\mathsf{f}(x), \mathsf{g}(\mathsf{f}(\mathsf{g}(x, \mathsf{c}), \mathsf{f}(\mathsf{g}(x, \mathsf{c})))))$.

paper [2] follows a different approach for associativity and commutativity.) To handle such cases we use the following well-known result.

▶ **Lemma 4.3.** *Let $\to \subseteq \rightarrowtail \subseteq \to^*$. Then confluence of $\rightarrowtail$ implies confluence of $\to$.* ◀

The following example contains rules for associativity and commutativity.

▶ **Example 4.4.** Consider the TRS $\mathcal{R}$ consisting of the following two rules

$$x \circ (y \circ z) \to (x \circ y) \circ z \qquad\qquad x \circ y \to y \circ x$$

All four critical peaks are joinable but the critical peak $(y \circ z) \circ x \leftarrow x \circ (y \circ z) \to (x \circ y) \circ z$ cannot be shown decreasing with our labeling functions. Let $\mathcal{S}$ be the TRS $\mathcal{R}$ augmented by the rule $(x \circ y) \circ z \to x \circ (y \circ z)$. All twelve critical peaks of $\mathcal{S}$ can be shown decreasing by the rule labeling and hence $\mathcal{S}$ is confluent. Confluence of $\mathcal{R}$ follows by Lemma 4.3.

5 Implementation

In this section we sketch how the results from this paper can be implemented.

Before decreasingness of critical peaks can be investigated, the critical pairs must be shown convergent. For a critical pair $t \leftarrow\!\rtimes\!\to u$ in our implementation we consider all joining sequences such that $t \to^{\leqslant n} \cdot {}^{\leqslant n}\!\leftarrow u$ and there is no smaller n that admits a common reduct.

To exploit the possibility for incremental confluence proofs by lexicographically combining labels (cf. Lemmata 3.8 and 3.14) our implementation labels rewrite steps with tuples of natural numbers. Since our labeling functions are implemented by encoding the constraints in non-linear (integer) arithmetic it is straightforward to combine existing labels (some partial progress) with the search for a new labeling that shows the critical peaks decreasing.

It is straightforward to implement Corollary 3.16. After establishing termination of $\mathcal{R}_{\mathsf{d}}/\mathcal{R}_{\mathsf{nd}}$ (e.g., by an external termination prover) any weak LL-labeling can be tried to show the critical peaks decreasing. In [1, 8] it is shown how the rule labeling can be implemented by encoding the constraints in linear arithmetic.

We sketch how to implement the labeling $\ell^{\mathcal{S}}_{\mathrm{sn}}$ from Lemma 3.7 as a relative termination problem. First we fix a suitable set $\mathcal{S}$, i.e., we extend the definition of critical pair steps to *critical diagram steps*: $\mathsf{CDS}(\mathcal{R}) = \{s \to t, s \to u, t_i \to t_{i+1}, u_j \to u_{j+1} \mid t \leftarrow s \to u$ is a critical peak in $\mathcal{R}$, $t = t_0 \to \cdots \to t_n = u_m \leftarrow \cdots \leftarrow u_0 = u$, $0 \leqslant i < n-1$, $0 \leqslant j < m-1\}$. Facing the relative termination problem $\mathsf{CDS}(\mathcal{R})/\mathcal{R}$ we try to simplify it according to Theorem 2.1 into some $\mathcal{S}_1/\mathcal{S}_2$. Note that it is not necessary to finish the proof. By Theorem 2.1 the relative TRS $(\mathsf{CDS}(\mathcal{R}) \setminus \mathcal{S}_1)/\mathcal{R}$ is terminating and hence by Lemma 3.7 $\ell^{\mathsf{CDS}(\mathcal{R})\setminus\mathcal{S}_1}_{\mathrm{sn}}$ is an L-labeling. Let $\geqslant := \to^*_{\mathcal{R}}$ and $> := \to^+_{(\mathsf{CDS}(\mathcal{R})\setminus\mathcal{S}_1)/\mathcal{R}}$. Since $\geqslant$ and $>$ can never increase by rewriting, it suffices to exploit the first decrease with respect to $>$. Next we show how critical diagrams are labeled with natural numbers. Consider a rewrite sequence $v_1 \to_{\mathcal{R}} v_2 \to_{\mathcal{R}} \cdots \to_{\mathcal{R}} v_l$. If $v_1 \to^*_{\mathcal{S}_1} v_l$ then all steps are labeled with 1. Otherwise take the largest $k < l$ such that $v_1 \to^*_{\mathcal{S}_1} v_k \to_{\mathcal{R}} v_{k+1} \to^*_{\mathcal{R}} v_l$. Then we set $\ell_{\mathrm{sn}}(v_i \to v_{i+1}) = 1$ for $1 \leqslant i \leqslant k$ and $\ell_{\mathrm{sn}}(v_i \to v_{i+1}) = 0$ for $k < i < l$. Note that $v_k \to v_{k+1}$ is the first step that causes a decrease with respect to $>$, i.e., $v_1 \to_{(\mathsf{CDS}(\mathcal{R})\setminus\mathcal{S}_1)/\mathcal{R}} v_{k+1}$. We demonstrate the above idea on an example.

▶ **Example 5.1.** Consider the following TRS $\mathcal{R}$ from [3]:

$$\mathsf{I}(x) \to \mathsf{I}(\mathsf{J}(x)) \qquad \mathsf{J}(x) \to \mathsf{J}(\mathsf{K}(\mathsf{J}(x))) \qquad \mathsf{H}(\mathsf{I}(x)) \to \mathsf{K}(\mathsf{J}(x)) \qquad \mathsf{J}(x) \to \mathsf{K}(\mathsf{J}(x))$$

We show how the labels for the critical peak $\mathsf{H}(\mathsf{I}(\mathsf{J}(x)))\ {}_1\!\leftarrow \mathsf{H}(\mathsf{I}(x)) \rightarrow_1 \mathsf{K}(\mathsf{J}(x))$ and the joining sequences $\mathsf{H}(\mathsf{I}(\mathsf{J}(x))) \rightarrow_1 \mathsf{K}(\mathsf{J}(\mathsf{J}(x))) \rightarrow_0 \mathsf{K}(\mathsf{J}(\mathsf{K}(\mathsf{J}(x))))\ {}_1\!\leftarrow \mathsf{K}(\mathsf{J}(x))$ can be established by $\ell^{\mathcal{S}}_{\mathrm{sn}}$. Let $\mathcal{S}$ be the TRS generated by the steps in the critical peak and the joining sequences above. The interpretation $\mathsf{K}_{\mathbb{N}}(x) = \mathsf{H}_{\mathbb{N}}(x) = \mathsf{J}_{\mathbb{N}}(x) = x$ and $\mathsf{I}_{\mathbb{N}}(x) = x + 1$ allows to "simplify" termination of the problem $\mathcal{S}/\mathcal{R}$ according to Theorem 2.1. Since the rules that reduce the number of $\mathsf{I}'s$ are dropped from $\mathcal{S}$ (and $\mathcal{R}$), those rules admit a decrease in the labeling.

The above trick does not work to implement Corollary 3.25, since $s \rightarrow_{\mathcal{R}} t \rightarrow_{\mathcal{R}} v$ does not ensure $\ell_\star(s \rightarrow t) \geqslant \ell_\star(t \rightarrow v)$. Here the solution is to employ only techniques (for proving the relative TRS $\star(\mathcal{R})$ terminating) that can label a rewrite step with a concrete number. To this end we will recall matrix interpretations [5] which are a very powerful method for proving termination of relative rewrite systems that allow to compute a variant of $\ell_\star$.

An $\mathcal{F}$*-algebra* $\mathcal{A}$ consists of a non-empty carrier A and a set of interpretations $f_{\mathcal{A}}$ for every $f \in \mathcal{F}$. By $[\alpha]_{\mathcal{A}}(\cdot)$ we denote the usual evaluation function of $\mathcal{A}$ according to an assignment α which maps variables to values in A. An $\mathcal{F}$-algebra $\mathcal{A}$ together with a well-founded order $\succ$ and a quasi-order $\succsim$ on A is called a *monotone algebra* if every $f_{\mathcal{A}}$ is monotone with respect to $\succsim$ and $\succ$ and the inclusion $\succsim \cdot \succ \cdot \succsim\ \subseteq\ \succ$ holds. Any monotone algebra $(\mathcal{A}, \succsim, \succ)$ induces a well-founded order on terms: $s \succ_{\mathcal{A}} t$ if for any assignment α the condition $[\alpha]_{\mathcal{A}}(s) \succ [\alpha]_{\mathcal{A}}(t)$ holds. The quasi-order $s \succsim_{\mathcal{A}} t$ is similarly defined.

Matrix interpretations $(\mathcal{M}, \succ)$ (often just denoted $\mathcal{M}$) are a special kind of monotone algebras. Here the carrier is $\mathbb{N}^d$ for some fixed dimension $d \in \mathbb{N} \setminus \{0\}$. The orders $\succsim$ and $\succ$ are defined on $\mathbb{N}^d$ as $\vec{u} \succsim \vec{v}$ if $(\vec{u})_i \geqslant_{\mathbb{N}} (\vec{v})_i$ for all $1 \leqslant i \leqslant d$ and $\vec{u} \succ \vec{v}$ if $\vec{u} \succsim \vec{v}$ and $(\vec{u})_1 >_{\mathbb{N}} (\vec{v})_1$. Here $(\vec{v})_1$ denotes the first element of the vector $\vec{v}$. If every $f \in \mathcal{F}$ of arity n is interpreted as $f_{\mathcal{M}}(\vec{x_1}, \ldots, \vec{x_n}) = F_1\vec{x_1} + \cdots + F_n\vec{x_n} + \vec{f}$ where $F_i \in \mathbb{N}^{d \times d}$ for all $1 \leqslant i \leqslant n$ and $\vec{f} \in \mathbb{N}^d$ then monotonicity of $\succ$ is achieved by demanding that the top left entry of every matrix F_i is non-zero. Let α_0 denote the assignment with $\alpha(x) = \vec{0}$ for all variables x.

▶ **Lemma 5.2.** *Let $\mathcal{R}$ be a TRS and $\ell^{\mathcal{M}}_\star(s \rightarrow_{p,l\rightarrow r} t) = ([\alpha_0]_{\mathcal{M}}(\star(s,p)))_1$ for some matrix interpretation $\mathcal{M}$. Then $(\ell^{\mathcal{M}}_\star, \geqslant_{\mathbb{N}}, >_{\mathbb{N}})$ is a weak LL-labeling, provided $\mathcal{R}^\star_> \cup \mathcal{R}^\star_= \subseteq\ \succsim_{\mathcal{M}}$.*

Proof. That $\ell^{\mathcal{M}}_\star$ is a labeling follows from the fact that $(\succsim_{\mathcal{M}}, \succ_{\mathcal{M}})$ is a monotone reduction pair and $\mathcal{R}^\star_> \cup \mathcal{R}^\star_= \subseteq\ \succsim_{\mathcal{M}}$. The latter also ensures that $\ell^{\mathcal{M}}_\star$ is a weak LL-labeling. ◀

To establish progress with Lemma 5.2 the implementation demands $\succ_{\mathcal{M}} \cap\, \mathcal{R}^\star_> \neq \varnothing$. By repeated applications of Lemmata 5.2 and 3.8 weak LL-labelings are combined lexicographically until they form an LL-labeling. This is exactly the case if termination of $\ddagger(\mathcal{R})$ can be established using matrix interpretations.

Finally, we explain why $\ddagger(\mathcal{R})$ need not be computed explicitly to implement Corollary 3.25 with the labeling from Lemma 3.27. The idea is to start with $\star(\mathcal{R})$ and incrementally prove termination of $\mathcal{R}^\star_>/\mathcal{R}^\star_=$ until some $\mathcal{S}_1/\mathcal{S}_2$ is reached. If all left-hand sides in $\mathcal{S}_1$ are distinct then they must have been derived from different combinations (l, x) with $l \rightarrow r \in \mathcal{R}$ and $x \in \mathcal{V}\mathsf{ar}(l)$. Hence they are exactly those rules which should be placed in $\mathcal{R}^\star_=$. We show the idea by means of an example.

▶ **Example 5.3.** We revisit Example 1.1 and try to prove termination of $\star(\mathcal{R})$. By an application of Theorem 2.1 with the interpretation given in Example 1.1 the problem is termination equivalent to $\mathcal{R}_\dagger/\mathcal{R}^\star_=$ and by another application of Theorem 2.1 the same proof can be used to show termination of $(\mathcal{R}^\star_> \setminus \mathcal{R}^\star_\dagger)/(\mathcal{R}^\star_= \cup \mathcal{R}^\star_\dagger)$ which is a suitable candidate for $\ddagger(\mathcal{R})$ since the rules in $\mathcal{R}^\star_\dagger$ have different left-hand sides.

	without Lemma 4.3				with Lemma 4.3			
method	pre	CR(ℓ_{rl})	CR(ℓ_{sn})	CR	pre	CR(ℓ_{rl})	CR(ℓ_{sn})	CR
rule labeling	40(0.2)	35(0.2)	–	–	40(0.2)	37(0.3)	–	–
Corollary 3.16	45(0.3)	40(0.6)	37(1.4)	42(1.5)	45(0.3)	42(0.7)	40(1.4)	44(1.5)
Corollary 3.25$\star$	45(0.3)	42(0.3)	34(1.1)	42(1.2)	45(0.3)	44(0.3)	38(1.1)	44(1.2)
Corollary 3.25⁑	48(0.3)	45(0.3)	36(1.1)	45(1.2)	48(0.3)	47(0.4)	40(1.1)	47(1.3)
ACP	–	42(0.1)	–	48(0.1)	–	–	–	–

Table 1 Experimental results for 53 left-linear TRSs.

6 Experiments

The results from the paper have been implemented and form the core of the confluence prover CSI [18]. For experiments[4] we used the collection from [1][5] which consists of 106 TRSs from the rewriting literature dealing with confluence. From these systems 67 are left-linear, but 14 of these are known to be non-confluent which gives a theoretical upper bound of 53 systems for which the proposed methods can succeed. Our experiments have been performed on a notebook equipped with an Intel® Core™2 Duo processor U9400 running at a clock rate of 1.4 GHz and 3 GB of main memory.

Table 1 shows an evaluation of the results from this paper. The first column indicates which criterion has been used to investigate confluence. A $\star$ means that the corresponding corollary is implemented using $\star(\mathcal{R})$ whereas ⁑ refers to ⁑$(\mathcal{R})$. The column labeled pre shows for how many systems the precondition of the respective criterion is satisfied, e.g., for rule labeling the precondition is linearity while for Corollary 3.16 the precondition is termination of $\mathcal{R}_{\mathsf{d}}/\mathcal{R}_{\mathsf{nd}}$. The columns labeled CR(ℓ) give the number of systems for which confluence could be established using labeling ℓ. (For Corollary 3.25 implicitly $\ell_\star$ is also employed.) The column labeled CR corresponds to the full power of our approach, i.e., when the lexicographic combination of all labelings is used. In Table 1 the numbers in parentheses indicate the average time for establishing the precondition (column pre) and finding a confluence proof (remaining columns) in seconds, respectively. These timings show that establishing the precondition is fast and the same holds for ℓ_{rl}. The most costly criterion is ℓ_{sn} which is also used in column CR. All tests finished within 60 seconds.

From the table we draw the following conclusions. Depending on the labeling function (ℓ_{rl} versus ℓ_{sn}) either Corollary 3.16 or Corollary 3.25 can handle more systems. When both labelings are used, Corollary 3.25⁑ subsumes Corollary 3.16 on this testbed. Corollary 3.25$\star$ does not subsume Corollary 3.16 since $\star(\mathcal{R})$ is non-terminating for the TRS in Example 1.1 which is contained in this testbed. For the systems where Corollary 3.25⁑ succeeded but Corollary 3.25$\star$ failed the corresponding relative TRS $\star(\mathcal{R})$ is non-terminating. The three systems where the precondition of Corollary 3.25 is satisfied but confluence could not be shown (without Lemma 4.3) contain rules for associativity and commutativity. To cope with (two of) these systems we exploit the ideas from Example 4.4. The corresponding numbers are given in the right part of Table 1.

For reference we also give the data for ACP [3], a powerful confluence prover which implements various confluence criteria from the literature. According to [1] their tool can

[4] Details available from `http://cl-informatik.uibk.ac.at/software/csi/labeling`.
[5] `http://www.nue.riec.tohoku.ac.jp/tools/acp/examples/crexamples-100410.tgz`

method	1.1	3.18	3.20	4.1	5.1
rule labeling	×(0.2)	×(0.2)	×(0.3)	×(0.2)	×(0.2)
Corollary 3.16	✓(4.6)	✓(0.5)	×(0.5)	✓(0.6)	✓(1.9)
Corollary 3.25⋆	×(1.4)	✓(0.5)	✓(1.5)	×(0.3)	✓(1.8)
Corollary 3.25⁑	✓(4.6)	✓(0.5)	✓(1.3)	✓(0.5)	✓(1.9)
ACP	×(11.5)	×(0.1)	×(0.7)	×(0.1)	✓(0.1)

Table 2 Experimental results for the examples from the paper.

show 42 systems confluent by (their extensions of) the rule labeling and using its full power ACP can prove 48 systems confluent. In the collection considered for Table 1 there is one system (Example 1.1) which ACP cannot handle but where our approach (because we consider ⁑($\mathcal{R}$)) succeeds and two systems that we can show confluent by adding rules as proposed in Example 4.4. Consequently we miss four systems compared to the full power of ACP which handles them by considering development closedness [11] and parallel critical pairs [16, 7]. Note that these criteria investigate confluence of $\mathrel{-\!\!\circ\!\!\!\rightarrow}$ and $\mathrel{+\!\!\!\!\rightarrow}$, while our approach considers $\rightarrow$. Since for these systems neither $\mathcal{R}_{\mathsf{d}}/\mathcal{R}_{\mathsf{nd}}$ nor ⁑($\mathcal{R}$) is terminating there also is not much hope that our current approach can be extended to handle these systems. Hence as future work we will study properties on labeling functions that allow to investigate confluence of $\mathrel{+\!\!\!\!\rightarrow}$ and $\mathrel{-\!\!\circ\!\!\!\rightarrow}$.

Table 2 does a similar evaluation as Table 1 on the examples from the paper. Here a ✓ indicates that the tool could establish confluence while a × means that the tool failed. The numbers in parentheses give the time (in seconds) the tool spent on the respective example.

From Tables 1 and 2 we conclude that our framework admits a state-of-the-art confluence prover for left-linear systems. For the sake of completeness we remark that ACP also supports confluence analysis for non-left-linear systems.

7 Conclusion

In this paper we studied how the decreasing diagrams technique can be automated. We presented conditions (subsuming recent related results) that ensure confluence of a left-linear TRS whenever its critical peaks are decreasing. The labelings we proposed can be combined lexicographically which allows incremental proofs of confluence and has a modular flavor in the following sense: Whenever a new labeling function is invented, the whole framework gains power. We discussed several situations (Examples 1.1, 3.18, 3.20, 4.1) where standard confluence techniques fail but our approach easily establishes confluence.

Currently all our investigations are aimed to show confluence of $\rightarrow$. As motivated in Section 6 one obvious issue for future work is to study conditions on the labelings such that $\mathrel{+\!\!\!\!\rightarrow}$ (or $\mathrel{-\!\!\circ\!\!\!\rightarrow}$) can be shown confluent. This would allow to handle the systems which we currently lose against ACP in Table 1. Furthermore, if the recent developments in the termination community will also reach confluence, then automatic certification of confluence proofs by means of a theorem prover is inevitable. Since our setting is based on a single method (decreasing diagrams) while still powerful it seems to be a good starting point for certification efforts.

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