18 Unconventional essays on the nature of mathematics

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Bridging the communities of mathematics and mathematics education: Is reconceptualizing the philosophy of mathematics an answer?

The mathematics education community has sporadically investigated fundamental philosophical questions pertaining to the nature of mathematics important for mathematics education. Individuals like René Thom, Hans-Georg Steiner, Imre Lakatos, Jean Piaget, and Hans Freudenthal, among others periodically engaged in deep philosophical musings about the basis for an epistemology, methodology and philosophy of mathematics education. Lerman (2000) suggested that today's interest in the philosophy of mathematics stems from the impact of Lakatos' (1976) classical book *Proofs and Refutations*, particularly its emphasis of mathematics as a humanistic, quasi-empirical activity subject to fallibility. There also exists a body of work in the mathematics education literature (Davis and Hersh, 1981; Dawson, 1969; Ernest, 1991; Kitcher, 1983; Sriraman, 2006; Tymoczko, 1986), which build on the Lakatosian approach of viewing mathematics teaching and learning as a humanistic and quasi-empirical activity. The considerable number of studies in mathematics education on the nature of proof further builds on some of these previously cited studies. This fruitful area of exchange between mathematicians and mathematics education researchers is in need of being developed if we wish to bridge the communities of mathematicians and mathematics educators. The book under review here provides a basis for cultivating such an exchange.

Reuben Hersh's book 18 *Unconventional essays on the nature of mathematics* presents rich possibilities for the community of mathematics educators to initiate a dialogue with mathematicians on the nature and purpose of mathematics and as a consequence mathematics education. The book as its title promises is a delightful collection of essays written by mathematicians, philosophers, sociologists as well as an anthropologist, a cognitive scientist and a computer scientist. When reading the contents of Hersh's book, I could not help but compare some of the "unconventional" and humanistic reflections of the various authors on doing mathematics to those of the great mathematicians in Raymond Ayoub's (2004) *Musings of the Masters*, which is an anthology of essays written by prominent 20th century mathematicians on eclectic topics such as mathematics and its relations to human intellect, the arts, science, culture, society, theology and literature. The interested reader may want to read Ayoub's book as well since both books question what constitutes a philosophy of mathematics and re-examine foundational questions without getting into Kantian, Quinean or Wittgensteinian linguistic quagmires.This review will summarize and analyze the contents of Hersh's book and discuss the myriad connections to other literature of interest to the community. In particular, the style of this review is discursive with relevant connections to the mathematics education literature pointed out to the (interested) reader.

In his introduction to the book, Hersh writes that his main criteria for including these particular essays are *"Nothing boring! Nothing trite, nothing trivial!"* and the book delivers on this promise. Readers may be familiar with earlier works of Reuben Hersh such as *The Mathematical Experience*, co-authored with Phillip Davis (see Davis & Hersh, 1981) and *What is Mathematics, really?* (see Hersh, 1997), which have both become classics in the literature. The current book is destined to become a classic as well for numerous reasons which will soon become transparent.

The first chapter by Alfréd Rényi, the co-creator of the theory of random graphs, is an imaginary Socratic dialogue between Socrates and Hippocrates spanning 16 pages. The dialogue is nothing short of gripping with Hippocrates seeking Socrates' counsel on becoming a pupil of Theodoros and pursuing the study of mathematics. Socrates', true to form, applies the Socratic method to counsel Theodoros and help him arrive at a decision whether or not to pursue mathematics. The dialogue poses the rhetorical question at the end as to the nature of the world and life, if ordinary people were to adopt the same standards of precision in arguments as that found in mathematics. Perhaps this type of provocation was instrumental in the well chronicled demise of Socrates!

The second chapter by Carlo Celluci entitled *Introduction to Filosofia e Matematica* shreds to pieces 13 dominant views/assumptions about mathematics. Celluci asserts that mathematics is a messy human endeavor, and not the playground for purely speculative and abstract philosophical questions and solutions alà Frege. The claim of philosophers that the *philosophy of mathematics* is a specialized area of philosophy in which problems occur in a "pure or especially simplified form" (p.17) marginalizes the scope of philosophical problems that occur in mathematics. For instance, Celluci in his demolition of dominant view #3 writes "another important problem in the philosophy of mathematics is the existence of mathematical objects….the view asserted in this book is instead that the problem of the existence of mathematical objects is irrelevant to mathematics…[m]athematical objects are simply hypotheses introduced to solve specific problems. To speak of mathematical objects is misleading because the term 'object' seems to designate the things the investigation is about, whereas hypotheses are tools for the investigation. The latter is intended not to study properties of mathematical objects but to solve problems" (pp.19-20). Celluci also gives convincing arguments to the messy and social nature of proof, far removed from dominant view that deduction is the holy grail of mathematical thinking.

The third chapter by William Thurston, the 1982 fields medal winner, outlines for the lay person (1) what mathematicians do, (2) how (different) people understand mathematics, (3) how this understanding is communicated , (4) what is a proof, (5) what motivates mathematicians, and finally (6) some personal experiences. Thurston stresses the human dimension of what it means to do and communicate mathematics. He also gives numerous insights into the psychology of mathematical creativity, particularly in the section on what motivates mathematicians. This chapter which is aptly entitled *On Proof and Progress* contains a first person account from an eminent mathematician on how mathematical ideas germinate and spread, as well as the nature of proof. Mathematics educators can draw great satisfaction from Thurston's writings, particularly on the need for a community and communication to successfully advance ideas and the very social and variant nature of proof which depends on the sophistication of a particular audience.Readers may be interested in revisiting Leone Burton's book (2004) *Mathematicians as Enquirers* which examines empirically many of the domain specific issues arising in the life of working mathematicians, some of which are pointed out by Thurston in his reflections on the profession (also see Sriraman, 2005a, 2005b).

Chapter 4 by Andrew Aberdein addresses the informal logic of mathematical proof by using Toulmin's (1958) classical work on argumentation. Mathematics educators familiar with the summary and analysis conducted by Sriraman & Kaiser (2006) of CERME4 research reports will recall that many researchers in the working group of argumentation and proof used Toulmin's work as a theoretical framework for studies involving proof analysis. Sriraman & Kaiser (2006) pointed out several common issues for researchers in the domain of proof which are particularly relevant in the new context of Hersh's book. These issues were: (a) The use of Polya style heuristics/conjectures : i.e., how does one arrive at a mathematical result in the first place before embarking on proving its truth? (b) Uses of evidence: What types of evidence are used in order to consolidate the truth of a result. For instance, the use of empirical data, examples, verification on computer via manipulation or checking cases ; (c) reasoning/argumentation/generalization (inductive, inferential, logico-deductive (axiomatic), informal, visual etc); (d) epistemology (questioning the very nature of proof). The types of proof accepted by the particular community one is working within, which would take into consideration (b) and (c). This aspect involves examining the place and use of definitions, levels of evidence etc. , and (e) pedagogical and curricular implications of (a)-(d) (Sriraman & Kaiser, 2006, p.35). Table 3 presented by Aberdein in this chapter gives a classification of seven different types of proof dialogue with the specific outcomes sought by the prover. This classification may well be adaptable by mathematics education researchers to analyse student proofs.

Chapter 5, which happens to be one of my favourites in the book is an essay by Yehuda Rav, a polymath working in numerous domains. In this chapter Rav examines the philosophical problems of mathematics from the point of view of evolutionary epistemology. This chapter will be of great interest to mathematics education researchers who seek metaphors from biology such as recapitulation theories to explain cognitive difficulties students' encounter with particular mathematical constructs. Rav argues that the human ability to create and use mathematics, and mathematization can be understood in Darwinian terms as a result of natural selection. Rav also demolishes the romantic notion of Platonism particularly its incompatibility with evolutionary epistemology, as well as discusses portions of Réyni's delightful dialogue from the point of view of evolutionary epistemology. In Rav's eloquent words

> …Mathematics does not reflect reality. But out cognitive mechanisms have received their imprimatur so to speak, through

> dealing with the world. The empirical component in mathematics manifests itself not on the thematic level, which is culturally determined, but through the logico-operational and logico-mathematical schemes. As the patterns and structures that mathematics consists of are molded by the logico-operational neural mechanisms…[they] acquire the status of potential cognitive schemes for forming abstract hypothetical world pictures…*[M]athematics is a collective work of art that derives its objectivity through social interaction* (p.81).

The domain of semiosis has recently started receiving more attention from mathematics education researchers for researching teaching and learning processes in the classroom. This is evidenced in the recent special issue of *Educational Studies in Mathematics* (vol 61, nos 1&2, February 2006) with numerous papers which address researchable topics in semiotics of mathematics, as well as an increasing number of references in the math ed literature to Charles Pierce's seminal work. In Chapter 6, Brian Rotman provides the semiotic basis of what mathematicians do when they publish their results in the language of mathematics. This chapter which is based on an earlier article that appeared in *Semiotica*, suggests that three levels of semiotic participation occurs in mathematical writing. Hersh summarizes these in his Introduction as "the disembodied pure thinker…,[t]he imaginary automaton, …[a]nd the actual live flesh-and-blood human being."(p.xii). Rotman's gives a superb presentation of what constitutes a semiotic analysis of mathematical signs as well as a semiotic model of mathematics. Chapters 5 and 6 are by no means a light read and require time and effort to digest the deep ideas in the exposition.

Chapter 7 by Donald Mackenzie is entitled "Computers and the sociology of mathematical proof". The author explores the relationship between proof and the computer and what constitutes "certainty" if proofs are automated. Mackenzie addresses "the question of variation: the way in which demonstrations that are convincing to some are unconvincing to others; the way in which "mathematical proof" can mean different things to different people" (p.129). The appendix of this chapter gives a chronology of the development of theorem-proof automation starting from the Logic theory machine of Newell, Shaw and Simon (1955-56) onto Thomas Hales' computer assisted proof of Kepler's conjecture in 1998.

In chapter 8, Terri Stanway investigates the question of "ownership of ideas" and "knowledge management" in mathematics. Stanway chronicles the changing needs and expectations of the mathematical community and how knowledge management practices affect the community from the time of Hardy & Littlewood onto today's age of MAPLE. Next, Rafael Nuñez contributes a chapter "Do real numbers really move?: The embodied cognitive foundations of mathematics" in which mathematics is presented as a human conceptual system and embodied in nature. This chapter presents ideas on the important role of gestures during mathematics teaching and learning which extend those presented in his earlier work *Where Mathematics Comes From* with George Lakoff (see Lakoff & Nuñez, 2000)

William Timothy Gowers, the fields medal winner of 1998, poses the question "Does mathematics needs a philosophy?" in the title of chapter 10, and proceeds to answer it by saying mathematics both needs and does not need a philosophy. One of the main arguments made in this chapter is that mathematics would proceed pretty much the way it always has irrespective of any brand new –isms developed in philosophy. The important point made by Gowers is that questions which philosophers consider important are too far removed from the actual business of doing mathematics. In chapter 11, Jody Azzouni argues how and why mathematics is a social practice.

In chapter 12, Gian-Carlo Rota writes on "The pernicious influence of mathematics upon philosophy" which is bound to provoke philosophers. Rota who is a distinguished applied mathematician and a philosopher writes that while philosophy can never truly answer its fundamental questions definitively, these "answers" are more connected with problems of our existence. The basic questions in philosophy have pretty much stayed the same starting with the Greeks. Answers posed by one generation of philosophers end up being revised or rejected by the next generation. There are no definitive answers per se. On the other hand, mathematics seems to be the envy of philosophy because mathematicians are able to give definitive answers to the problems that occur within mathematics. This has influenced philosophy to adopt mathematical standards of rigor for the study of philosophical problems. Rota writes:

> It has not occurred to our mathematizing philosophers that philosophy might be endowed with its own kind of rigor, a rigor that philosophers should dispassionately describe and codify, as mathematicians did

> with their own kind of rigor a long time ago. Bewitched as they are by the success of mathematics, they remain enslaved by the prejudice that the only possible rigor is that of mathematics and that philosophy has no choice but to imitate it. (p. 224).

Logic which used to traditionally be a part of philosophy has mutated into mathematical logic and consequently become subsumed as one of the many areas of mathematics. Does philosophy want to meet the same fate? This is Rota's provocation to philosophers. In Chapter 13, Jack Schwarz, Rota's mentor at Yale, examines "the pernicious influence of Mathematics on Science". This is another provocative piece of writing which the reader will find both entertaining and stimulating.

Alfonso C. Avila del Palacio gives a social scientists' answer to "What is philosophy of mathematics looking for?". Del Palacio shows the differences between the "mathematical reasoning about mathematics" and the "philosophical reasoning about mathematics" to better understand the limits of the answer given by philosophers and mathematicians to the question "what is mathematics?" Are the answers complementary or not? The interested reader can look up Del Palacio's interesting verdict in chapter 14.

Chapter 15 contains a detailed narrative of Rowan Hamilton's invention of the quaternions. Hersh in his introductory remarks Andrew Pickering's chapter as one of the first efforts to align the philosophy of mathematics and science. However, "Concepts and the mange of practice constructing quaternions" is one of the longer chapters in the book and most certain to please those interested in how mathematical ideas develop over history. In chapter 16, Eduard Glas argues that Popper's contributions to the philosophy of science are both relevant and adaptable to a good working philosophy of mathematics without the entrapments of the current –isms in use. Chapter 17 is written by the eminent anthropologist Leslie White. Although White calls this chapter "The Locus of mathematical reality: An anthropological footnote", his answer to the question of what is mathematics is both direct and simple. Towards the end of his compelling essay, White writes:

> Thus we see, there is no mystery about mathematical reality. We need not search for mathematical "truths" in the divine mind or in the structure of the universe. Mathematics is a kind of primate behaviour as languages, musical systems and penal codes are. Mathematical concepts are man-made just as ethical values, traffic rules and bird cages are man-made" (p.319).

Chapter 18 is written by Reuben Hersh and entitled "Inner Vision, Outer Truth". Hersh reflects on mathematical ideas that struck him as particularly beautiful, and the strange fit between mathematics and physics. His explanation of the sometimes miraculous fit between available mathematics and the problems that physicists are grappling with lead the reader to wonder whether this fit is natural or actually imposed simply because of historical precedence.

Reuben Hersh is to be commended for orchestrating this masterful collection of essays from distinguished individuals, which bring down commonly held prejudices about mathematics as well as the "tower" of ideas constituting the philosophy of mathematics. This "tower" has been left unchallenged so far as no one has dared to challenge the status quo. I predict that this book will become a classic for the coming generations of mathematics philosophers as well as for mathematics educators interested in changing dominant conceptions of *what is mathematics, finally!* (pun intended alà Hersh, 1997).

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