Functions: a modelling tool in mathematics and science

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Abstract: *It is difficult for the students to transfer concepts, ideas and procedures learned in mathematics to a new and unanticipated situation in science. An alternative to this traditional transfer method stresses the importance of modelling activities in an interdisciplinary context between mathematics and science. In the paper we introduce a modelling approach to the concept of function in upper secondary school is introduced. We discuss pedagogical and didactical issues concerning the interplay between mathematics and science. The discussion is crystallized into a didactical model for interdisciplinary instruction in mathematics and science. The model is considered as an iterative movement with two phases: (1) the horizontal linking of the subjects: Situations from science are embedded in contexts which are mathematized by the students, (2) the vertical structuring in the subjects: The conceptual anchoring of the students' constructs from the horizontal linking in the systematic and framework of mathematics and science respectively.*

ZDM-Classification: I20, M10, M50

1. Mathematics education and the problem of isolation

Mathematics is often seen as a very specific subject by students - and even by mathematicians themselves. Although a growing number of subjects include ingredients from mathematics, it is still difficult for both teachers of mathematics and teachers of other subjects to see the use of mathematics in other subjects - partly due to the use of concepts and language. This paradox of relevance leads to the *problem of isolation*, which works to the disadvantage of both mathematics and other subjects, which could profit from a conscious inclusion of mathematical competences When teachers discuss students' difficulties in learning, they seldom consider which problems are shared in common across disciplines, and which issues are specific to the specific discipline. As a role teachers know very little about other subjects. A permanent theme in science education is the adequate level of mathematics language. Mastering mathematical formalism is often a prerequisite for understanding in science and for many newcomers in upper secondary school this formalism act as a barrier, too high to overcome. A source of the problem might be that it is often by the teachers presumed obvious that the basis mathematical facts must be apprehended before studying science.

1.1 Mathematics and science

Mathematics plays a crucial role in science and practice too. This role is brought about predominantly through the building, employment, and assessment of mathematical models. Currently scientific research is undergoing a major transition from disciplinary approaches to interdisciplinary work. Mathematics is a gatekeeper to many mainstream studies and careers, thus a certain transparency of the transition to mathematics beyond upper secondary education is necessary if we want the students to realize the importance of mathematics. This role should also be reflected in educational settings. The instruction should promote the students' motivation for and interest in technology, natural sciences and scientific ways of thinking by including authentic interdisciplinary complexes in the instruction. But this is not the rule at upper secondary education, where focus in mathematics instruction is on relatively specialized algebraic techniques that students rarely remember beyond the final exam. One might ask the question if contemporary mathematics education prepares the students well for the application of mathematics in other subjects, their civic obligations, their future employment and personal needs. In this paper we advocate for reforms in secondary education mathematics and science education aiming at a more coherent structure of the organization of the subjects. We here use the notion science education for education related to the subjects of science, biology, chemistry and physics.

The traditional border between the school subjects of mathematics and science do not reflect the interdisciplinary work of modern science and the usefulness of mathematical competences in practice. The challenge is to replace the current monodisciplinary approach, where knowledge is presented as a series of static facts disassociated from time with an interdisciplinary approach, where mathematics, science, biology, chemistry and physics are woven continuous together. We present a framework for interdisciplinary instruction based on a discussion of pedagogical and didactical problems concerning the interplay between mathematics and science. The framework emerges from research that we have lead in the field of learning the concept of function in interdisciplinary settings between mathematics and science in upper secondary education. The paradigm for our research is a view of mathematics and science as related disciplines, and we see them as subjects that attempt to describe phenomena by models in order to act and deal with them in a sensible way. As a consequence we propose the development of an innovative integrated mathematics and science curriculum as an alternative to the traditional subject oriented curriculum.

2. The concept of function in upper secondary education

The function concept has become one of the fundamental ideas of modern mathematics, permeating virtually all the areas of the discipline. Placed in the centre of attention of upper secondary education the concept of function has a central and organizing role around which many other important mathematical ideas resolve. The mathematical discussions about the function concept have produced a gradual evolution in its accepted meaning from a correspondence to a formal set definition. Cooney & Wilson (1993) point at that a similar pedagogical evolution regarding the importance of the concept to school and how the concept should be taught. This evolution has to some extent paralleled the changes in the mathematical definition of function, but has also included other issues. In the textbooks of the 19th century until the middle of the 20th century a function was considered as a change, or as a variable depending on other variables. Influenced by the rise of abstract algebra, of which Bourbaki was the most famous proponent, changes occurred from the middle of the 20th century toward a in almost all school curricula definition of function by two sets A and B with a rule which assigns exactly one member of B to each member of A. Yet, despite being a powerful foundation for the final edifice of mathematics organized in a formal Bourbaki style many researchers in mathematics education find this approach is excessively abstract, especially at secondary education. Literature from the 1990s stress that a less abstract notion of functions as rules allows students to gain a strong conceptual background in functional thinking before progressing to the more abstract notion of functions as sets of ordered pairs (Eisenberg 1992, NCTM 1989, Sfard 1991). As a consequence the concept of functions presently at upper secondary mathematics education is introduced as a correspondence between the elements of two sets of real numbers, such that for each element of the domain there is exactly one element of the range. This approach amounts to the definition of Dirichlet given in 1837, and it appears to be more easily grasped by the students than the Bourbaki definition although they are very similar technically. But the dominance of the correspondence approach is by no mean an indicator for a general agreement upon a conceptual and pedagogic framework for the concept of functions.

2.1 Upper secondary students and functions

Students participating in a one of our research project reflect the complexity of the concept of function and the associated difficulties students experience in learning functions. 70 grade 10 students from Danish gymnasium (general upper secondary school) wrote an essay about their conceptions of function and the utility of the concept (Michelsen 2001). The following extracts from two of the essays illustrate the majority of the students' conception of function:

Function is something about a formula, which describes how the function should look like. You can always draw a function in a coordinate system, which is 2-dimensional. Every point therefore has two values, an x- and a y-value. This means that the function is something flat. In a function there is always a variable. But there must only correspondent one value to the variable of the function else it is no longer a function. The formula for the function is used for calculations and it can at times be considered as an equation. In nearly everything that we have learned in mathematics the ulterior motive was the utility for the functions. Mathematics teachers are obsessed by functions, they do not think about anything else and they expect the same from their students. Grade 10 student (Michelsen 2001, p. 80).

When we look at formulas with several terms then it is really difficult to see in which situations you can use them. And you are confirmed be the answers you get from the teacher when you ask him. In fact it is something you will be responsible for at the final examination. And after the examination only a few of us will encounter situations where you can use such formulas. It is quite enervating for students to learn something, which is useless. In fact it is very difficult for my teacher to give examples of application of formulas with a quadratic term. Altogether the missing examples of application are a problem in mathematics instruction. Grade 10 student (Michelsen 2001, p. 129)

As a rule the students evidently have difficulties of ascribing a reasonable meaning to functions. The students ask for examples of the usefulness of functions. But the essays show, that the students are aware of the important role that the concept plays an important role serving as models for understanding phenomena in science. This is illustrated by the following extracts:

I am quite sure that my understanding of the concepts would be improved if the teaching of mathematics and physics were more coordinated. Then the student would se that there is a method in the method. Then you could start with theory and that see how it is used in practice. Very often you learn something from mathematics, which is applicable in physics. But often you learn something in mathematics several months after you have used it in the physics lessons. That is wrong. Grade 10 student (Michelsen 2001, p. 143).

I think, that from the moment we all realize that what we have learned is applicable in "everyday life" it will be easier for the teacher to involve all the students. That is not the case today. Actually it is in physics and chemistry we find most of the examples of application of the mathematical rules. That is by no means wrong, but it implies that the teachers of mathematics, physics, and mathematics must talk and coordinate. Grade 10 student (Michelsen 2001, p. 145).

The students would like to see slightly more coordination between the mathematics and the science lessons. Only very few students are dismissive of a closer instructional coordination between mathematics and the subjects of natural sciences:

I think that a closer coordination between mathematics and physics would end up in a terrible mesh. I like the way it is now. Grade 10 student (Michelsen 2001, p. 145).

3. Expanding the domain as an alternative to transfer

In the current curriculum links between phenomena from science and representations of functions are introduced only after the students have been taught to think of functions as interpretations of algebraic expressions. And there are very few examples of textbooks and teaching materials that combine mathematics and science in real interdisciplinary contexts. Such an approach accentuates the island problem – the gap between the island of formal mathematics and the mainland of real human experience. Kaput (1994) elucidates the gap with the difference between mathematical functions that are defined by algebraic formulas, and empirical functions that describe every-day-life phenomena. The problem of transferring formal mathematical knowledge to a new context is one of the biggest challenges in education. It is well known that it is difficult for the students to apply concepts, ideas and procedures learned in mathematics in a new and unanticipated situation, either in or out of school. Learning is inherently associative, and it is not surprising that students have difficulties applying a competence that they associate with one setting to something entirely new.

3.1 Transfer and domain specifity

According to Niss (1999) a significant example of the major findings of research in mathematics education is the key role of domain specifity. The student's conception of a mathematical concept is determined by the set of specific domains in which that concept has been introduced for the student. When a concept is introduced in a narrow mathematical domain, the student may see it as a formal object with arbitrary rules. This results in the recognised difficulty of application of the concept in new settings. As an alternative we want to point at that interdisciplinary activities between mathematics and science offer a great variety of domain relations and context settings that can serve as a basis for developing a more practical and coherent structure of a mathematical concept. By *expanding the domain* with contexts from science the problem of domain specifity is transcended and the curriculum is presented as a cohesive program. Lobato (2003) addresses the central educational issue of transfer learning and criticises the traditional models of transfer for relying too heavily on the determination of transfer from an expert's point of view. She argues for an actor-oriented framework, where one assumes that the students are making connections between situations nearly all the time, guided by the aspects they find personally salient. As a consequence the critical issue is to design an instructional environment that supports the students' construction of personal relations of similarities across situations.

3.2 The differences in terminology and notational systems between science and mathematics

Even though the mathematical tools underlying situations in science lessons are often quite elementary the application of the tools are surprisingly sophisticated. To be useful the mathematical competences learned by the students need to be applied or reinforced in more sophisticated contexts throughout the educational system.

One of the sources of this problem is the differences in terminology and notational systems between science and mathematics. Ellermeijer & Heck (2002) describe how the differently use of mathematical entities in mathematics and disciplines of science comes salient when it comes to the design of integrated learning environments. In mathematics the students think of a graph as a representation of a function, i.e., as a representation of a single object, so that it suffices to work with one variable. In science and applied mathematics graph represents a relation between quantities, and the one must work with a least two variables. In mathematics the concept of variable has several meanings, and it is almost impossible to rigorously define the term. In science a variable is most often used as a name for a quantity that can vary and that in many cases can be measured. Freudenthal (1983) classifies the various appearances of variables: (i) a placeholder (ii) a polyvalent name and (iii) a variable object. In mathematics variables are used mostly as placeholders and polyvalent names. Emphasis is on generalized pure arithmetic and on the concept of function defined as a special correspondence between to sets. In science the third kind of variable, the variable object, comes into play, and one is involved with functional relationship between varying quantities. Research (Schoenfeld & Arcavi 1988, White & Mitchelmore 1996) shows that a major source of students' difficulties in applying functions is an undeveloped concept of variable. In particular, the students often treat variables as symbols to be manipulated, rather than as quantities to be related. Malik (1980) calls attention to, that the algebraic approach to the concept of function appeals to the discrete faculty of thinking and lacks a feel for the variable. But for calculus and other practical sciences the requisite instruction should enable students to feel for smooth change of the variables in phenomena. We point at, that when students are involved in a modelling activity they gain practice in identifying and representing variables. Representing variables in a model requires students to practice looking for structures that both extra- and intramathematical entities have in common. In an interdisciplinary activity between mathematics and science variables represent quantities that change, and functions are the tool to study the relationships among the changing quantities.

4. Function as modelling tool in mathematics and science

According to Selden & Selden (1992) modelling real-world to help organize the physical world is one of the most common uses of functions. Sierpinska (1992) suggests that functions should first appear as an appropriate tool for mathematizing relationships between physical (and other) magnitudes. An awareness of the possible use is a *sine qua non* condition for making sense of the concept of function at all. In *Principles and Standards* (NCTM 2000) it is recommended that students of upper secondary school expand their repertoire of familiar functions for mathematical modelling. This provides them with a versatile and powerful means for analyzing and describing their world. And *as students' knowledge of mathematics, their ability to use a wide range of mathematical representations, and their access to sophisticated technology and software increase, the connections they make with other academic disciplines, especially the sciences and social sciences, give them greater mathematical power* (NCTM 2000, p. 354).

In our view the importance of school mathematics should be justified by the fact that it provides the students with powerful tools for dealing with the quantitative aspects of the world. This role is brought about predominantly through the building, employment, and assessment of mathematical models. The models are the results of mathematizing situations, which people wish to study. That is, to identify the elements of the situation that is considered important for the purpose at hand and the relationships among them. Here it is appropriate to refer to the famous quotation of Galileo that the laws of Nature are written in the language of mathematics (Kline 1972). The notion of mathematics as a language is not new. Arianrhod (2005) points at that the grammar of mathematics has powerful implications and focuses on events in the use of mathematics, where the language of mathematics grows in front of our eyes until it reveals a brand-new piece of physics.

4.1 Modelling by mathematization

Freudenthal (1991) emphasises phenomenological exploration, and argues for that the starting point for mathematics education is those phenomena that beg to be organized. In science education it is often accentuated that many phenomena and their patterns of interaction are best described in the language of mathematics, which then becomes a bridge between the students verbal language and the scientific meaning we seek to express (Osborne 2002). Modelling by mathematization treats specifically the role of mathematics in science, and of the link with mathematics in various fields of science education. Pointing at the dramatically change in the nature of problem solving activities during the past twenty years and at the difficulties to recruit students capable of graduate level in interdisciplinary such as mathematical biology and bio-informatics Lesh & Sriraman (2005) suggest a bottom up solution. That is, initiate and study the modeling of complex systems that occur in real life situations from the early grades. The current calculus-driven high school curriculum is unlikely to produce a quantitatively literate student population. According to Steen (2004) it is therefore time to recognize the distinctive quantitative requirements of universal education in the computer age in the form of quantitative literacy. The essence of quantitative literacy is the use of mathematical and logical thinking in context. A cornerstone of quantitative literacy is the ability to apply quantitative ideas in unfamiliar contexts.

4.2 Integration of mathematics and science

Issues related to the integration of topics from mathematics and science are complex, because they comprise two apparently different components, an extra-mathematical and a mathematical context. The integration must take into account, that didactical problems with the various topics have more consequences than the respective curricula might suggest. There exist numerous general models of integration that offers a theoretical grounding for the diverse perspectives associated with integration of mathematics and science. These models most often take form of a continuum reaching from the discipline of mathematics over balanced mathematics and science to the discipline of science. Generally speaking the models are based on two assumptions: (i) teaching mathematics in relation to science supports students' learning by providing meaningful contexts in which the students can see the application of abstract mathematical concepts - and as a side effect this helps students instil positive attitudes towards mathematics (ii) mathematics offers science education the tools for quantifying, representing, and analyzing phenomena from science – and as a side effect fosters a sense of objectivity in science. This often results in connections between mathematics and science, which is imbedded within the context of real-world applications, and as such the connections tend to be contextual rather than conceptual (Berlin 2003). It is of great importance for the students of mathematics to have opportunities to apply their mathematical skills to the solution of scientific problems. But a consequence is that mathematics is learned separately from its applications. Contextual problems should therefore also be exploited as meaningful starting points from which conceptual structures in mathematics and science can emerge.

4.3 Contextual and conceptual approaches

There exist several frameworks, which include both contextual and conceptual approaches. The concept of emergent model suggested by Gravemeijer (1997) has as the departing point situation specific problems, which are subsequently modelled. Context problems first offer the students the opportunity to develop situation-specific methods and symbolisations. Then the methods and symbols are modelled from a mathematical perspective and in this sense the models emerge from the students' activity. The models first come to the fore as a model of the situation, and then the model gradually becomes an entity on it own right and starts to serve as a model for mathematical reasoning. The shift presented from 'model of' to 'model for' should concur with a shift in the way students perceive and think about the models, from models that derive their meaning from the context situation modelled to thinking about the mathematical content.

The transition from a 'model of' to a 'model for' has certain similarities with Sfard's (1991) description of the process-object duality of mathematical concept based on historically and epistemologically analysis. With the concept of function as the leading illustration, Sfard sees a dual nature to mathematical conceptions; they have an operational (process) and a structural (object) mode. Students start by engaging in computational processes, which leads them to an operational conception. From these, structural conceptions usually develop by going through three stages: interiorization, condensation, and reification. While interiorization and condensation are gradual, reification is a sudden qualitative jump from what was initially perceived as a process or procedure to a mental object, which is seen as a single entity that possess certain properties and that can be operated on by other higher level processes, such as transformations or compositions. Reification proves difficult and Sfard deduces two didactic principles of her description: (i) new concepts should not be introduced using their structural descriptions, and (ii) structural conceptions should not be introduced as long as students can do without them.

4.4 Functions as an essential means of describing, explaining, and predicting real world phenomena.

Although Sfard does not link the process conception to aspects of application, one could argue that problems of applications should be included to motivate the students. The historical origin of the function concept is rooted in the study of natural phenomena, and mathematical functions play a significant role in the study of natural sciences, social sciences, engineering and technology. Functions constitute an essential means of describing, explaining, and predicting real world phenomena. O'Callaghan (1998) presents a conceptual framework for teaching the concept of function, which firmly rooted in a problem-solving environment consists of four component competences: modelling, interpreting, translating, and reifying. The model accommodates the process-object duality by linking explicitly the components of modelling, interpreting, and translating to the process conception and the reifying component to the object conception.

5. Horizontal linking and vertical structuring

A modelling approach has the potential to encompass both contextual and conceptual aspects of the function concept. We agree with the suggestion by Lester & Kehle (2003) to subsume problem solving within the much broader category of mathematical activity centered on model eliciting tasks. They see a fruitful blurring of task, person, mathematical activity, learning, applying what have been learned, and other features of mathematical problem solving. This blurring metaphor does not express less precision. On the contrary it reflects a richer view and a more authentic view on activities in a classroom, where the students must express, test, modify, revise, and refine their own ways of thinking during the process of designing powerful conceptual tools that embody constructs that students are intended to develop. Further the artificial division in a mathematical world and reality is abolished.

5.1 Interdisciplinary competences

Many mathematics and science educators are in favor of a more realistic education, where modelling activities are used to treat concepts in realistic, everyday life contexts (Gilbert & Boulter 1998, de Lange, 1996). But although there is an implicit interdisciplinary aspect in a modelling approach, only little attention has been paid to the potential of such activities to bridge the gap between mathematics and science. The Berlin-White Integrated Science and Mathematics Model (Berlin & White 1998) exceptionally focuses on both mathematics and science, and offers a typology to describe and understand the complex nature of integration of mathematics and science from both a content and a pedagogical position. The model identifies the following six aspects of integration: ways of learning, ways of knowing, content knowledge, process and thinking skills, attitudes and perceptions, and teaching strategies. In our view this multidimensional view of the interplay between mathematics and science offer a very practicable framework for developing and implementing integrated mathematics and science programs embedded in a modelling context.

As an alternative to overemphasizing algebraic symbolism we suggest an approach to the function concept aiming at using modelling activities to bridge the gap between mathematics and science. The idea is to look for models that can be generalized and formalized by the students to develop into entities of their own, which can become models for reasoning in science and mathematic. This draws attention to new ways of describing a modelling curriculum. In the so-called *KOM-report* (Competences and Learning in Mathematics) the notion of competence is introduced as a basis for describing and analyzing mathematics instruction from kindergarten to tertiary level. The KOM-report defines a mathematical competence as an insightful preparedness to act properly in situations that contain a particular kind of mathematical challenge. Eight mathematically competences are identified aiming at describing what it means to master mathematics: (1) mathematical thinking competence (2) problem solving competence (3) modelling competence (4) reasoning competence (5) representational competence (6) symbolizing and formalizing competence (7) communication competence and (8) auxiliary tools competence (Niss & Jensen, 2002). The modelling competence includes structuring an intra- or extra-mathematical situation to be modelled, mathematizing the situation, analysing and tackling the model, interpreting the results, validation of the model, communicating about the model, monitoring the modelling activity.

Almost all forms of learning involve some ways of representing information. Modelling can be considered a form of mathematical challenge that supports the emergence of representations of functions. When modelling in an interdisciplinary context the students are confronted with information from multiple sources that is presented and communicated in different forms. As the students face more complex problems they must develop an increasingly large repertoire of representations and knowledge of how to use them productively. Lesh & Doerr (2003) introduce the model and modelling perspective based on the assumption that models for making sense of complex systems are some of the most important components of mathematical knowledge and understanding. This approach gives the students a more accurate portrayal of what mathematicians spend most of their time doing – constructing and investigating structurally interesting systems – or using them to make sense of real-life applications. Especially it is emphasized that in the process of mathematizing the students focus on representations that have the greatest power and usefulness and go beyond thinking with models and representation systems to also think about the similarities and differences, and strengths and weakness, for a variety of purposes.

Our point is that the modelling and representation competencies overlap each other, and they are by no means independent in practice. We therefore call them *interdisciplinary competences*. The representational competence includes handling of representations of a diversity of matters from mathematics and science, understanding and application of different form of representations, knowledge of strength and weakness of a representation, and selection among and translation between different forms of representations. Considering the representational competence as an interdisciplinary competence makes it possible to transcend the above-mentioned problem of the differences in terminology and notational systems between science and mathematics

5.2 A model for integrated mathematics and science courses

According to Freudenthal's (1991) mathematizing is the key process of mathematical activity. There are two types of mathematization in an educational context – horizontal and vertical mathematization. In horizontal mathematization, the students come up with mathematical tools, which can help to organize and solve a problem located in a real-life situation. Vertical mathematization is the process of reorganization within the mathematical system itself, like finding shortcuts and discovering connections between concepts and strategies, and the application of these discoveries. This approach espouses mathematics as an activity without loosing sight of mathematics as a product. The strong emphasize on mathematization makes the approach a promising theoretical framework to design instructional sequences that strengthens the relation between mathematics and science. This does not mean that that all models are or should be mathematical, but mathematization makes it possible to transfer the modelling process from one place to another.

We propose an extension of the distinction between horizontal and vertical mathematization to a didactical model for integrated mathematics and science courses consisting of two phases: *horizontal linking* and *vertical structuring*. The description of the modelling competence as an interdisciplinary competence of course entails that situations from science are embedded in the contexts to be mathematized, and this is a horizontal linking of mathematics and science. Also the vertical mathematization must include a vertical structuring, that is the conceptual anchoring of the general model in the systematic and framework of mathematics and science respectively (Michelsen 2005).

The didactical model emphasizes mathematizing and modelling as ideas that cut across mathematics and science, and encompasses both contextual and conceptual aspects. In the horizontal phase thematic integration is used to connect concept and process skills of mathematic science by modelling activities. Also in this phase explicit connections is established between the process skills of mathematics and science. For example the facility with graphical, tabular, algebraic and written representations of functions is explicitly linked to data analysis. The vertical phase is characterized by a conceptual anchoring of the concepts and process skills from the horizontal phase by creating languages and symbol systems that allow the students to move about logically and analytically within mathematics and science, without reference back into the contextual phase. The shift from the horizontal to the vertical phase thus might concur with a shift from integrated instruction to subject oriented instruction. It should be stressed that the didactical model is iterative. Once the concepts and skills are conceptual anchored in the respective subjects, they can evolve in a new interdisciplinary context, as a part of a horizontal linking.

6. Integrated modelling courses

If reform of mathematics and science education is the aim, prototypes of instructional sequences with learning materials that are in harmony with new perspectives must be available to the teachers. At University of Southern Denmark we are a team of educational researchers and teachers from upper secondary school, who try to find out how the idea of a modelling approach to function in interdisciplinary context can be implemented. With interdisciplinary competences as a frame and the horizontal linking - vertical structuring model as a didactical tool we develop, implement, and evaluate courses with integrated tasks for mathematics and science. The idea is to look for models that can be generalized and formalized by the students to develop into entities of their own, which can become models for reasoning in science and mathematics.

6.1 Trigonometric functions and exponential growth in context

In the traditional approach of to functions there is a preference for linear functions as the students' first encounter with the function concept. We agree with the argument of Yerushalmy & Schwartz (1993) that this tradition derives from the computational simplicity of evaluating them, and there is no particular need to concentrate to concentrate on them in the early stage. According to Dooren et al (2004) the extensive attention paid to linear relations in primary and secondary school lead to "illusion of linearity", the tendency in students to see and apply the linear model "anywhere". Already when the students at the first time meet proportional and linear relationships in a more formal way they should be confronted with counterexamples of situations where linearity does not work. We think that trigonometric functions and exponential growth are well suited to give students a feel for smooth change of the variables in phenomena, and in that way help the students to develop their understanding of variables.

We have developed and implemented two courses, where the concept of function is introduced to grade 10 students in an interdisciplinary context. The first course focuses on movement and trigonometric functions. The motion of bicycle wheel is used to introduce the trigonometric functions. The students videotape the motion of a fixed point on the wheel. The motion of the point is explored and modelled by students' with a software program providing different forms of representations and possibilities for transforming the constructed models. The other course is centred on exponential growth and radioactivity. The exponential behaviour of the accumulation of the variable number of nuclei is a very common phenomenon in our physical, biological and social environments and therefore the same model, structure and behaviour can be found in the study of other parts of the real world.

The overall curricular units are designed to integrate three components; the gathering of data from a physical experiment, development and exploration of mathematical models (verbal, symbolic, graphical and tabular) and a reflective discourse about the mathematical and science content of the models. The idea is to involve the students in expressive modelling, where the students can express the evolving concepts. To make the structure and content of mathematics and science as explicit as possible, the course material is organised around a small number of basic models.

The courses are organized as a steady movement between horizontal linking and vertical structuring. For example in the exponential growth course the students simulate the radioactive decay process, and have to make their own representation of a physical event, choose variable, and pose relationships among them. Then in the vertical structuring phase the students' models are anchored conceptual as exponential growth models and their mathematical characteristics are investigated. Exponential growth models can subsequent be applied in a new modelling context, for example the investigation of cooling of a hot liquid.

An important aspect of modelling is that it allows the students to visualize abstract concepts by creating structures through which they can explore and experiment. Concepts can be worked on, modified according to the situations where they are called upon to be used and this might cause new concepts to emerge, which can be worked on, interpreted, modified and generalised. In the trigonometric functions course the students experience the emergence of model when the software transforms the video of the movement of the fixed point on the bicycle wheel to a graph in a coordinate system. This model now serves as a basis for developing formal mathematical knowledge about the function $f(x) = a \cdot sin(b \cdot x+c)+d$. Using the software the students can investigate the meaning of the parameters a, b, c and d by stating hypothesis about the meaning and then checking them. Thus the model of the wheel's movement changes character from a context-specific model of a situation to an entity of its own and the movement from the horizontal linking to the vertical structuring is carried out. The following extracts from a student's report gives an expression of how the investigations took place:

Hypothesis: The parameter d together with c govern the interception with the y-axis in some complicated

way.
The hypothesis is confronted with the graph on the computer screen and the student proposes a more precise hypothesis:
d is "the centre" of the curve and it can raise and lower the curve.
Then the student states the final conclusion:
c can displace the curve to the right or the left, and determines together with d, which can raise and lower the curve, the interception with the y-axis. Furthermore d is the centre of the curve, and the curve cannot span over more than from d-a to d+a. In the same way a determines the oscillation of the curve. b is the width of the wave - an increase in the numeric value of b gives a narrower wave. (Rasmussen, 2003, p. 32)

The use of context problems is very significant in the two courses approach. The context problems not only function as a field of application, but must also be designed to help the students to capture and identify the basic processes underlying a functional situation. For example the students' investigation of the simultaneous decay of two different radioactive sources in the context problem *Activated silver* naturally leads to the idea of summation of two functions. The contexts from physics thus provide the students with an experience of the actions that create the need for the application of mathematizing:

> **ACTIVATED SILVER**
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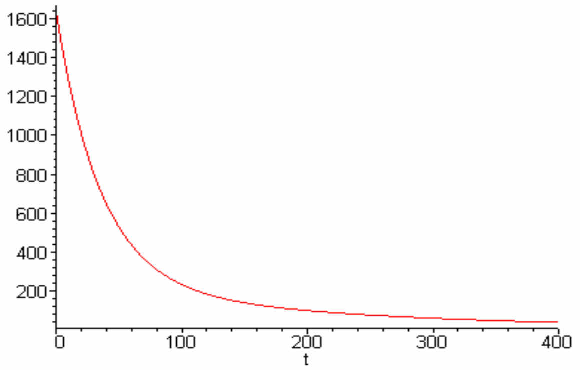
> A well known experiment to investigate the phenomena of radioactivity is the activation of silver. A small silver probe is placed in a neutron source and is irradiated for about 15 minutes. Then the activity is measured for 400 seconds. The graph below shows how the activity decreases with time



> The silver probe consists of two stabile silver isotopes Ag-107 and Ag-109. In the neutron source they are both by neutron capture changed to respectively Ag-108 and Ag-110.
>
> a) Write the nuclear reactions for the change of Ag-107 and Ag-109.
>
> Ag-108 and Ag-110 are both radioactive isotopes of silver.
>
> b) Use the a nuclear card to identify which radioactive kinds isotopes Ag-107 and Ag-109 are, and write nuclear reactions for the change of respectively Ag-108 and Ag-110.
>
> This means that there in the experiment above is measured on two decay processes, and the graph therefore shows the activity from both of the isotopes.
>
> c) Assume that at *t* = 0 the number of Ag-108 nuclei is 50.000 and the number of Ag-110 nuclei is also 50.000. The half-life of Ag-108 is 24.76 s and for Ag- 110 it is 144.41 s. Set up an algebraic formula for the total activity as a function of time. Does this activity decrease exponentially with time?

(Michelsen 2001, p. 168)

The above approach to the summation of two functions should be contrasted to the traditional structural and abstract definition. In the framework of process-object duality, one might say that the students get the first process-oriented experiences with summation of two functions. With Gravemeijer & Doorman (1999) we note a reflexive relation between the use of context problems and the development of the student's experiental reality. At one hand, the context problems are rooted in this reality at the other hand, solving these context problems helps the students to expand their reality.

To illuminate how the students' engagement in modelling activities contributed to experience and manifest their interdisciplinary competences, we point on that the students' made reference to both mathematics and physics and used representations transcending the boundaries between the two subjects. The students' combination of perspectives from both subjects often helped them to overcome problems in the modelling process. When modelling the absorption of light in water a group of students was blocked by the problem of assigning the correct value to a^0. According to the students the value was 0, but then reference to the intensity of light at the surface of the water made them change to the value 1, which then was checked on their calculator. And the following from a student's written report shows that the models constructed by the students had both a formal and concrete status: *We saw that cooling of the hot liquid resulted in a decreasing exponential* *function. Therefore the graph has the form of* $y = ba^x + c$*. In the formula the value of c must be the room temperature; else we had to cool the liquid with something colder than room temperature. The graph of heating the ice water was a graph that was opposite to cooling graph. If you place the temperatures from both graphs at the same time you get a horizontal graph.* (Michelsen 2001, p. 220)

In this description the function represented by a formal expression, but it is not considered as a static entity but as a manipulative object that can be transformed. And through the transition to the more formal and generalized model, the extended model does not become detached from the original model.

6.2 Including new perspectives on mathematics and science

According to Mason (2001) a major contribution towards effective teaching of modelling lies in enculturating students into what it is like to perceive the world like a modeller. Teaching modelling is more than simply rehearsing established models from mathematics and science. It is also displaying what it means to interrogate the world and to construct models to explain the phenomena we identify. Anchoring students' activities in specific contexts often opens up for new perspectives and often these are personal and political. In one of our courses the students' explored models of population growth. In the introduction to their report a group of students on their own initiative wrote: *We are fortunate being born in a continent, where we all get food (and plentiful of it). But others are not so fortunate. And that despite calculations show that our advanced technology makes it possible to support more than all people on Earth. We think it is only politics and old controversies between the continents that block a heavy reduction of the number of people living below the subsistence level. And that is wrong*. Group of grade 10 students (Michelsen 2001, p. 202).

Ethical issues of mathematics and science are not being given considerable attention at upper secondary education. But when more applied mathematics and science are taught in school it must be acknowledged and taught as being embedded in social contexts. And there is considerable evidence that students would like ethical issues to be more widely addressed in mathematics and science than is often the case. Furthermore research suggests that ethical reasoning can be assed without making excessive demands on the teachers (Reiss 1999).

6.3 Perspectives

The experiences from the two integrated courses are the base for our development of new prototypes of interdisciplinary courses and the basis of the course *Modelling as a tool for interplay between the disciplines of mathematics and natural sciences* for in-service upper secondary teachers. The aim of the course is to prepare the teachers for structural the reform introduced at Danish upper secondary education from August 2005. In the course the teachers are introduced to the horizontal linking - vertical structuring model as a didactical framework for coordination and mutual interaction of the subjects of mathematics and natural sciences. As a part of the course the teachers develop and implement integrated modelling courses, which are presented at a seminar for teachers.

7. Conclusion

Making connections and transferring ideas to a new context are difficult processes that many students cannot accomplish by their own. A focus on model and modelling avoids the problems of transfer and domain specifity, because models make sense of complex situations, and the purpose of the models is to provide meaningful ways for students to construct explain, describe, explain, manipulate, or predict patterns and regularities associated with complex situations. In this article the starting point is a view on mathematics and science as two naturally connected disciplines that should attempt to develop students' modelling and representational competences. As a consequence of this the traditional border between the two school subjects should be reconsidered. We do not argue that students should realize all aspects of the concept of function in interdisciplinary contexts. There are aspects, which are the result of structuring within the mathematical system. And in such cases the teacher must point this out to the students, so they are not looking for connections. But we want to emphasize that the more connections there exist among facts, concepts, and procedures, the better the students' understanding, and the more the curriculum is presented as a cohesive program with a range of tentacles, the more likely the students will have a rounded, effective and meaningful education.

The isolation problem in mathematics education is transcended when concepts, skills and principles of mathematics and other subjects – are weaved together into a unified whole. The students' interpretation of the mathematical concepts in terms of other disciplines gives a richer approach and a consistent meaning to the concepts that helps learning. If we as mathematics and science educators take the stance that our subjects have value of solving meaningful problems or even improving society, then we have to design learning environments that are meaningful to and value for the students. Knowledge is not inherently divided into categories, and the separation in subjects is artificial. Problems of today are multidisciplinary and they often handled by bringing together teams of people in different disciplines.

In this article the starting point is a view on mathematics and science as connected disciplines, and as a consequence of this that the traditional border between the school disciplines mathematics and science should be reconsidered. We argue that the central goal of mathematics and science education is to help students to use the languages of mathematics and science to construct and interpret meaning. Mathematics and science education must be reformed to prepare the students to think mathematically and scientifically beyond school. Meeting this need requires development of an appropriate educational strategy involving design of model courses along with supporting teaching materials. There are practical problems to overcome, but there is also room for considerable improvement. Researchers of mathematics and science education must continue to explore the effects of integrated approaches on students understanding, attitudes towards and interest in mathematics and science. Implementation and evaluation of integrated programs within the framework presented in this paper is needed to shed light on strengths and weaknesses of the didactical model. A curriculum is not integrated if a variety of unrelated activities are presented or if many strands taught unrelated to each other. Interdisciplinary competences could be the generic methodology that acts as a common denominator for subjects, such as mathematics and science. An identification and clarification of the interdisciplinary competences is therefore needed.

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