Solving Methods of Combinatorial Geometric Problems

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Abstract. *There is considerable experience of organization and management of mathematical contests and interest groups in Latvia. It is necessary to analyse solutions of different mathematical challenges in out-door activities for to develop students' skills of solving non-standard problems. For this reason collections of thematically related problems with references to applicable methods are useful as valuable manuals for teachers and also as a source of original ideas for the students' independent work.*
Frequently students' attention is attracted by problems which do not require complicated mathematical formulae. For instance the combinatorial geometric problems deal with systems of geometric objects requiring to estimate dimensional quantities, or to investigate the features of decomposition as well as covering or colouring of geometric figures. Although these problems seem rather simple, various methods are used in their solutions, where general geometric regularities are supplemented with results from different fields of mathematics in combination with general thinking methods such as mathematical induction, the method of invariants and the mean value method. An effective auxiliary in problem solution is the mean value method, which allows making qualitative estimates of given objects by dealing with their quantitative properties.

ZDM - Classification: U40, K20, G90

School-children Mathematical Contests in Latvia.

The origin of Latvian students' mathematical contests dates back to 50 years ago when Latvia was incorporated in the USSR. Therefore, it will be interesting to find out how the mathematical contests in the Soviet Union started. In 1934, the first mathematical contest of school-children took place in Leningrad inspired by Professor Boris Delone and organized by Leningrad State University (Чистяков, 1994). The organizers set a political objective based on the communist ideology of that time: train young professionals as quick and best as possible for their further involvement in the attainment of far-reaching goals of building socialism. They clearly understood the decisive role of mathematical achievements in the development of economy. The analysis of contest results provided an opportunity not only to pick out the most talented mathematicians among school-children but also draw conclusions regarding the maths education level among secondary schools of Leningrad, which proved to be insufficiently high. Therefore the organizational committee of the contest pointed to the necessity to upgrade the qualification of the teachers of maths and address issues like development of exercise collections and reference books in maths, as well as organization of children societies and mathematical contests at different levels. This plan was implemented in the course of several coming years. During successive years, the school-children mathematical contests were run also in other cities of the Soviet Union and this practice gradually expanded into organization of the All-Russia mathematical contests. Since 1967, the All-Union Mathematical Contests were organized involving individual and corporate participants chosen among the best young mathematicians from the 15 republics of the Soviet Union.

The first mathematical contest for secondary schools in Latvia was organized in 1946 by the Maths Chair of the Latvian State University (Andžāns, Ziļicka, & Treilibs, 1977). Since 1950, these contests have been organized on regular basis, while their enumeration started in 1951, when Riga Pioneer Palace was involved in their organization. After restoration of independence, since 1992, the best young mathematicians of Latvia take part in International mathematical contests. Of late years, the National Mathematical Contest is run in four rounds, namely: preparatory contest, district contest, national contest and elimination contest. Preparatory contest takes place at schools. District contest is run in 26 districts and five largest cities of Latvia. On that level, the focus is on two age groups particularly: grade 5 – 8 and grade 9 – 12. The winners of the elder group are invited to participate the National Contest, the winners of which in their turn continue to compete for inclusion in the national team of the young mathematicians.

Taking a critical attitude to the nature of national (formerly: republic) contests, where only students from higher grades could participate, the Students Research Society of Faculty of Physics and Maths of the Latvian State University, proposed to organize alternative mathematical contests, where any interested student could participate. Consequently, the first Open Mathematical Contest was organized in 1974, attracting 316 participants. Recently, the popularity of this contest has grown considerably. Up to now, the largest number of participants took part in 2004: there were 3293 school-children from grade 3 to 12.

After the success of the first open contest, the idea was generated to organize mathematical contests by correspondence through the newspaper. Thus in 1974, newspaper „Pionieris" (Pioneer) started to publish *„Exercises of Professor Littledigit's Club"* (PLC), the solutions of which were mailed to the newspaper. At this stage, the *School of Maths by Correspondence* (SMC) of the University of Latvia has undertaken the organization of this mathematical contest (homepage: http://www.liis.lv/NMS/). SMC runs also some other correspondence contests as well as full-time and correspondence classes for secondary school students.

Training materials for mathematical contests.

In order to be successful at the mathematical contests, school-children are obliged to input a great deal of individual effort in solving non-standard problems. Practical skills can be acquired taking part in the school of maths by correspondence or participating contests. Teachers of maths societies can provide good advice and useful hints.

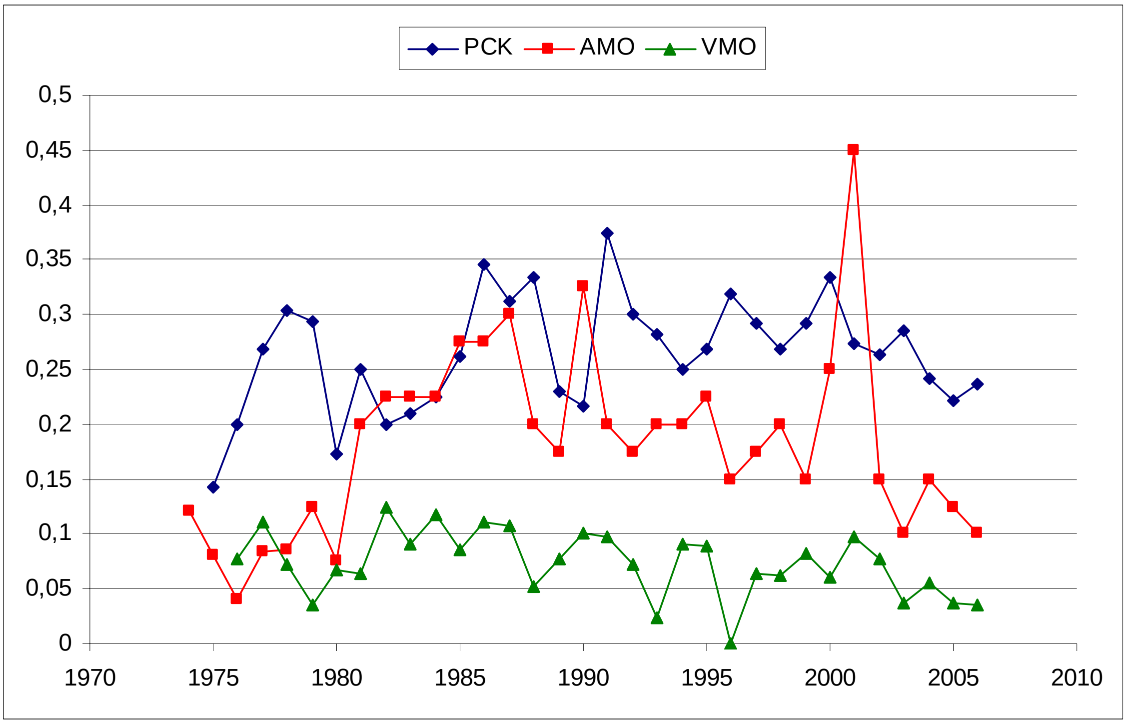
The course of development of Latvian maths contests has provided for accumulation of an extensive and versatile selection of non-standard mathematical exercises. The problems of previous years actually form the training basis for students. Therefore the development of mathematical materials like that is of great importance. In Latvia (formerly: Latvian SSR) collections of mathematical context exercises have been issued in print since 1960. In similar collections printed by USSR centralized publishers, complete solutions of mathematical contest problems used locally and overseas were included, often supplemented with detailed comments on the origin of the given problem and related questions (Яглом & Яглом, 1954; Морозова, Петраков, & Скворцов, 1976; Васильев, Гутенмахер, Раббот, & Тоом, 1981). Contrary to that, the collections of maths exercises from the Latvian contests and competitions were mostly accompanied by brief solutions or indications (Andžāns, Ziļicka, & Treilibs, 1977; Andžāns, & Bērziņš, 1998; Andžāns, Seile, & Zvirbule, 2000). To comprehend these clarifications and improve their personal problem solution strategies, the students needed a thorough knowledge in maths. Therefore, the systematization of exercises is of great importance.

Nowadays, the relevant training materials in Latvia are widely accessible also in electronic form within the framework of Internet project LEIS (Latvian Education Information System) operational since 1997 (http://www.liis.lv). The LEIS website is very popular: there are about 1500 visits per day on the average. Among training materials provided by LEIS, there are many solutions of mathematical contest problems supplemented with appropriate introductory exercises. There are collections of problems available on this website dedicated to particular fields, e.g. algorithmic problems, problems with elements of graph theory, problems on extremes, problems about quadrilaterals, about pentamino and other topics. Other training materials comprise problems to be solved by general solution methods: mathematical induction, the method of invariants, the mean value method, etc.

It should be noted that the progressive role of mathematical contests in Latvia as well as all over the world has been appreciated just recently obliging the relevant parties to pay much more attention to different methods of problem solving. The role of general thinking methods like mathematical induction, extremal element method, the method of invariants, interpretation method and mean value method start to be especially appreciated. The textbooks and exercise collections of elementary mathematics of today, alongside with thematically related problems include also the examples of application of the above mentioned methods (Горбачев, 2004; Lovasz, Pelikan, & Vesztergombi, 2005).

Specific features of Latvian mathematical contest exercises.

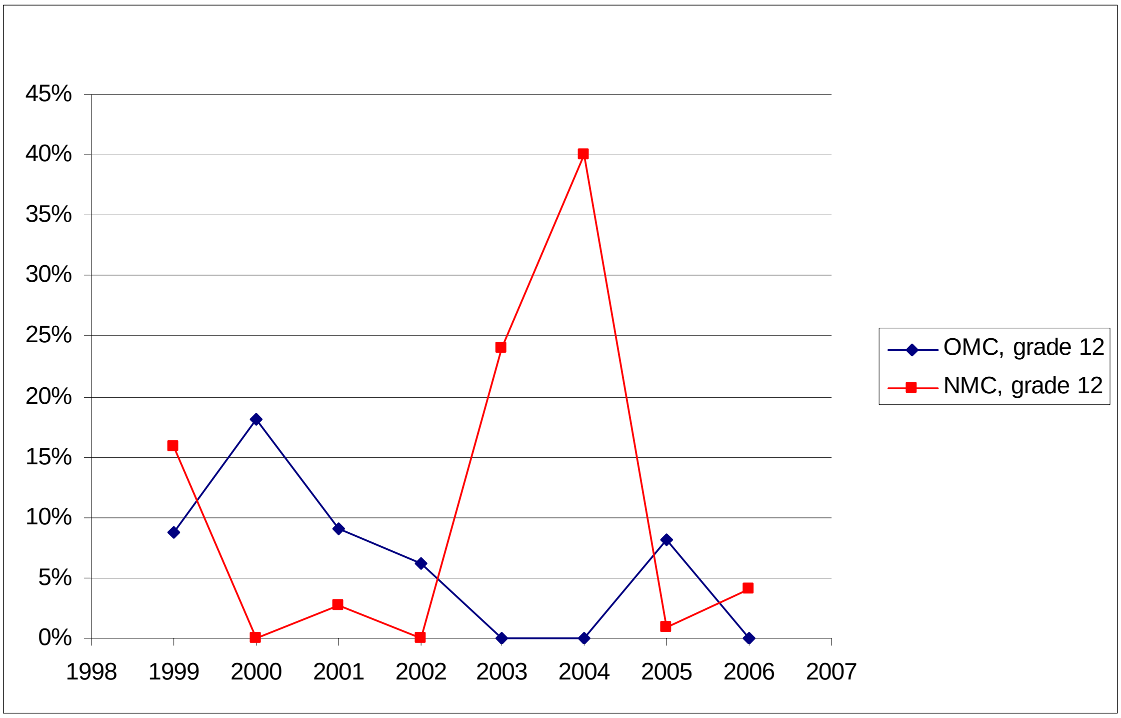
For the republic level maths contests the exercises were put together on the basis of the school curriculum. Thus, for instance, in earlier contests problems from the mathematical analysis introductory course were included dealing with limits, derivation and integration. The amendments in the school curriculum brought about changes in the subject matter of the contest problems. Over the recent years, the National Contest includes more problems from the field of discreet maths, withdrawing from algebraic transformations, geometric construction and identity demonstration problems or at least significantly reducing their number. At open contests and competitions of Professor Littledigit's Club, a lot of different combinatorial problems were used which did not require specific knowledge in maths. Studying the problem topics of these contests (Andžāns, & Bērziņš, 1998; Andžāns, Seile, & Zvirbule, 2000), one can see that combinatorial geometric problems were especially popular. Graph 1 shows the percentage proportion of problems of this type versus the total number of contest problems.



Graph 1. Percentage proportion of combinatorial geometric problems versus the total number at mathematical competitions of Professor Littledigit's Club (PLC), Open Mathematical contests (OMC) and National Mathematical contests (NMC).

Surveying the protocols of mathematical contests of the recent years (NMO and OMC) accessible in SMC and summarising information on the number of points acquired by every participant in evaluation of solutions of separate problems (at the contest, participants of all grade groups have to solve 5 mathematical problems evaluated on 10 points scale), the overall conclusion is that the average solution results of combinatorial geometric problems could be better (see graph 2).

It should be noted that between 1999 and 2006 the percentage of combinatorial geometric problems of OMC and NMC contests took up 40% of the share of more complicated problems, supposing that the most complicated problems were the ones for the solution of which the lowest number of points was acquired.



Graph 2. Average numbers of points acquired for the solution of combinatorial geometric problems versus the eventual number of points for grades 12 at Open Mathematical contests (OMC) and National Mathematical contests (NMC).

Combinatorial geometric problems.

Combinatorial geometric problems address the properties of systems of geometrical objects. Therefore they can be classified according to the type of objects addressed. Consequently, their types are as follows:

- Problems on point systems where the points are laid out:
  - in general position, i.e., no three points lay on the same straight line;
  - as the lattice points of orthogonal lattice;
  - in a special way, e.g. circumference or apices of a polygon.
- Problems on systems of straight lines, line segments or vector systems.
- Problems on overlapping of plane figures, decomposition or combination of figures.
- Problems on tables or squared rectangles and placement of objects within them.
- Problems on colouring.
- Problems addressing process analysis.

One of geometrically combinatorial problem selections is accessible online, included in the list of LIIS training materials (Andžāns, Ramāna, & Šulce, 1998). It comprises over 300 systematized problems intended for school-children from grade 5 to grade 9. The selection discusses intersecting possibilities of straight lines, polygons and circles and includes estimation of number of plane regions, evaluates colouring features of different geometric objects, considers various combinations of configurations of squares or poliomino, analyses examples with chess and checkers elements and solves various other problems. Great part of the solutions provided in this selection includes extensive explanations and covers a vast range of problem solving approaches. Apart from such general issues like development of geometric imagination and mastering principles of combinatorics, the methods offered by the authors instruct to:

- Make precise proofs like set up a hypothesis, verify it by means of some special case and prove its truth.
- Discuss all significantly different features of a given combination in solution of a problem or

provide a general solution that covers all possibilities.

- Systematize information and order it in an obvious way such as a table or a graph.
- Divide given objects by their equivalence categories in order to determine common or different features of particular objects.
- Take into account invariant features of a given configuration in order to conclude about position of geometric objects placed in it.
- Decompose a given geometric configuration into smaller congruent parts and use features of one part for expressing a general hypothesis.
- Also learn to apply results of other mathematical fields.

Also textbooks contain separate chapters addressing geometrically combinatorial issues (Яглом & Яглом, 1954; Lovasz, Pelikan, & Vesztergombi, 2005). Very interesting, various issues about mathematical challenges and comprehensive problems are found in scientific magazine of physics and mathematics for school-children „Квант" that is published in Russian Federation and whose all first issues from 1970 to 2003 are available at their Internet home page (http://kvant.mccme.ru/).

The mean value method.

Variety of topics of combinatorial geometric problems and solution methods stimulates to systematize these problems not only by their content, but also by the method of solution. *The mean value method* is one of combinatorial reasoning methods having significant role in proving many important mathematical results. It may be formulated very generally like this:

If a given object is divided into „a small" number of parts, at least one part will be „sufficiently" big.

This method may be applied to problem solving in algebra, geometry, number theory, graph theory and other mathematical fields. One of the special cases of the mean value method - *Dirichlet's box principle* (DP) is rather simple in substance:

Theorem 1. If nm+1 pigeons have been housed in n pigeonholes, at least one pigeonhole will have at least m+1 pigeons in it.

Such significant theorems like Ramsey's theorem on colouring (Грэхем, 1984), van der Waerden's theorem on arithmetic progressions (Хинчин, 1979), Minkowski's theorem on convex region that contains points with integer coordinates (Проданов, 1998), Dilworth's lemma for partially ordered sets (Andžāns, Čakste, Larfeldts, Ramāna, & Seile, 1996) etc. have been proven by means of generalizations of Dirichlet's box principle.

Geometric generalizations of DP may be formulated as theorems on overlapping line segments or areas. There is *Dirichlet's box principle for line segments:*

Theorem 2. If length of a line segment is n, and there are several line segments located within it whose total length is more than n, at least two of located line segments are overlapping.

If solution of a problem includes numerical estimation of some element or a set of elements, for instance, quantity of a point subset, length of line segments, size of an angle, qualitative conclusions may be drawn by comparing the mentioned value and an arithmetic mean value of given set of elements. For example, one of such theorems is the theorem on *the arithmetic mean value of a set of numbers:*

Theorem 3. In a given set of numbers at least one number is not more than the arithmetic mean value of all elements and at least one number is not less than the arithmetic mean value.

For solution of combinatorial geometric problems such classical algebra results like *Cauchy's theorem* on the arithmetic mean and the geometric mean values may be applied:

Theorem 4. The arithmetic mean of n positive numbers is not less than the geometric mean of these numbers:

$$\frac{a_1 + a_2 + ... + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdot ... \cdot a_n} \ .$$

Wide category of problems for mathematical contests is the so-called Ramsey type problems that may be interpreted as combinatorial geometric problems. These problems mainly are rather simple generalizations of Ramsey theorem therefore DP or more general mean value methods may be used for solution similarly to the proof of Ramsey theorem.

Examples of combinatorial geometric problems.

Several examples of combinatorial geometric problems shall be discussed where the mean value method will be used. The chosen examples are problems where one of the basic principles is proof by contradiction. Application of the mean value method is one of the most important arguments in order to get the contradiction. The problems are chosen among materials of Latvian mathematical contests and competitions for the most part.

Problems on point system.

Possible distances between given points are analyzed, the largest or the smallest possible number of points is determined that satisfies problem conditions, possible locations of points are studied as well as other challenges are solved.

Problem 1. Length of one square edge is 10 units. There are 102 points in general position given inside the square. Prove that among them are 3 points that form a triangle with area not greater than 1 unit.

Solution. We will consider a convex hull of point system that according to problem conditions is completely located inside a square, thus area of this polygon is less than 100 square units. Let us cut this convex hull into triangles whose apices are the given points. Number of triangles inside such polygon depends on type of a polygon. Suppose that the convex hull contains *m* points, but there are *102-m* points inside it. Number of triangles will be numbered with *n*. Let us sum up all inner angles of triangles in two different ways: separately in each triangle and inside the convex hull in total:

$$\alpha = 180° \cdot n$$

$$\alpha = 180° \cdot (m-2) + 360° \cdot (102-m)$$

Then number of triangles is $n = 202 - m$.

If all 102 points inside the given square are located on apices of the convex hull, number of triangles is the least. Then the least possible number of triangles is 100. According to theorem 3 on arithmetic mean, there is at least one triangle whose area is less than 1.

When applying DP in solution of some problem, one must choose which elements or their properties would be considered „cages" and which ones would be considered „hares". Line segments of particular length („hares") will be classified by their endpoints („cages") in the next example.

Problem 2. Let us consider all possible line segments defined by 10 given points. Given that length of each line segment is not less than 1 cm prove that there are no more than 30 line segments whose length is 1 cm exactly.

Solution. Suppose that there are more than 30 1 cm long line segments. It will be calculated for each given point how many 1 cm long line segments originate there from. Then every such line segment will be marked twice at each of both endpoints. The total list will included more than 60 line segments. According to DP at least 7 line segments of 1 cm length originate from some point. Their endpoints are located on a circumference. These points divide the circumference into at least 7 parts. According to theorem 3, at least one of the arcs does not exceed the mean value of all given arcs that is not longer than one seventh of the circumference. A centre angle corresponding to such arc is less than 60º, thus its endpoints define a line segment that is shorter than 1 cm. But this contradicts the problem conditions.

Evaluation of integer numbers' even-odd properties provides good results in solving combinatorial geometric problems where elements are located orthogonally.

Problem 3. 9 lattice points are chosen in an orthogonal three dimensional lattice. Prove that there are at two points such that line segment joining these points passes through another lattice point.

Solution. Let us place a three dimensional orthogonal frame of reference in an orthogonal lattice so that coordinates of each lattice point would be integers. Let us consider all 8 possible combinations of coordinate parities:

$$(e;e;e) \quad (e;e;o) \quad (e;o;e) \quad (e;o;o)$$

$$(o;e;e) \quad (o;e;o) \quad (o;o;e) \quad (o;o;o)$$

Classifying given 9 points by their coordinates parities, at least 2 points, let us say A and B, coordinates will coincide by parity according to DP. The coordinates of A and B are:

$$A(x_a; y_a; z_a), \quad B(x_b; y_b; z_b).$$

Midpoint M coordinates of line segment AB will be integers:

$$M = (\frac{x_a + x_b}{2}; \frac{y_a + y_b}{2}; \frac{z_a + z_b}{2})$$

Thus, point M is a point of orthogonal lattice.

Problems on systems of straight lines and line segments.

In these combinatorial geometric problems, number of points of intersection, minimum or maximum numbers of plane or figure parts arising due to intersecting straight lines or line segments, properties of some set of figures intersected by straight lines are estimated. One of characteristic methods to solve such problems is to project line segments or sets of figures on one or several straight lines allowing to estimate the projection by means of theorem on line segment overlapping and to conclude about location of such line segments or figures.

Problem 4. In a circle of radius *n,* there are *4n* line segments placed, each of length 1. Prove that a straight line parallel or perpendicular to the given straight line can be drawn so that it intersects at least two of the given line segments.

Solution. A straight line perpendicular to the given straight line is drawn and all line segments located in the circle are projected on both straight lines. Sum of both projections of one line segment will be either equal to 1 if a line segment is parallel to one of the straight lines, or more than 1 according to the triangle law. Therefore total sum of all *8n* projection lengths is not less than *4n.* Let us say that sum of projection lengths on one straight line is more than *2n*. Given that common line segment of projections does not exceed diametric length *2n,* it can be concluded that projections of at least two line segments overlap according to the mean value principle or theorem 2. It means that these two line segments can be intersected by one straight line being parallel to the projection line. If the sum of projection lengths is the same on both straight lines and it is *2n*, it is concluded that a half of line segments is parallel to one projection straight line, but the other half is parallel to the second straight line. Let us look at projections on one of the two straight lines. If at least two projections overlap, corresponding line segments can be intersected by one straight line being perpendicular to the projection straight line. Otherwise at least two line segments are placed on diameter parallel to the projection straight line. A straight line that contains both line segments can be drawn through diameter.

Problem 5. There are 50 line segments on a straight line. Prove that there are either 8 line segments that all contain at least one common point, or 8 such line segments that no two of them contain any common point.

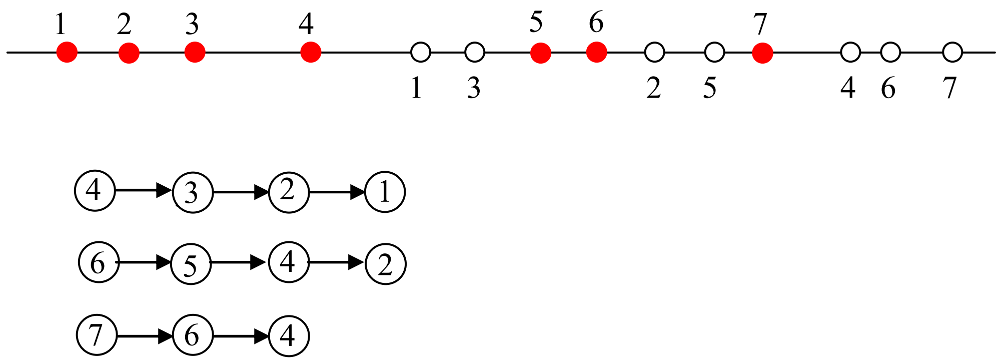
Solution. Let us mark all endpoints of the line segments on the line. Left endpoints of the line segments will be coloured red whereas right ones will be coloured white. It is possible that we can find some points that have 2 colours. Then we suppose that such point is a line segment with zero length. Let us consider all contiguous pairs of points whose left point is red whereas right one is white. All those line segments for each such pair are marked wherein this pair of points is included. If some of these lists have 8 line segments, statement of the problem is proven.

Let us presume that each list does not have more than 7 line segments. The least possible number of lists is estimated. According to DP 50 line segments will be divided into at least 8 lists. One line segment from each list can be chosen so that no two of the segments will have any common point.

Discussed problem is a special case of Dilworth's lemma.

Dilworth's lemma. If partially ordered set contains mn+1 element, it contains a chain of length m+1 or an antichain of length n+1.

In solution structure of this problem, partial order is obtained creating lists (chains). There are all such line segments in the same list that all of them have at least one common point. Antichains consist of such line segments none two of them have any common point. See picture for partially ordered example of 7 line segments overlapping. Here chain consists, for instance, of line segment 6, 5, 4 and 2 whereas antichain consists of line segment 3, 5 and 7.



Picture 1. Example of partial order establishment of a line segment system.

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Problems on placement and figure overlapping.

Problems on placement and figure overlapping study issues concerning placement properties of figures of particular type, assess common size of overlapping area or how much placement of given figures is spaced out. Separate problems on placement of chess figures, dominoes, stones, and other elements in a squared rectangle can also be included here.

Problem 6. Several ink spots fell on a square sheet of paper with edge length a. It is known that area of each ink spot does not exceed 1 and any straight line drawn parallel to the edges of the sheet cuts not more than one ink spot. Prove that total area of the ink spots is not greater than a.

Solution. Similarly to solution of problem 4, projections of ink spots on two inter-perpendicular edges of a square will be considered. Area of one spot of ink S will be assessed. Therefore projections of square edges will be marked x and y. The ink spot is not greater than the rectangle created from projections of this spot:

$$S \le xy .$$

This proportion can be assessed algebraically by means of Cauchy's theorem:

$$\frac{x+y}{2} \ge \sqrt{xy} \ge \sqrt{S} \ge S .$$

Then sum of all ink spot areas does not exceed a half of sum of all projections. Considering the problem conditions, projections of ink spot on one edge of a square do not overlap. Thus, sum of their length does not exceed length of square edge a. Summing up all the projections on both edges of square, estimation of total spot area is obtained:

$$\sum_i S_i \le \sum_i \frac{x_i + y_i}{2} \le \frac{2a}{2} = a .$$

Problems of colouring.

Problems of colouring are often included in various mathematical competitions. These are combinatorial geometric problems on colouring of points, line segments, figures or their parts, on colouring surface of polyhedrons or composition of polyhedrons from smaller polyhedrons with coloured faces. If we compare these combinatorial geometric problems with others, we see that these problems are generally solved by means of combinatorial calculations and results of graph theory.

Problem 7 (Bulgarian Competition of Mathematic, 1998). There are n points $A_1, A_2, A_3, ..., A_n$ in two colours given on a straight line so that $A_1A_2 = A_2A_3 = .. .= A_{n-1}A_n$. Find the least natural number n ($n>2$) if at any point colouring in two colours such 3 points A_i, A_j, A_{2j-i} ($1 \le i < 2j - i \le n$) can be found that are coloured in the same colour.

Solution idea. The solution consists of two parts. First of all, it can be proved that if the number of points is less than 9, there always exists colouring that contradicts the problem conditions. In the second part of the solution one presumes the opposite, i.e., that in the case of 9 points such colouring can so be found that the problem conditions are not met. Analysis of such case can be done by different approaches. For instance, symmetrical placement of the fifth point on a line segment can be used and various colouring possibilities can be analyzed that lead to a contradiction. Another approach can be applied if the given indices of points are considered and it is taken into account that they form an arithmetic progression. If we denote $j = i + d$, then the indices of corresponding points will be

$i;\ i + d;\ 2(i + d) - i = i + 2d.$

The given problem is one of the simplest cases of van der Waerden's theorem that includes the so-called van der Waerden's number W(3, 2)=9. The problem is comparatively easy to solve by colouring points directly and trying to avoid any one-colour arithmetic progression case of length 3.

Process analysis.

Process analysis is done for the problems where changes of different object placement are discussed in length of time. Movement can be continuous or take place in fixed moments of time. In majority of combinatorial geometric problems sets of objects of finite number are discussed. Then one of the important arguments in solution of such problems will be the fact that placement possibilities of finite number of elements are limited.

Problem 8. A circumference is divided into $2n$ equal parts. There are $2n+1$ grasshopper on its arcs. In a moment of time exactly 2 grasshoppers placed on one arc hop to different contiguous arcs. This process repeats in every moment of time. Prove that grasshoppers will be placed on at least n arcs in some moment of time.

Solution. Notice that the number of grasshoppers is more than the number of circumference arcs, thus according to DP there will always be more than one grasshopper on some arc. It means that this process never stops. Suppose that there are at least $n+1$ empty arc in any moment of time. Then according to DP at least 2 empty arcs will be located next to each other. Under the problem conditions, a new pair of contiguous empty arcs cannot occur. Therefore no grasshopper ever hops in this pair of empty arcs. Number of grasshoppers on arcs being next to this pair can only increase. Grasshoppers cannot get out from these arcs anymore. At particular moments of time a number of active grasshoppers decreases, therefore it can be concluded that the process will cease. Contradiction of process continuity is obtained.

We can see in this example that reasoning applied in the solution is similar to so called *method of infinite descent* applied for determining the greatest common divisor of two integers under Euclidean algorithm.

Conclusions.

Combinatorial geometric problems are very different. They always generate interest among the school-children; therefore such problems are suitable not only for students especially interested in mathematics, but they can be solved also during math lessons within regular school curricula. Combinatorial geometric problems are good supplement for to acquire the geometric and combinatorial regularities. Several problems that have already been discussed in the article demonstrate that school-children can learn a wide range of solution and thinking methods. Solving such problems kindles mathematical imagination of the students. Combinatorial geometric problems are applicable for school-children of different age: school-children of first grades can experiment with problems, classify them and discover interconnections among objects. The mastering of different thinking methods can be a good foundation in scientific research work of senior grades' students.

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