

Reconstructing basic ideas in geometry – an empirical approach

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Abstract: *“Basic ideas” (or “fundamental ideas” etc.) have been discussed in mathematical curriculum theory for about forty years. This paper will centre on the hypothesis that this concept can only be applied successfully by using it as a category for the analysis of concrete mathematical problems. This hypothesis will be illustrated by means of a sample problem from the Austrian Standards for Mathematics Education (“Bildungsstandards”). In this example, basic ideas are used in a content matter analysis which takes students’ solutions to the problem as a starting point for the creation of a potentially substantial learning environment in trigonometry.*

Kurzreferat: *„Grundlegende Ideen“ (bzw. „fundamentale Ideen“ etc.) werden in der Mathematikdidaktik seit etwa 40 Jahren diskutiert. In diesem Aufsatz wird die These vertreten, dass dieser Begriff nur dann erfolgreich angewandt werden kann, wenn er als Kategorie zur Analyse konkreter mathematischer Problemstellungen genutzt wird. Diese These wird anhand der Analyse einer Beispielaufgabe aus den österreichischen Bildungsstandards konkretisiert. In diesem Beispiel werden grundlegende Ideen als Kategorie einer didaktisch orientierten Sachanalyse genutzt, die Lösungsansätze von Lehramtsstudierenden zu dieser Aufgabe als Ausgangspunkt für die Konzeption einer potenziell produktiven Lernumgebung zur Trigonometrie begreift.*

ZDM-Classifikation: D20, D30, D50, G30

1. Introduction

In the mid 1960s, Jerome S. Bruner’s “The Process of Education”, emphasized the important role of the “structure of the discipline”(Bruner 1962)¹. Since then, the principle of concentrating teaching and learning of mathematics on some meaningful “basic ideas” or “fundamental ideas” has been widely discussed – especially in Germany and Austria. Over the past forty years a wide range of different approaches to identify such ideas and to put them

¹ To some extent (mis-)leading to a rather strict and ambivalent interpretation at best by the “New Math”-movement.

into practice has been discussed in German and Austrian mathematics education. These conceptions range from rather philosophical approaches with little impact on actual mathematics teaching to mere pragmatical ones. The latter sometimes raise the question whether there is more to them than simply coining new buzzwords.

In my opinion, the only way to overcome both problems is to use basic ideas as an analytical category at a level more closely related to actual mathematics teaching: to use such basic ideas for the analysis of concrete sample problems. In this paper, basic ideas are used to classify and analyse different approaches taken by students to solve a given open ended geometrical sample problem. The results of this analysis are then used to frame a genetic approach to one of trigonometry’s finest theorems: the law of cosines.

2. Group of subjects and sample problem

The following analysis is based on the investigation of a group of 28 subjects: students (prospective teachers for either primary or secondary schools). The subjects were taking my course “Problem Solving & Heuristics” last winter semester. The actual material analysed are the students’ pieces of homework for the course which were handed in electronically (via email). The students were not introduced to any conception of basic ideas during the whole course. The original purpose of the analysed pieces of homework was to solve some sample problems from OECD-PISA as well as from German and Austrian standards for mathematics education at 8th/ 9th grade level. The students were told to do this according to George Polya’s famous steps for solving mathematical problems². The sample problem to be discussed here is the following:

“Chris and Angela are lying at the beach. Chris has to go 30 meters to reach the ice cream parlour, Angela has to go 40 meters. What is the distance between Angela and Chris? Discuss different positions (in a group of 3 to 4 pupils) and make a poster presentation for the class. Which positions allow a simple calculative solution?” (Bundesministerium für Bildung, Wissenschaft und Kultur 2004, p. 105)³

² As introduced in his book “How to solve it” (Polya 2004).

³ The sample problem and the pieces of homework were originally written in German, all translations have been done by the author.

The students were explicitly told to try and solve this problem by mathematical means an 8th-/9th-grader would have normally been taught so far.

3. Method

The material was analysed in three steps:

1. First, all pieces of homework were roughly scanned through and then classified into a limited number of idealized approaches.
2. Second, connections between these idealized approaches and such basic ideas that had been selected beforehand were explored. In other words, the influences of these basic ideas on the approaches were examined.
3. Third, the underlying content matter was analysed irrespective of the actual solutions discussed before. This step could be described as “widening the content matter analysis”. Its main goal is to draw conclusions from the analysis in Step 2, to modify and broaden the sample problem, so it becomes a possibly “substantial learning environment”(Wittmann 2001, p. 2).

During the second step I had a rather wide collection of *potentially connected* basic ideas in mind. Hence, there was no fixed set of ideas which necessarily had to be linked to the problem only because of the mathematical content matter. In other words: Relevant basic ideas weren't postulated theoretically but their relevance had to be proven empirically.

Concerning the actual presented solutions, it became apparent that three basic ideas had the most significant influence on the student's solutions: the idea of geometrical structuring, the idea of measurement, and the idea of functional reasoning. To see the importance of such basic ideas, it is not only crucial to regard them as meta-conceptions⁴ but also to discuss related local sub-conceptions. Those sub-conceptions could be both of heuristical and of conceptual kind. In my opinion this methodology opens up the possibility of bridging the gap between qualitative empirical research on the one hand and traditional ways of analysing the content matter in itself on the other hand⁵.

⁴ Which – regarding to Peschek (2005) – consist of a network of typical strategies, operations and techniques linked to those ideas.

⁵ The latter one seems to have dramatically gone out of fashion at least in German didactics of mathematics over the past twenty or maybe thirty years.

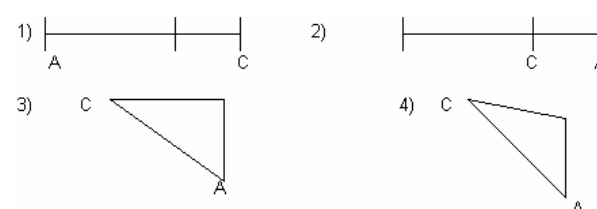
4. Idealized approaches to the problem

4.1 Calculative approach using the Pythagorean Theorem

In this approach students realize the supposedly underdetermined nature of the problem in the step “understanding the problem”. They state that “for the calculation of a triangle there have to be three given parts (SAS, SSS, and SSA). The problem definition gives only two pieces of information and is thus insufficient.” When students “devise a plan”, they suggest “to change the problem definition, so that there is a right angle at the corner with the ice cream parlour”. “Carrying out the plan” can then easily be reduced to “the correct use of the Pythagorean Theorem”. All other cases are classified as “unsolvable”. This pure calculative approach was only taken by four students.

4.2 The “two straight lines and two triangles”-approach

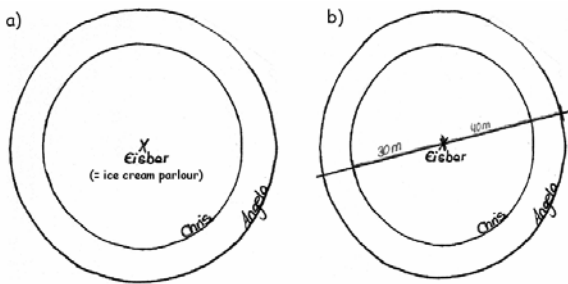
Figure 1. Two straight lines and two triangles



In this approach – which was taken most often with 19 subjects using it as the only or as a supporting approach – students follow Polya's advice for “understanding the problem” by drawing a figure of the situation. Early in their approach they take into account that “the problem definition leaves open different options for the actual position of Angela, Chris and the ice cream parlour”. One student wisely states that “a figure will help to examine those options which may lead to solutions of different calculative complexity”. She – and nearly every other student who took this approach – then concludes “the actual positions presented in case 1) and 2) allow a simple calculative solution, case 3) relies on previously achieved specific knowledge (Pythagorean Theorem), and case 4) must be regarded as unsolvable with the details given in the problem definition”. In some of the students' solutions the fourth case (no right angle) was even missing completely; nonetheless, there were two subjects who realized, that the right angle could be at another corner of the rectangle, also allowing a simple calculative solution by means of the Pythagorean Theorem.

4.3 The “two circles and extreme values”-approach

Figure 2. Two circles and extreme values



Only a single student used this approach as their only approach, all the other four subjects who suggested this approach also took one of the former two as their main approach. Nonetheless, this approach opens up a new and interesting perspective on the sample problem. When trying to “understand the problem” students identify the core of the problem as the answer to the question “whether Chris and Angela are as far as possible away from each other or as close as possible to each other”. In Figure 2, the characteristic of the circle⁶ is used correctly to both identify the two simple cases from the last-mentioned approach⁷ and answer their question: “10 meters and 70 meters are the shortest/longest possible distance between Angela and Chris, also allowing easy calculation as in those cases all three points are on a straight line and the distances can simply be added or subtracted”.

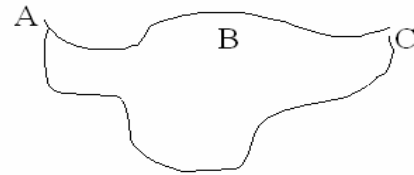
4.4 Interesting cases of ignoring the implicit “Do What I Mean” of the problem definition

The implicit “Do What I Mean (DWIM)”-phenomenon was introduced as a phrase to German mathematics education by Jahnke (2005). Jahnke states, that each and every problem definition not only gives students concrete data to calculate with, but also a more or less explicit or implicit list of “Do’s and Don’ts”; and which applicable solving techniques to be chosen, as in inherent conditions that are not mentioned explicitly as they can be taken for granted in the context of mathematical instruction (see Jahnke 2005, pp. 9-11). A few of the students solutions’ are clear cases of ignoring those implicit DWIM-instructions in the problem definition or the advice given by me during the course. These solutions are particularly interesting in so far as some of those approaches are led by a desire to have calculative control over all possible

⁶ The circle is defined as the line that is equally distant to a fixed point, in this case the ice cream parlour.
⁷ Figure 1, case 1 and 2.

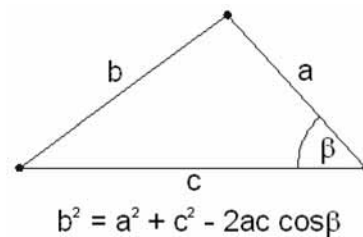
positions⁸. Exactly that desire opens up the door to trigonometry!

Figure 3. No straight lines



The approach shown in figure 3 does not bring us closer to a solution, yet it shows us the rigid “Don’ts” of the problem definition: If one wants to solve this task by mathematical (especially calculative) means, one has to assume, that the given distances are measured on straight lines. If we take this task seriously as a real world problem, figure 3 might be closer to reality. But the problem definition implicitly suggests using much simpler models. Such as models that allow for easy geometrical structuring and arithmetic calculation⁹.

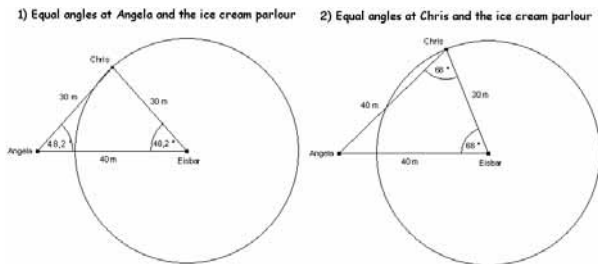
Figure 4. The law of cosines



Another student mimics the lazy but clever pupil: When “looking back on his work” he tries to find a better way by looking up a theorem related to the problem in a formulary: “I found a universal triangle and made a new sketch of the situation”. Needless to say, the authors of Austrian Standards for Mathematics Education would not be too pleased with that answer.

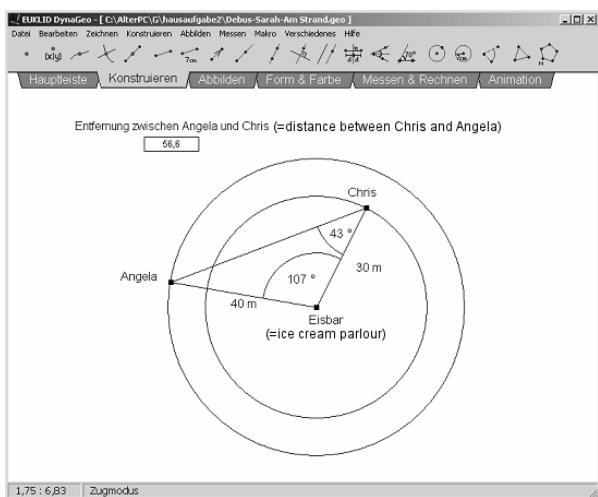
⁸ Especially the fourth case/ universal case in figure 1.
⁹ In German mathematics education such tasks are called “eingekleidete Aufgaben”, which literally translates as “dressed up tasks”, as their authors try to dress up a plain mathematical task with the help of a realistic context that should not be taken too serious.

Figure 4. Isosceles triangles



The last approach to be discussed here does not only show straight lines and both cases in which the Pythagorean Theorem is applicable, but also the solution presented in figure 4. Assuming isosceles triangles is a very simple way of solving the problem: Nothing needs to be calculated; the distance has to be either 30 or 40 meters. Yet this assumption is in no way more far-fetched than assuming a right-angled triangle.

Figure 5. Dynamic geometry solution



The student who chose this approach also sent me a file for a dynamic geometry system, where you could move the position of Chris and Angela on the two circles (see figure 5). The dynamic geometry software directly allows us to measure the distance between Chris and Angela for all possible positions. Again, nothing is directly calculated. Measurement is done by the software which internally uses a numerical approximation.

5. Inherent basic ideas

The diagram in figure 6 (see next page) tries to categorize the approaches discussed earlier based on the underlying (implicit) basic ideas. If we discuss our problem from a synthetic geometry point-of-view, the problem is rather simple: All distances

between 10 and 70 meters could be the questioned distance, as we can easily construct all possible triangles considering the SSS-law of congruence. The data given in the problem definition does not allow for a more precise answer from this point-of-view: There is no hint about any angle in the triangle, so every angle is equally possible. If one wants to construct a possible triangle, there are no easy “special” cases: Every triangle is equally well constructible. This conclusion is represented in the “two circles”-approach. The underlying basic idea is the idea of geometrical structuring: We use triangles and circles to structure the situation described in the problem definition. Considering the characteristics of triangles and circles leads us to the conclusion that any distance between 10 and 70 meters is possible, no particular distance between 10 and 70 meters is more likely than any other distance.

Our conclusions change dramatically when we switch to an arithmetical or algebraic point-of-view: Based on the typical knowledge of an 8th- or 9th-grader there are only a few cases that allow for easy calculation of the possible distance. If we want to calculate the distance, we have to assume the angle β to equal either 0° , 90° or 180° . Only in these cases is our knowledge about geometrical laws and theorems, which can be used as “formulas”, sufficient to calculate the questioned distance by means of the given distances. In this case, the underlying idea is the idea of (exact) measurement: Geometrical laws and theorems might be read as “formulas” to calculate specific parts of a geometrical figure by means of other parts. This calculation relies on measuring experience which is virtually accumulated in those formulas (see Baireuther 1990, p. 84 & p. 99-103, see also Vohns 2005, pp. 77-79).

Both the synthetic geometry point of view and the arithmetical / algebraic point of view are found partly unsatisfactory by the students. If we rely on SSS, the questioned distance can only be found through the act of construction. If we rely on our applicable knowledge about laws and theorems which can be used as formulas, only some special cases can be discussed.

There is a desire to achieve a formula which lets us to calculate the distance depending on every possible angle in the centre of the circle. If we use a dynamic geometrical system to solve the problem, our solution is of empirical-numerical kind. The internal system uses a numerical approximation to solve the problem, yet allowing an immediate solution for every manipulation of the position of Chris and Angela on the circumference of the two

Figure 6. Approaches and inherent basic ideas

**synthetic geometry
point of view**

Every distance between 10 and 70 meters can be constructed easily.

Knowledge from synthetic geometry lessons of prior school years (SSS law of congruence)

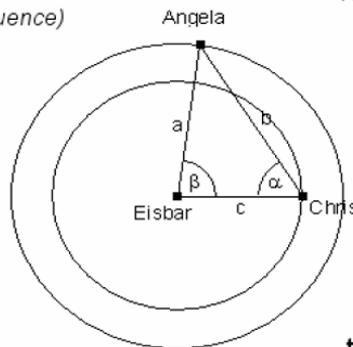
Idea of geometrical structuring

**arithmetic / algebraic
point of view**

If β equals $0^\circ, 90^\circ$ or 180° the distance can be calculated easily by means of known formulas.

Knowledge about geometrical formulas (Pythagorean theorem)

Idea of measurement



**empirical-numerical
approach**

All possible distances can be "measured" easily within a dynamic geometry system

**trigonometrical
approach**

Every distance might be calculated easily by means of the law of cosines

quasi-functional point-of-view

circles. Thus an exact arithmetical solution for all possible cases can only be achieved by means of the law of cosines. If two sides of the triangle are given, one can read the law of cosines as a function of the angle β . The last two approaches adopt a (quasi-) functional point of view: Dynamic geometry software allows diachronic examination of all possible distances through interactive manipulation of the positions of Angela and Chris. The law of cosines additionally allows for a synchronic overview over the whole definition range.

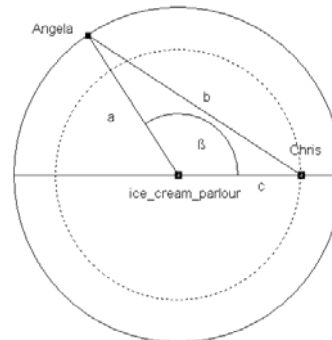
**6. Broadening the sample problem:
A genetic pathway to the law of cosines**

If we sum up our findings so far, we must classify the law of cosines as the only fully satisfactory solution. That raises the question whether we are able to broaden our sample problem to use it to form a substantial learning environment for the teaching of the law of cosines. The approach of using dynamic geometry software might be a good starting point, as it approximates the functional approach best.

Reconsidering the solution presented in figure 5, we see that it does not allow for a simple functional model. There are too many points that can be manipulated.

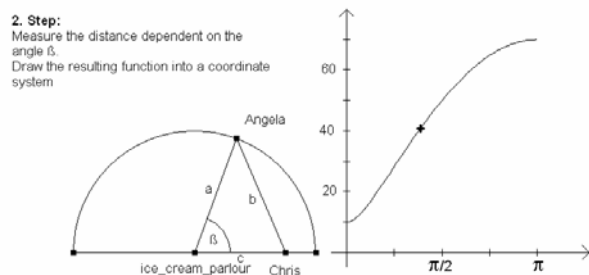
Figure 7. First Step

1. Step:
Fixing the position of Chris and the ice cream parlour



This evokes a characteristic strategy linked with the idea of functional reasoning: try to reduce the number of variables. In our problem the number of variables can be reduced by fixing the positions of e.g. Chris and the ice cream parlour. By fixing their positions – while still leaving the position of Angela to be manipulated – no possible distance is lost. This also allows for a second step of modifying the problem: Angela’s position can now be interpreted as a function of the angle at the centre of the circle; and also the distance between Chris and Angela might be interpreted as a function of that angle.

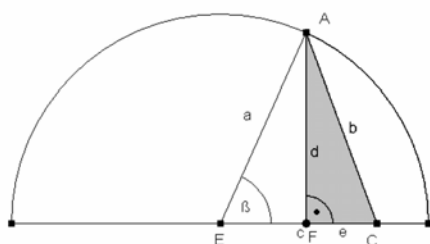
Figure 8. Exploring the function



If we use the dynamic geometry system’s capabilities to measure the distance and to plot the locus of the measured distance depending on the angle β , the resulting function is not an easy guess. If we want to extend the solution from a mere empirical-numerical to an exact algebraic one, we have to change our point of view again.

Figure 9. Using an auxiliary problem

3. Step:
 Seeing similarities to the definition of sine / cosine
 => Construct a perpendicular from A



When devising a plan to solve a problem Polya (2004) suggests asking oneself: “Do you know a related problem? Do you know a theorem that could be helpful?” (Polya 2004, p. xvi). The important heuristic of auxiliary problems can be very helpful here. The triangle ECA is inscribed into a semicircle. If only the angle at C was a right angle, our figure would match the figure used for the definition of sine and cosine in terms of a unit circle. This knowledge can be used to restructure the geometrical situation: We should draw a supporting line. In our case this means constructing a perpendicular line from A. The result of that is shown in figure 9. This figure is the typical figure used to prove the law of cosine based on our knowledge about right-angled triangles. We know the length of side a, we know the angles at E and F, so we can calculate both the length of d and the distance between E and F. Afterwards we can calculate the length of e and again we know three parts of the grey triangle: the length of d and e and

the angle at F. Now we can calculate the length of b. The reader will easily see for himself, that in our case b equals $a^2 + c^2 - 2 \cdot a \cdot c \cdot \cos \beta$. So we found a satisfactory answer to our original question as well as a proof for the law of cosines.

If we take some time to look back on our solution, we can clearly see the implicit but nevertheless important role of locally interpreted basic ideas: Restructuring the problem is without any doubt related to both the idea of geometric structuring and the idea of measurement. Without sound knowledge of geometric structures, related laws and theorems one simply would not be able to find meaningful auxiliary problems. On the other hand measurement comes into consideration because finding auxiliary problems means finding simpler problems. A simpler problem means a problem for which our knowledge about geometric measurement is sufficient to solve the problem. This means, in particular, that we need to find subfigures that can be calculated by means of laws and theorems which can be used as formulas, in other words as most elaborate forms of exact geometric measurement.

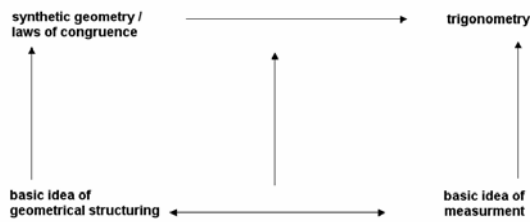
Our sample problem is also a broad illustration of the hypothesis that Polya’s rather universal heuristics are at least closely connected to local sub-conceptions of basic ideas, in fact they could be understood as such sub-concepts themselves. Drawing a supporting line is practically useless, if it doesn’t help to restructure the geometric situation. It simply is one of the most important ways of geometric structuring. However, it is also closely connected to one of the most elementary strategies of geometric measurement: reducing a figure to subfigures which can be measured more easily. It also evokes a more elaborate strategy of measurement in our case: “measured more easily” simply means “easily calculated by already known formulas”.

7. Conclusions

This paper started with the hypothesis that the concept of basic ideas can only be applied successfully by using it as a category for the analysis of concrete mathematical problems. To illustrate this hypothesis and give some sample evidence we applied basic ideas to categorize and analyse different approaches to a geometrical sample problem. Our analysis classified the two major approaches students took according to typical knowledge of an 8th- or 9th-grader as in some way unsatisfactory: If one wants to have calculative control over all possible solutions, the typical

knowledge about geometrical laws and theorems is insufficient.

Figure 10. Interconnections



From a synthetic geometric point-of-view all possible cases are equally difficult, but only a few special cases can be solved by arithmetical or algebraic means. To broaden our geometrical knowledge means to broaden our knowledge about elaborate ways of geometrical measurement: that is to gather calculative control over the universal triangle. Broadening our knowledge in this way marks the changeover from synthetic geometry to trigonometry as well as changing our point-of-view from mere geometrical structuring to (exact) geometrical measurement. Using well-known heuristics closely connected to those basic ideas, we found a genetic pathway from exploring the problem empirically with help of a dynamic geometry system to proving the underlying law of cosines.

In my opinion, this example of the application of basic ideas also shows the importance of linking traditional ways of content matter analysis with qualitative empirical methods. Empirical data was the source of our investigations, while widening the related content matter theoretically was the key point to see the potential of the sample problem as core problem for a genetic approach, as a possible substantial learning environment.

To close this paper, a last remark should be made: Current trends in German mathematics education pay very much attention to the role of sample problems. Nevertheless, sample problems alone will not ensure the quality of mathematics instruction. As researchers in mathematics education one of our major tasks is to provide mathematics teachers with examples of how to use sample problems to design substantial learning environments. For this task, the theoretical analysis of the mathematical content matter as well as the empirical analysis of the actual students' solutions is equally essential. Linking both ways has an enormous potential, which has yet to be investigated closer.

References

- Baireuther, P. (1990): *Konkreter Mathematikunterricht*. Bad Salzdetfurth: Franzbecker.
- Bruner, J. (1962): *The process of education*. Cambridge: *Harvard University Press*.
- Bundesministerium für Bildung, Wissenschaft und Kultur (Ed.) (2004): *Bildungsstandards für Mathematik am Ende der achten Schulstufe* (Version 3.0). Retrieved from http://www.gemeinsamlernen.at/siteVerwaltung/mOBibliothek/Bibliothek/Standards_Endversion_korr_25-10_eBook.pdf
- Jahnke, T. (2005): *Aufgaben im Mathematikunterricht*. Manuscript. Retrieved from http://www.math.uni-potsdam.de/prof/o_didaktik/a_mita/aa/Publ/mu
- Pesчек, W. (2005). Reflexion und Reflexionswissen in R. Fischers Konzept der höheren Allgemeinbildung. In K. Lengnink & F. Siebel (Eds.), *Mathematik präsentieren, reflektieren, beurteilen* (pp. 55-68). Darmstadt: Verlag Allgemeine Wissenschaft.
- Polya, G. (2004). *How to solve it*. Expanded Princeton Science Library Edition. Princeton: *Princeton University Press*.
- Vohns, A. (2005). Messen oder (Be-)Rechnen? Mit fundamentalen Ideen über Mathematik reflektieren. In K. Lengnink & F. Siebel (Eds.), *Mathematik präsentieren, reflektieren, beurteilen* (pp. 69-84). Darmstadt: Allgemeine Wissenschaft.
- Wittmann, E. Ch. (2001). *Drawing on the richness of elementary mathematics in designing substantial learning environments*. Report of the PME 25 Research Forum "Designing, researching and implementing mathematical learning environments". Retrieved from <http://www.mathematik.uni-dortmund.de/didaktik/mathe2000/pdf/rf4-2wittmann.pdf>

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