

## Experiments with diagrams – a semiotic approach

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*Abstract: A challenging task when doing research in mathematics education is the comprehensible description of activities shown by students and their construction of new knowledge as well when doing mathematics. Charles S. Peirce's semiotics seems to be a well promising tool for fulfilling this task. Since several years, Peirce's semiotics is well known and extensively discussed in the scientific community of mathematics education. Among the numerous research reports several papers dealing with Peirce's semiotics concentrate on the meaning of diagrams as a tool for gaining new knowledge. The aim of the following paper, where a case study will be presented, is to offer the usefulness of such a view on diagrams. In this study two students, which have to solve a problem from elementary geometry, are introduced. The question presented to them asked for a mathematical description of the movement of a rigid body. To answer this question they started experimenting with this rigid body and afterwards invented and used diagrams in manifold ways. Video-based data show these diagrams to be the source of new mathematical knowledge for these students. Therefore, this paper offers Ch. S. Peirce's semiotics as a successful theoretic frame for describing and interpreting the learning activities of students and their use of diagrams to solve a given mathematical task.*

**ZDM-Classification: C30, G40**

### 1. Introduction

Mathematics is a science, which is always interested in inventing and using signs (e.g. symbols, diagrams etc.). Since the end of the 1990s, numerous scientists in mathematics education have been investigating theoretical approaches to such signs, with the aim of using these theories as a tool for answering research questions in the field of learning and teaching mathematics. In order to become a "tool science" in mathematics education, semiotics has to prove its usability for establishing new and fruitful views on questions beside those sciences like

psychology, pedagogy or sociology already used and accepted in mathematics education.

One of these questions to which the semiotics of Ch. S. Peirce offers an answer (Hoffmann, 2005, Dörfler, 2006) will be focused in this paper. How does new knowledge come into being? Using Peirce's answer, I will analyse the activities and the results of two young students when solving a geometrical task. To do so I will accompany them through the solving of their problem, concentrating mainly on everything they draw and write. Following the sociologist Bruno Latour, I call this drawn and written "inscriptions". On the evidence provided by these inscriptions, Peirce's semiotics will be introduced as a successful instrument for describing aspects of learning mathematics. It will be shown that certain kind of inscriptions, which are diagrams in Peirce's sense, are valuable means for constructing new knowledge (see also Kadunz, 2003, 2004; Dörfler 2005; Hoffmann 2005a, Stjernfelt 2000).

My considerations are divided into five parts. The first gives a short review of the literature on semiotics and mathematics education. Then I will introduce the students and the geometrical problem given to them. In part four I will concentrate on some of Peircian semiotics concepts in order to analyse the students' activities. The fifth part is the center of my video-based data where a surprising solution will be shown. Finalling my paper I will apply the introduced semiotic concepts as a tool to analyse the students' activities.

### 2. Semiotics and mathematics education

In his "Handbook of Semiotics" (Nöth, 1995) Wilfried Nöth presents a thorough and comprehensive review of the mainstreams of modern semiotics. Nöth's handbook demonstrates that there is no universal semiotics, but a number of quite different ones. In addition to his presentation of well-known semioticians – from Peirce to Eco – Nöth also shows their semiotics to be valuable tools in different research areas. We can find semiotics for example in the field of linguistics, in aesthetics or in media theory, to name only a few. Semiotics seems to be a very "broad" concept. The use(s) of semiotics in mathematics education seem similarly "broad" as we will now see.

If we look at papers in mathematics education we can find numerous articles in journals and edited books treating questions from a semiotic point (Cobb, 2000; Anderson, 2003; Hoffmann, 2003, 2005; Educational Studies in Mathematics Education (ESM), special issue 2006). Because of its topicality I have chosen the special issue of ESM on semiotics from this list. Norma Presmeg and Adalira Saenz-Ludlow (Saenz-Ludlow, 2006a), the editors of this special issue, note the founding of a PME “discussion group on semiotics” at the 25<sup>th</sup> PME conference at Utrecht, and being continued on the following PME conferences (Norwich, 2002; Honolulu, 2003; Bergen, 2004). The outcome of this discussion group was a basis for this special issue. Among other topics presented in this issue the reader finds semiotics as a means of studying epistemological questions, or of planning mathematics lessons (in a very wide sense) or of interpreting classroom communication. As diverse as these research questions are the semiotics used (Ch. S. Peirce, Ferdinand de Saussure or Michael Halliday) to mention only a few.

In a reviewing paper Michael Hoffmann (Hoffmann, 2006) presents the highlights of all articles in this ESM special issue. He closes his text with an answer to the following question: “... is there a shared conception of “semiotics” behind all the “semiotic perspectives” delivered here, and should there be one?” (Hoffmann, 2006, p. 290).

Hoffmann denies that one universal semiotics can be established and warns against blending different semiotics.

“This variety, however, is not necessarily a problem. As long as the terminology is consistently defined and used so that communication and understanding are possible, several semiotic approaches can be used side by side. ... If we are interested in epistemological problems of learning and communicating mathematics, and if we need a highly differentiated semiotic terminology that allows very precise discussions of problems such as meaning, cognition, interaction, and interpretation in mathematics, Peirce’s semiotics is by far the best tool.” (Hoffmann, 2006, p. 290).

In my use of semiotics I will follow Hoffmann’s suggestion. In order to achieve a thorough and precise deliberation of a single problem in mathematics education I will focus on one semiotic approach. In part three I will concentrate

on Peirce’s semiotics as he developed not only a differentiated semiotic terminology but also used his semiotics to answer epistemological questions from mathematics. I will give a view on Peirce’s famous classification of signs into icons, indices and symbols. In particular the role of icons and diagrams in constructing new knowledge will be investigated. Before illustrating my view on icons and diagrams I will now move to the first presentation of video-based data.

### 3. Case study part I

In spring 2005 Manfred Katzenberger – a mathematics teacher at the Gymnasium of St.Paul/Carinthia – started a series of learning experiments with some of his students (7<sup>th</sup> to 9<sup>th</sup> graders). He investigated the impact of free hand drawing and sketching when solving a mathematical problem. To do so he produced numerous videos, and I am grateful to him for making one video available to me.



Figure 1

This video shows two 8<sup>th</sup> graders solving a geometrical task. In the following I will call them A and B. The students had been asked to describe the movement of given objects on the table. They were to use their mathematical/geometrical knowledge. Among the objects the teacher presented to them we can be found cylinders, cones, spheres or wooden objects similar to drumsticks. Figure 1 shows both students and one object they had to investigate. The question just to describe a movement seems to be formulated in a very open way. This openness was intended. The researcher’s aim was to establish a context where both students could themselves feel like researchers. The study was designed in such a

way that they could write down all their attempts without looking for an algorithmic solution. A more narrowly formulated task would have resulted such a strategy. The number of tools both participants of our case study were allowed to use also mirrored this openness. Besides their tools for doing geometry (ruler and compass), they could in addition use different measuring tools, this being a tape measure, vernier callipers (a tool for measuring a circle's diameter) or computer software (spreadsheet software and software for

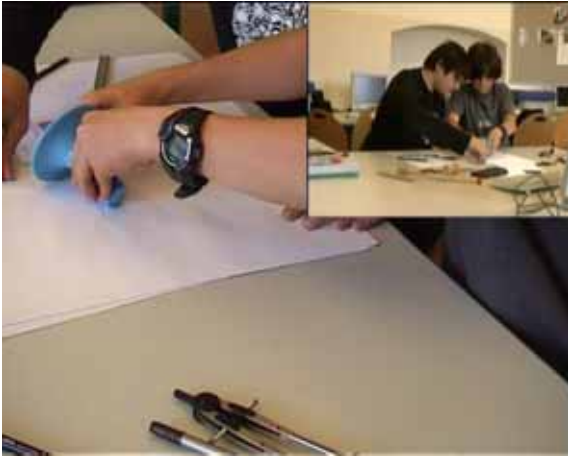


Figure 2  
dynamic geometry).

Their activities together were captured with two video cameras. For supporting the evaluation, the two video pictures were incorporated into a single picture (figure 1). One camera was fixed in one position, while the observer focused the other on interesting details. The students were given 90 minutes time to answer the questions presented to them. The video was taken in the afternoon when class had finished.

Following my own research interest I will focus on the students' inscriptions i.e. the icons and diagrams they invented and used. Therefore I will only reproduce their spoken comments if they seem necessary for explaining their activities.

Video data shows that both students investigated the objects presented to them in various ways during the first 40 minutes (starting phase). Among others we can observe: rolling the different objects on the table, sketching the rolling paths, estimating the radii of rolling paths or making other measurements.

In the documentation that follows I will present the students' activities after this starting phase. The comming starts with the presentation of a new task when a body of rotation (figure 1) was introduced to the students. Again they had to investigate the movement of this object and to

find out the rolling paths. The first successful attempt took some 17 minutes. It started from the observation of the movement of this object on the table. Several times the students in our case study, like young children playing with a toy, pushed the object to roll on the table and observed this rolling with great attention. Thereby they focused their interest on the points of contact where the rolling touched the table. As the movement of the object obviously describes circles – the contact points between object and table defined two circles with one common centre – student B suggested marking some contact points with his pencil to get some details about these circles. He failed as both students could not agree which contacting points had to be marked he failed.

The video data also revealed that neither of the students ever thought about investigating any other curves except the contacting curves, though this would have been easy to do. If we fix a single point at the surface of the rolling object then this point describes a curve similar to a cycloid, which is an important curve in connection with technical problems. During our experiment the object was always seen as one whole body. It never entered the students' minds to define the object as a set of different points.

As both students had failed to locate points to draw the contacting circles they estimated the position of the common centre. Again this attempt was not successful. Therefore, the question of



Figure 3

how to record history of this rolling remained. A clever strategy, invented by student B, brought them a step further. This strategy determined their remaining activities.

B: (*He holds the object with two fingers and looks at that paper the object had rolled on.*) "Hey, we

could just try to press it down firmly, couldn't we?"

A: "Where?"

B: "There (*He points at the paper.*) now you can see it!"

What had happened? When trying to find the circles the students had placed several sheets from a stock of paper on their table. While rolling their object on these sheets traces of the rolling object had come into existence. Student B recognized this fact and it became the starting point of their new strategy. They took some more sheets from the stock of paper to let their object roll on a "soft plane". While rolling the object on the sheets one student pressed it with great force into the paper (figure 2). The result was a partially visible but completely "sensible" engraving on the sheets. To strengthen this tactile impression and to make it more utilizable for their visual senses student B coloured the engraving of the smaller circle with his pencil (figure 3).

In the minutes which followed the students started processing the curves they had found. As there was no question of the discovered curves being circles the students began to look for the circle's common centre. They opened their geometrical toolbox and a circumscribed square was drawn around the engraved circle. The square's diagonals led to the circle's centre immediately (figure 4). It is worth recording that our students had never used the theorem of the circle circumscribed to a given triangle, although they

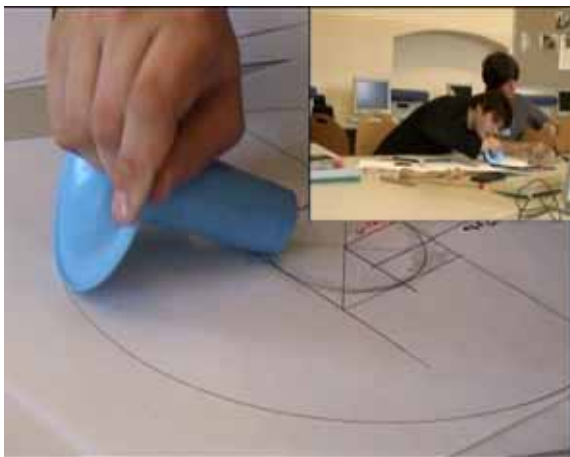


Figure 4

had learned about it in their geometry lessons. What I want to show is not only our students' success but also the way they found their solution. When watching the video one recognizes that inscriptions which the students had already produced or which they invented and drew on the

fly heavily influenced many steps on the way to a solution. Starting from their practically unsystematic playing with the given object, they followed a strategy which enabled them to literally feel the curves they were looking for. To strengthen this first tactile impression and to make it more utilizable for their visual senses one student coloured the engraving with his pencil. From the seen and the felt, both students conjectured that the curves they were looking for had to be circles. This paved the way to their first solution to the given problem. Thus our students used their inscriptions to achieve the target. Additionally to this clever use of inscriptions it is worth mentioning that the students had a thorough working knowledge of how to do a geometrical construction. They acquired this working knowledge during their last two years in school. We can call it contextual knowledge or following Ch. S. Peirce we call this knowledge collateral knowledge. It has its origin in a particular form of geometrical socialization of the students. Although this collateral knowledge was relatively insignificant for finding the first solution it will be at the heart of the students' second solution. Finally, looking at the students' first attempt at solving the given problem it is easy to recognize a connection between hand and eye or between inventing and using inscriptions. The use of such inscriptions, their meaning for constructing new knowledge, and the importance of the already mentioned collateral knowledge, all viewed from a semiotical position, are at the centre of the next part of my paper.

#### 4. Diagrams as means for thinking

Like other sciences, mathematics education also deals with the concept of "representations". As a representative example I refer to papers and research reports presented by researchers like Gerald G. Goldin or James J. Kaput (Goldin and Kaput, 1996; Goldin, 1998, 1998a). Generally speaking they investigate internal or mental representation and external or physical representation. This kind of separation between the mental and the physical brings up some epistemological and psychological difficulties I will not discuss here. For a detailed explanation I refer to Falk Seeger (Seeger, 2000) or Michael Hoffmann (Hoffmann, 2005a 4<sup>th</sup> chapter). A remarkable development, at least of the last 20 years describes, a *turn* where those

“representations” which are perceptible to our senses step into the centre of interest.

“Before attributing any special quality to the mind or to the method of people, let us examine first the many ways through which inscriptions are gathered, combined, tied together and sent back. Only if there is something unexplained once the networks have been studied shall we start to speak of cognitive factors.” (Latour, 1987, p. 258).

More from art theory than from sociology Thomas Mitchell (1994) diagnosed a *pictorial turn* and Gottfried Boehm introduced in 1994 his *iconic turn* (Boehm, 1994, p. 13). With these *turns* Mitchell, Boehm and other researchers express their interest in the epistemological importance of “representations” available to our senses. Similarly Frederik Stjernfelt formulated the importance of icons in semiotics:

“...this return of the iconic in semiotics is probably the main event in semiotic scholarship during the recent decades...” (Stjernfelt, 2000, p. 357).

In my deliberations I will concentrate on such perceptible signs on icons and on diagrams. These diagrams will be introduced later in detail.

With his semiotics Ch. S. Peirce introduced a far-reaching project to demonstrate the importance of signs. I will point at a “trademark feature” of this semiotics. First I only mention Peirce’s view of signs as a triadic relation. This relation consists of an object, a representamen, and an interpretant<sup>1</sup>. They are the corners of Peirce’s semiotic triangle. “... a ‘sign’ is *integrated* in a triadic relation whose most important feature is what he called the sign’s ‘interpretant’.” (Bakker, 2006, p. 336). As I will concentrate on the second “trademark feature”, I refer to papers which elaborate this triadic concept of sign (e.g.: Hoffmann, 2003; Bakker, 2006; Presmeg, 2006; Saenz-Ludlow, 2006).

Peirce’s “trademark feature” which I will examine, is his famous classification of signs into iconic, indexical, and symbolic.

#### *Icon*

An icon is a sign which represents relations. By definition it is a sign which is similar to its object.

<sup>1</sup> To give an example of a sign we can think of a barometer as representamen, which gives some information about its object the air pressure. A person recognizing a change of the air pressure can interpret this change. It is remarkable that the representamen itself can be called a sign. For more information see the above-cited literature.

This similarity can lead to some misunderstanding (Stjernfelt, 2000, p. 358). Critical remarks dealing with the concept of similarity can also be found in Nelson Goodman’s *Language of Art* (Goodman, 1976). As Stjernfelt indicates, it seems that Peirce himself had recognized some of the difficulties connected with similarity. Icons are not in themselves similar. The impression of similarity comes into existence from possible activities we can do with the icon.

“The icon is not only the only kind of sign involving a direct representation of qualities pertaining to its object; it is also – and this amounts to the same – the only sign by the contemplation of which more can be learnt than lies in the directions for its construction.” (Stjernfelt, 2000, p. 358).

These constructions and the activities with them may be the source of new knowledge, as I will show in the following using diagrams, which are intimately related to icons.

#### *Index*

Following Peirce a sign is an index, which focuses the attention of a person using this sign. We can find indexes in our everyday language when we use words indicating something. If we think of geometrical drawings then the labels on these drawings are indexes as they point to certain parts of the construction.

#### *Symbol*

A symbol is a sign, the use of which is given by definition. We can find symbols in words of a language as the meaning of a word, which has to be learned by definition. In mathematics, symbols are widely used. We can think of  $e$  or  $\pi$  to name the most famous ones. But also letters used as variables in an equation are symbols in this sense.

#### *Diagram*

Icons can be further classified following Ch. S. Peirce into images, diagrams and metaphors. From these three the diagram will have the greatest importance for the rest of my paper. Diagrams are icons, which are *constructed* following certain rules and may thereby show relations. When we look for diagrams, we can find them in geometry. Every drawing obeying the rules of geometry is a diagram. In the same sense, a written sentence is a diagram if it follows the grammar. On the other hand, the reader reading this sentence has to know the grammar to decide whether it is a diagram. Therefore, a diagram is not a diagram by itself!

However, diagrams are in most cases very complex signs. If we again take a diagram from

geometry, we see in it symbols, indexes and even other diagrams. As an example, we can imagine the drawing of a triangle and its circumscribed circle. The labels of its corners are indexes and symbols too. If we label the circle with “solution” then we have another symbol. The triangle itself is a diagram, as it is constructed using segments connecting three points in a special way.

Alongside this use of rules in constructing diagrams, the operational view on diagrams I mentioned previously for icons (Stjernfelt) will now be discussed. This operational view will be made use of in the interpretation of students’ activities to be presented in part 5. With diagrams as a special kind of icons, we can perform experiments when learning mathematics. Doing experiments and constructing new knowledge is called diagrammatic reasoning (Hoffmann, 2003; Bakker, 2006). How can we imagine such reasoning when learning mathematics?

In a first step, a diagram has to be constructed. To give some examples, this may be an equation from algebra, a geometrical drawing using software, or pencil and paper, or designing a graph to solve a problem from graph theory. Once construction has been finished, we can start experimenting. The algebraic equation may be transformed following the rules from algebra. If we have used software (DGS) for constructing the geometrical drawing, we can use the drag mode (Arzarello, 2002) to change the construction without destroying the geometrical relations of the drawing. However, we could also implement a new line or segment or even a new label into the drawing to gain a new view. This also means that when performing experiments we have to obey the rules governing the system.

“What makes experimenting with diagrams important is the rationality that is immanent to them... The rules define the possible transformations and actions, but also constraints of operations on diagrams.” (Bakker, 2006, p. 340).

In a final third step, the results of the experiment are explored. In front of the observers eyes new relations can become visible. A new configuration may show “itself”. A new pattern (Oliveri, 1997) may be visible within the algebraic equation. Making use of DGS drag mode the continuous movement of parts of the drawing may raise the idea of the equality of areas.

“As Peirce wrote, the diagram constructed by a mathematician “puts before him an icon by the observation of which he detects relations between

the parts of a diagram other than those which were used in the construction” (NEM III, 749)” (Bakker, 2006, p. 341).

With this citation, I close my remarks on diagrams and diagrammatic reasoning. I will finish this part with some hints on two further concepts Peirce presented. I will use them as a tool to “measure” the creativity of our students. In his semiotics, Peirce introduced two interesting concepts to describe logical deduction. “There are two kinds of Deduction; and it is truly significant that it should have been left for me to discover this. I first found, and subsequently proved, that every Deduction involves the observation of a Diagram (whether Optical, Tactical, or Acoustic) and having drawn the diagram (for I myself always work with Optical Diagrams) one finds the conclusion to be represented by it. Of course, a diagram is required to comprehend any assertion. My two genera of Deductions are first those in which any Diagram of a state of things in which the premises are true represents the conclusion to be true and such reasoning I call Corollarial because all the corollaries that different editors have added to Euclid’s Elements are of this nature. To the Diagram of the truth of the Premises something else has to be added, which is usually a mere May-be, and then the conclusion appears. I call this Theorematic reasoning because all the most important theorems are of this nature.” (Peirce, A Letter to William James, EP 2:502, 1909).

As we see, the corollarial deduction is the more simple form of deduction. It describes those logical activities we have to do when we draw a conclusion from observing a diagram without changing this diagram. Take, for instance, an isosceles triangle with its axis of symmetry drawn in. Then we can deduce corollarily that the base angles of this isosceles triangle are equal. If we draw a new or change a given diagram and we deduce a conclusion then we have done a theorematic deduction. Mathematical argumentations or the proving of theorems are in most cases examples of theorematic deduction. The now following data will present an instance for this kind of deduction. I return now to our two students and their interesting second solution to their geometrical problem.

## 5. Case study part II

The video data I will present now offers a new solution of a very different kind. The way to this

solution can be seen from three positions. From the first, we see free hand drawing where collateral knowledge, which I have mentioned above, plays a crucial role. As the second, I mention the collaboration between the students where they use one diagram together and from this diagram develop the main solving strategy. As the third, we will find in the students' activities

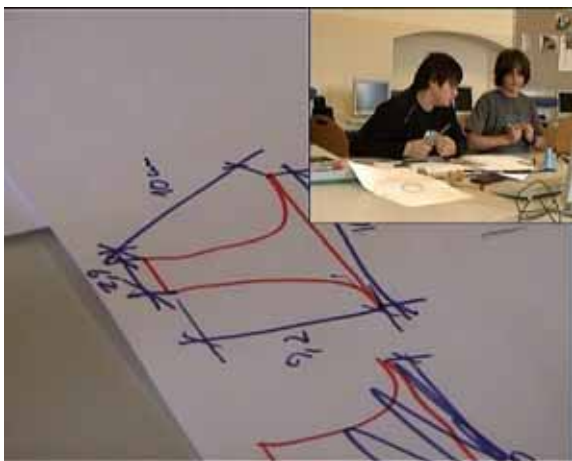


Figure 5 different kinds of inventing and using diagrams, in particular the rule governed transformations of an algebraic equation. We can now examine all three positions in detail.

After they had finished their first solution, the observer Manfred Katzenberger asked the students to search for a second way to answer the given problem. Thereby they were requested to apply the given measuring tools. After some two minutes of observing and measuring the object student B starts with a sketch of this object, which differs greatly from marked by their first "engraved" solution. After B had finished his sketch, he started to label it with measured values obeying labelling rules he had learned in school. Among the measured values we find: diameter of the base- and the top circle of the given object, the outer distance of these two circles and approximated height of the whole object. During sketching and labelling our three-dimensional object becomes an object of the drawing plane. A problem from geometry in the three dimensional space is transformed into a problem of plane geometry (figure 5).

Video data show that during the next few minutes both students observe the object and the visible sketch very carefully. We remember: They are investigating circles looking for diameters and centre points. Suddenly student A presents a far-reaching suggestion.

A: "Ah, mmh, I know how to reach the centre."

B: "Reach?"

A: "The way we can calculate it."

B: "How" (He looks at the paper with the sketch)

A: (Turns the paper) "Look! If we have there 10,5 (radius of the base circle), if we have this (A starts a new sketch), 10,5 there."

B: "Mhm (affirmative)."

A: (A draws a base line) "This is a kind of a triangle with a right angle."

B: "Yeah (affirmative)."

A: "We can make this longer (A draws the hypotenuse), OK? If we take the axis from this one (A points to the sketch B had draw and describes this axis with a movement of his pencil) then (A draws the second short side of the rectangular triangle) they have to meet. That is the distance..."

B: "Yes?"

A: "... from the centre point."

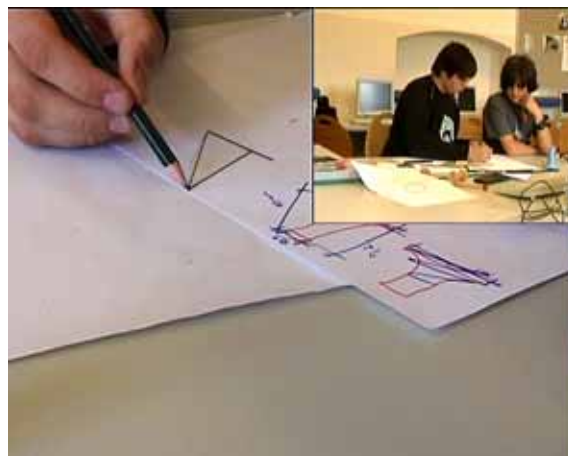


Figure 6

The visible result of these activities can be seen in figure 6. During the next few minutes, both students compare the given object and their sketches. This common activity ends with a comment by student A which will be remembered later in another context by student B. Student A want's to draw a vertical projection of the object to see it with "true measures" on the paper. He formulates:

A: "If we draw it obliquely then we need a 1:1 angle."

If we remember descriptive geometry, we know what has in his mind. Student A wants to draw the vertical projection of the object lying on the table (the object's axis is oblique to the tableplane). After this comment the students' common

activities, which were dominated by A ends. It takes some minutes before student B begins a further attempt.

B: "Just wait a moment."

In the meantime, student A had begun to draw his vertical projection.

B: "Now let me draw. Do you know what I have thought? It is the intercept theorem that means the relation!"

A: *(looks doubtful)*

B: "Now, look!"

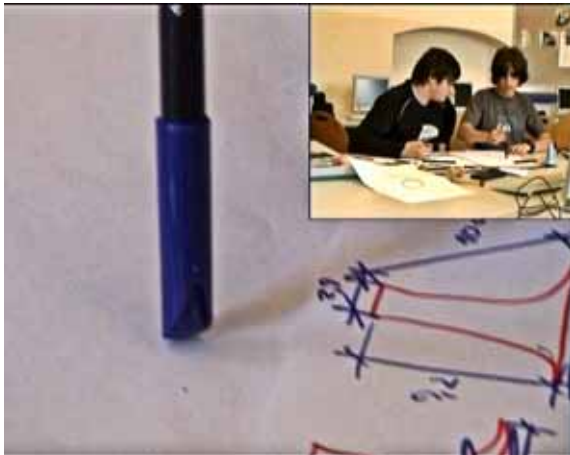


Figure 7

B starts his explanation with the aid of his labelled sketch. Then he begins to draw a new inscription. He labels it with all the measurements (figure 7 and 8) and uses this new inscription as a means to establish an algebraic equation (figure 8). Without any delay student B transforms this equation, even though he did not recognize the error he had

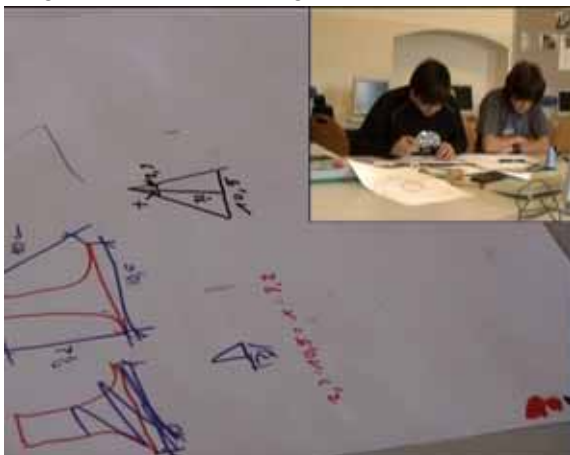


Figure 8

made. In this way, he gets a result, which he compares with the already existing "engraving" solution. As the result of his calculation differs from the measurements, he makes another attempt

using the intercept theorem. To do so he concentrates now on the inscription, which A had drawn. As the drawing seems to be too "pale" to B, he draws it again and labels the construction with measurements (figure 9). He obtains a second equation from this drawing, which leads

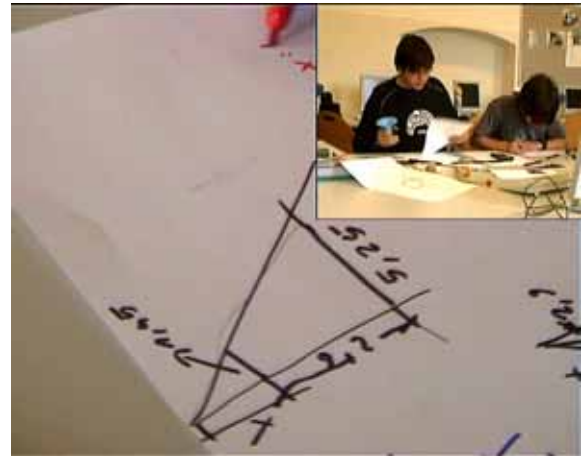


Figure 9

after some transformations to another numerical solution (figure 10) which fits with the "engraved" solution.

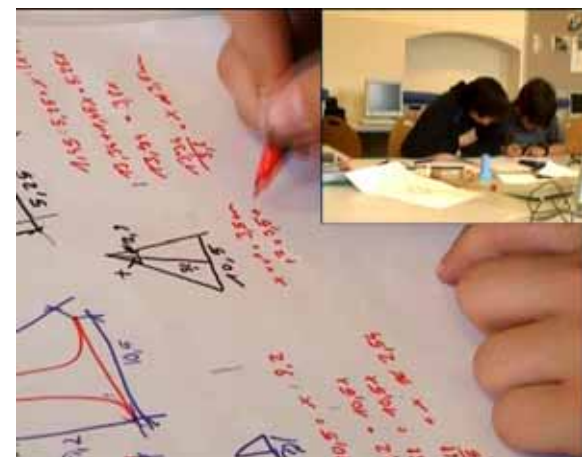


Figure 10

## 6. Student activities from a semiotic perspective

I now return to the goal of my paper, to discover how new knowledge come into existence? Peirce's semiotics as introduced will now provide a successful theoretical frame for describing and interpreting the learning activities of the students and their use of diagrams to solve a given mathematical task.

If we remember the first data, I presented in part 2 the successful idea for finding a solution started

from rolling the object on the table. The students had already used such a kind of movement when they investigated other bodies of revolution. Finding their interesting strategy of pressing the object into the sheets of paper emerged from a rather random observation. The students' achievement was their connecting of the engraving and the given task. This engraving was just a necessary requirement for finding a step to the solution. Memorizing the colouring of the engraved curve, we can say that the first solution was determined by their senses. Hand and eye, the sense of touch and the visual sense organize the students' activities. Beginning from the first inscriptions (figure 3 and figure 4), which are diagrams from geometry, the solution developed step by step. The students' collateral knowledge, in this case knowledge from geometry, was a handicraft-like prerequisite for their first solution. In some sense, the activities after colouring the engraved curve seem mechanical. Caused by the students' geometrical knowledge their activities seemed like an algorithm. One might say that the geometrical construction was not drawn but was written.

Comparing with the data given in part 2 the data from part 4 seems to be more profitable for my enterprise. After having done a series of measurements B started to draw a sketch from the axial section of the object, which he labelled carefully. The labelling with all its details was an easy job for student B. This ability has its root in his geometrical socialization. On the other hand, this construction of the sketch and all related activities were in some sense like the already mentioned "mechanical" activities. They are done by the way or we can also say collaterally. These activities offer on the one hand no direct support for gaining a new idea to the students, but on the other the constructed diagram provide fertile ground for the new ideas that follow.

How did student B invent the idea of using the intercept theorem? To begin with, we could suggest that B could read this theorem from his diagram. If B had offered an argument for his suggestion then in Peirce's words he would have made a corollary deduction. However, he could not even formulate a suggestion from his diagram. Using Peirce's words again student B could not even create an *abduction*. On the contrary, B needed support from his colleague student A.

Similarly to the "engraved" solution, where the starting point to the solution arose from marks on the paper, which had emerged unintentionally,

something unintentional was again the source of a successful idea. We can find this source in student B's activities, when he labelled his sketch with measurements (figure 5). Labelling a sketch or any other geometric drawing was a well-known practice for both students. This is another example of the use of collateral knowledge leading to unexpected results. Student A did not see just a section of the given object when he looked at the labelled sketch. His engagement with the given object and observing the measurements labels - which does not belong to Euclidian Geometry - caused A's intention to draw a right-angled triangle (figure 6). We can say that A abused these measurement labels. When (ab)using these labels A always had the context in mind as he referred all sides of his right-angled triangle to the given object. However, after this activity A lost interest in this diagram. Video data show that in the meantime student B had A's activities very carefully followed. Now two diagrams are drawn on the paper. There is the right-angled triangle as the result of A's "abuse", and B's own sketch. If one lays the first diagram over the other and additionally knows the intercept theorem then it is imaginable that such a person would have the idea of using this theorem. That is exactly what B did. One remark A made might have influenced B. A uttered the intention to draw vertical projection and he uses the wording "1:1". The wording "something to something" is part of the formulation of the intercept theorem.

However, that was not all! Not only did B suggest the intercept theorem by abduction but also he was able to give some arguments for why it was correct to use it. In figure 7, we see the tip of his pencil. With this tip, B marks the imagined corner of a right-angled triangle above his sketch. At this moment, he explains to his colleague why the intercept theorem was to be used. This means that by combining two diagrams a deduction was done. Following Peirce's concepts, this was a *theorematic deduction*.

The remaining activities can also be seen in the light of diagrammatic reasoning. In a first attempt, student B formulated an algebraic equation, which is a diagram. He used it to explore his solving strategy and to prove it empirically. As B had deduced this theorem with the aid of two geometrical diagrams, the geometrical intercept theorem had to pass the test. And this did not happen! In his first try, student B made a mistake when establishing his equation. We can see it in figure 8. However, as the calculation of one

variable was the only task B had to fulfil he could easily test his calculated result against the already existing "engraved" solution. We can say that a rule-governed transformation of a diagram supported the exploration. When he recognized his error, B constructed a new diagram and with a correct equation, he succeeded (figure 9 and figure 10).

## 7. Conclusion

The case study presented has shown the importance of inscriptions and in particular diagrams when solving a mathematical problem. Constructing and using diagrams can be seen as a possible source of new knowledge. At several points of their attempts to find a strategy and to answer the task, the students invented and transformed diagrams. These diagrams heavily influenced their learning activities. Thus we can say that the concepts of icon, diagram, theorematic and corollarial deduction found in Peirce's semiotics are valuable tools for describing the learning of mathematics.

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