Modeling conceptions revisited

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Abstract: *The previous issue of ZDM raised several fundamental issues on the role of modeling in the school curricula at micro and macro levels In this paper we complement the approaches described there by discussing some of the issues and the barriers to the implementation of mathematical modeling in school curricula raised there from the perspective of the on going work of the models and modeling research group. In doing so we stress the need for critical literacy as well as the need to initiate a new research agenda based on the fact that we are now living in a fundamentally different world in which reality is characterized by complex systems. This may very well require us to go beyond conventional notions of modeling.*

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The Situation Today

In comparison to the school curricula in the U.S in the 1950's and 60's, many contemporary reform based curricula are incorporating a data driven modeling approach to the teaching of mathematics in which context plays an important role in getting students interested in the material. Two recent examples of NSF funded high school curricula which make use of this approach are the Core Plus Mathematics Project (CPMP), and the Systemic Initiative for Montana Mathematics and Science (SIMMS). A data driven modeling approach has also been incorporated into middle school curricula (see various examples in Senk & Thompson, 2003). Although the situation is encouraging in comparison to the status quo of previous decades, authentic modeling based curricula like SIMMS have been difficult to implement at a macro level due to the various reasons, namely systemic inertia, professional development constraints and general societal resistance to a curriculum which goes against popular conceptions of what constitutes school mathematics (Burkhardt, 2006)

Pure versus Applied mathematics

While the aesthetic nature of mathematics remains a mutually agreeable focal point for both pure and applied mathematicians, the question of what type of mathematics should be part of the school curriculum remains a contentious issue. Given the changing nature of mathematics used in emergent professions, recent initiatives to emphasize the applied nature of mathematics are more than important. Steen (2001) argued that in spite of the rich and antiquated roots of mathematics, mathematicians among others should acknowledge the contributions of researchers in external disciplines like biology, engineering, finance, information sciences, economics, education, medicine etc who successfully adapt mathematics to create models and tool kits with far reaching and profound applications in today's world. These interdisciplinary and emergent applications have resulted in the field of mathematics thriving at the dawn of the 21st century. We prefer to use the term "design scientists" for such researchers and "design science" for such research. In the U.S the urgency of preparing today's students adequately for future oriented design science fields is increasingly being emphasized at the university level. More recently, Steen (2005) wrote that "as a science biology depends increasingly on data, algorithms and models; in virtually every respect it is becoming… more mathematical" (xi). Both the National Research Council (NRC) and the National Science Foundation (NSF) in the U.S is increasingly funding universities to initiate inter-disciplinary doctoral programs between mathematics and the other sciences with the goal of producing design scientists adept at using mathematical modeling in inter-disciplinary fields. Yet many institutions in the U.S are finding it difficult to recruit students capable of graduate level work in interdisciplinary fields such as mathematical biology and bio-informatics. This suggests that undergraduates feel under prepared to pursue careers in these emerging fields. Any educator with a sense of history foresees the snowball effect or the cycle of blaming inadequate preparation to high school onto middle school onto the very elementary grades, which suggests we work bottom up.

We contend that in mathematics and science:
(a) modeling is primarily about purposeful description, explanation, or conceptualization (quantification, dimensionalization, coordinatization, or in general mathematization) - even though computation and deduction processes also are involved.
(b) models for designing or making sense of complex systems are, in themselves, important "pieces of knowledge" that should be emphasized in teaching and learning – especially for students preparing for success in future-oriented fields that are heavy users of mathematics, science, and technology. Therefore it is important to initiate and

study modeling, particularly those of complex systems that occur in real life situations from the very early grades.

The Chicken or the Egg Problem

Another major issue we now discuss is the schism between the most well intentioned curriculum and classroom practices which ultimately affects student learning and the value derived by students *about mathematics*. Teaching the pedagogy of mathematical modeling is more an exception than the rule in elementary teacher education programs. Mathematics courses taken by prospective elementary school are typically focused on understanding the four arithmetic operations (+, - , ×, ÷), developed for the natural, whole, rational and real numbers. A course in basic mathematical modeling in addition to the "traditional course" would complement the mathematics learned by prospective teachers in such programs and conceivably also change their conceptions of what mathematics is. The argument against such courses has been that prospective elementary teachers lack the mathematical background necessary to take a course in mathematical modeling. Doerr (this issue) shows that secondary teachers are quite open to learning the pedagogy of mathematical modeling. In her contribution, Doerr presents the results of a case study of practice involving four in-service secondary teachers as they engage their students in the initial development of mathematical models for exponential growth.

The "Math wars" (in the U.S.) and similar differences of curricular opinion elsewhere show that two predominant schools of thought exist about the role of mathematics in the curriculum. On the one hand traditionalists emphasize the necessity of teaching and learning the "basic skills" of mathematics, i.e., rudimentary arithmetic, algorithms, etc in order to further develop the mathematical tools necessary to investigate open-ended problems. That is, why set up students for failure by having them tackle or pose real world problems for which they do not have the necessary mathematical tools? The reformists on the other hand de-emphasize excessive focus on "basic skills" and call for a greater emphasis on the teaching and learning of the concepts underlying procedures and algorithms. The retort to setting up students, without the mathematical tools for failure in problem-solving situations is typically: such situations will call for the development and the teaching of contemporary mathematical tools. The traditionalists typically have numerous rejoinders to this retort such as questioning the use of technological tools by students to run regressions etc without any awareness of the theoretical foundations of such procedures. In our opinion, this debate has proven to be inextricable as each side is entrenched in what they believe is "Good" for the students and yet each side emphasizes elements which constitute a Hegelian dialectic so to speak.

Estimation is natural: A Case for Fermi problems

The first question to ask ourselves is what types of elementary mathematical ideas/thinking (if any) are useful in the day to day world. We contend that (1) Reasoning in ratios, (2) Estimation and (3) mathematical modeling are arguably the three most important types of mathematical thinking for K-8 prospective teachers. The NCTM (2000) writes:

> *"Estimation activities encourage students to make connections among the mathematics concepts they are learning and the skills they are developing…the class discussions and the decisions the teacher makes contribute to students' opportunities to connect their understandings of number, measurement, geometry, and data in order to make estimates." (E-chapter 4, section 4.6).*

The NCTM further states that purposeful activities along with skillful questioning promote the understanding of relationships among mathematical ideas. In fact, this recommendation can be pushed a step further and estimation activities can be used as a way to initiate mathematical modeling, to promote the usefulness and development of mathematical concepts (and procedures) and cultivate critical thinking and critical literacy (D'Ambrosio, 1998; Gutstein, 2006; Michelsen, 2006, Skovsmose, 2000)

In science, particularly in physics Fermi problems are estimation problems used with the pedagogical purpose of clearly identifying starting conditions or assumptions and making educated guesses about various quantities or variables which arise within a problem with the added requirement that the end computation be feasible or computable by hand. The classical problem which states how many piano tuners are there in Chicago is attributed to the physicist Enrico Fermi after whom such problems are named. We argue that Fermi problems which are directly related to the daily environment are more meaningful and offer more pedagogical possibilities than purely intellectual exercises such as computing the number of piano tuners in a city or the number of grains of sand in a glass. For instance Fermi problems involving estimates of fresh water consumption, gasoline consumption, wastage of food, amount of trash produced etc have the potential to lead to a growing awareness of ecological problems related to the environment we live in as well as provoke critical thought when checking the accuracy of computations with different governmental and corporate resources (Sriraman, Adrian & Knott, pre-print). Such activities also present the possibility for interdisciplinary activities with other areas of the elementary curriculum and cultivating critical literacy. There are numerous possibilities for Fermi problems involving estimates of ecological parameters that have arisen due to the overuse and misuse of natural resources. For instance, the first author has incorporated the use of Fermi problems in mathematics courses taken by prospective elementary school teachers in which students make Fermi estimates of fresh water consumption and daily trash production, which inevitably lead to a growing awareness of resource usage and impact. This is brought about by critical thinking about the different estimates, their accuracy in comparison to various information sources and their impact on day-to-day life (Sriraman, Adrian & Knott, pre-print).

Another natural outcome of these Fermi activities is the necessity to simplify assumptions (simplification) before embarking on computations, the wide variety of possible approaches to generate an estimate, which then affords opportunities to discuss variations in student estimates. Analyzing different approaches to the same problem and questioning the accuracy of estimates also creates a natural opportunity to discuss the basic components of a modeling cycle, viz., simplification and improvement loops.

Models and Modeling Perspectives

Many of the recommendations outlined in the papers in the previous issue of ZDM (e.g., Burkhardt, 2006; Galbraith & Stillman, 2006; Kaiser & Schwarz, 2006) on the teaching and learning of mathematical modeling have been pursued in the recent research of the models and modeling group (e.g., (Lesh & Doerr, 2003; Lesh, Kaput & Hamilton, 2006,in press). Models and modeling (M&M) research investigates the nature of understandings and abilities that are needed in order for students to be able to use what they have (presumably) learned in the classroom in "real life" situations beyond school. M&M perspectives evolved out of research on concept development more than research on problem solving; and, rather than being preoccupied with the kind of word problems emphasized in textbooks and standardized tests, the focus is on (simulations of) problem solving "in the wild"(Lesh & English, 2005).

We find it important to emphasize here the foundation-level assumptions of the *models & modeling perspectives (MMP)*. We begin by rejecting the notion that only a few exceptionally brilliant students are capable of developing significant mathematical concepts unless step-by-step guidance is provided by a teacher. MMP research is filled with transcripts showing examples of *model-eliciting activities* in which the models (and other conceptual tools) that students develop for making sense of specific problem solving situations also result in significant developments of powerful, sharable, and re-useable constructs or conceptual systems (for examples, see Iversen & Larson, this issue). In fact, MMP research suggests that, if the goal of instruction is to make significant changes in a student's underlying ways of thinking about important conceptual systems in mathematics, then virtually the only way to induce significant conceptual change is to engage students in situations where they must express their current ways of thinking in forms that can be tested and revised (or rejected) (Lesh & Sriraman, 2005a, 2005b). MMP research suggests that the models that students develop involve a series of Iterative Design Cycles similar to design science professionals. In order to develop artifacts and designs that are sufficiently powerful, sharable, and reusable, it usually is necessary for designers to go through a series of design cycles in which trial products are iteratively tested and revised for specified purposes. Then, the development cycles automatically generate auditable trails of documentation which reveal significant information about the products that evolve.

The need to study complex systems

As we enter the 21st century, many of the most powerful "things" that impact the lives of both professionals and ordinary people are systems - ranging from communication systems, to economic systems, to ecological systems, to the kind of systems that underlie the design of complex artifacts such as continually adapting learning organizations. Some of these systems occur naturally, while others are created by humans. Some are mathematically complex, while others are not. And, some go beyond being complex to also involve self-regulation and continual adaptation. That is, they are both complex and adaptive systems. … But, in any case:

- In an age of increasing globalization, local decisions often impact remote locations where reactions may lead to feedback loops whose second-order effects dwarf local first-order effects. So, people who understand and anticipate such situations are less likely to be victimized by unforeseen events.
- In knowledge economies, the most important resources that many companies or individuals possess often consist of conceptual systems for creating, manipulating, predicting, and (perhaps) controlling a variety of different kinds of complex systems. For example, in the past, a key attribute of a prototypically successful business was to have a large inventory of resources on hand, whereas today, an essential goal of many knowledge industries is to have absolutely no merchandise sitting in warehouses or on shelves. The goal is to make the connection as rapid, efficient, and effective as possible between suppliers and consumers. So, powerful models (and the underlying conceptual systems that they embody) for designing or making sense of complex systems are, in themselves, important "pieces of knowledge" that are valued highly.
- In learning organizations, the conceptual systems that humans develop to make sense of their experiences also are used to mold and shape the world in which these experiences occur. In other words, humans are continually projecting their conceptual systems into the world. So, the world that needs to be understood is not one that has remained unchanged since the beginning of recorded history. It is a world that is continually and rapidly changing; and, for these reasons: (a) the understandings and abilities that are needed to make sense of such situations also are changing, and (b) the kind of knowledge and information that is most powerful often involves models for creating, manipulating, and making sense of complex systems.

All of the preceding situations involve emergent properties of complex systems. That is, they involve properties that only become meaningful in the context of some functioning system-as-a-whole. For example, if we study traffic patterns in a city, then gridlock (or the wave-like patterns of traffic flow) are properties of the system-as-a-whole. They occur because of the way automobiles interact. Similarly:

- Objects such as leverage points, discontinuity points, and attractors acquire significance only due to their functions within systems-as-a-whole.
- Irreversibility/reversibility, feedback loops, resonances are characteristics of systems-as-a-whole; they are not characteristics of isolated agents or objects within the systems.
- Patterns, regularities, invariance properties, and force fields are properties of systems-as-a-whole; they are not properties of isolated agents or objects within the systems.

From Vision to Reality: Implications for educational research

We argued previously that currently in mathematics education, very few research studies are aimed at developing tools that build infrastructure (so that complex problems can be solved in the long run); and, our funding agencies, professional organizations, research journals, and doctoral education have largely ignored their responsibilities to build infrastructure – or to support those who wish to try (Lesh & Sriraman, 2005b). In fact, they largely emphasize simplistic "quick fix" interventions that are precisely the kind practitioners do NOT need.

The USA's Department of Education says: *"Show us what works!!!"* … Yet, when discussing large and complex curriculum innovations, it is misleading to label them "successes" or "failures" - as though everything successful programs did was effective, and everything unsuccessful programs did was not effective. In curriculum development and program design, it is a truism that: *"Small treatments produce small effects; and, large treatments do not get implemented fully."* "*Nothing works unless you make it work!*" … Consequently, when developing and assessing curriculum innovations, it is not enough to demonstrate THAT something works; it also is important to explain WHY and HOW it works, and to focus on interactions among participants and other parts of the systems. This is why the underlying design (which describes intended relationships and interactions among parts of the relevant systems) is one of the most important components of any curriculum innovation that is designed; and, it is why useful designs are those that are easy to modify and adapt to continually changing circumstances. So, in successful curriculum innovations, modularity, modifiability and sharability are among the most important characteristics to design in – and assess

All programs have profiles of strengths and weaknesses; most “work” for achieving some types of results but “don’t work” for others; and, most are effective for some students (or teachers, or situations) but are not effective for others. In other words, most programs "work" some of the time, for some purposes, and in some circumstances; and, none “work” all of the time, for all purposes, in all circumstances. So, what practitioners need to know is when, where, why, how, with whom, and under what circumstances are materials likely to work. For example:

When the principal of a school doesn’t understand or support the objectives of a program, the program seldom succeeds. Therefore, when programs are evaluated, the characteristics and roles of key administrators also should be assessed; and, these assessments should not take place in a neutral fashion. Attempts should be made to optimize understanding and support from administrators (as well as parents, school board members, and other leaders from business and the community); and, during the process of optimization, auditable documentation should be gathered to produce a simple-yet-high-fidelity trace of continuous progress.

The success of a program depends on how much and how well it is implemented. For example, if only half of a program is implemented, or if it is only implemented in a half-hearted way, then 100% success can hardly be expected. Also powerful innovations usually need to be introduced gradually over periods of several years. So, when programs are evaluated, the quality of the implementation also should be assessed; and, again, this assessment should not pretend to be done in a neutral fashion. Optimization and documentation are not incompatible processes. In fact, in business settings, it is considered to be common knowledge that *“You should expect what you inspect!”* … In other words, all assessments tend to be self-fulfilling. That is, they are powerful parts of what educational testing enthusiasts refer to as “treatments”.

Similar observations apply to teacher development. It is naive to make comparisons of teachers using only a single number on a “good-bad” scale (without identifying profiles of strengths and weaknesses, and without giving any attention to the conditions under which these profiles have been achieved, or the purposes for which the evaluation was made). No teacher can be expected to be "good" in "bad" situations (such as when students do not want to learn, or when there is no support from parents and administrators). Not everything "experts" do is effective, and not everything "novices" do is ineffective. No teacher is equally “experienced” across all grade levels (from kindergarten through calculus), with all types of students (from the gifted to those with physical, social, or mental handicaps), and in all types of settings (from those dominated by inner-city minorities to those dominated by the rural poor). Also, characteristics that lead to success in one situation often turn out to be counterproductive in other situations. Furthermore, as soon as a teacher becomes more effective, she changes her classroom in ways that require another round of adaptation. So, truly excellent teachers always need to learn and adapt; and, those who cease to learn and adapt often cease to be effective. … Finally, even though gains in student achievement should be one factor to consider when documenting the accomplishments of teachers (or programs), it is foolish to assume that great teachers always produce larger student learning gains than their less great colleagues. What would happen if a great teacher choose to deal with only difficult students or difficult circumstances? What would happen if a great teacher choose to never deal with difficult students or difficult circumstances?

The need for a brand new research agenda

Before we initiate systemic changes in the curriculum at the elementary and middle school level, we need to ask the mathematics education research community whether we already know what it means for a younger student to understand models and modeling? We also need to take into account the claim (which we hear from many) that the nature of problem solving (and “mathematical thinking”) has changed dramatically in the past 20 years (see Lester & Kehle, 2003; Lesh, Hamilton & Kaput, 2006, in press). We also think there is also a real need for research about: (i) the nature of new “real life” situations where some type of mathematical thinking in needed for success, (ii) what it means to understand relevant knowledge and abilities, (iii) how these ideas and abilities develop, and (iv)how development can be documented and assessed. There is a also greater need to focus on a call for research which takes into consideration what we already know about concept development in children.

The following trends are important to consider.

- <u>New ways of thinking about old situations</u>: Emerging new technologies are creating new multi-media, interactive, and dynamic ways to think about old situations. For example, we only need to look at daily newspapers such as *USA Today* to see ample evidence that modern mathematics is becoming multi-media mathematics. In sections of the newspaper ranging from editorials, to sports, to business, many of the articles are coming to resemble multi-media computer displays which are filled with graphs, tables, diagrams and hyperlinks to other resources. Consequently, even in situations that involve nothing more sophisticated than buying and selling groceries or automobiles, situations are created and described using computer-based, multi-media, computational models – which are creating completely new ways of thinking about problems that involve optimization, stabilization, and other goals that used to require algebra, calculus, or other topics in advanced mathematics. But today, such issues often are handled using computational, interactive, multi-media models which are based on extensions of basic ideas in measurement and arithmetic - rather than models which are based on single algebraic functions. Therefore, even though such models may be associated with topics such as discrete mathematics, systems theory, game theory, complexity theory, or mathematical modeling, they often are well within the limits of elementary mathematics.

- New types of situations to understand and explain: New kinds of situations also are emerging that need to be understood and explained. Again, this is because a distinguishing characteristics of a technology-based *information age* is that the same tools that provide new ways to think about existing worlds of experience also enable completely new worlds of experience to be designed. Consequently, complex systems - ranging from communication systems, to economic systems, to transportation systems, to ecological systems - are coming to be among the most powerful "things" that impact the lives of increasing numbers of people. And, such systems are especially significant in countries that are developing knowledge economies - where increasing globalization typically leads to feedback loops and second-order effects which often overpower local actions – especially when interactions involve multiple agents with partly conflicting goals.

- New types of problem solvers and problem solving: Modern jobs increasingly involve "learning organizations" which need to adapt rapidly in response to continually changing circumstances; and, the most important assets of these "learning organizations" often consist of knowledge and networks rather than large warehouses filled with physical goods and resources. Consequently, "problem solvers" often are not isolated individuals whose only tools consist of pencil and paper. Instead, they often are teams of diverse specialists representing a variety of different practical and theoretical perspectives, and having access to a wide range of rapidly evolving technical tools. Similarly, the knowledge and abilities that productive groups and individuals possess often do not reside within the minds of isolated individuals. Instead, knowledge and abilities tend to be distributed – and often are off-loaded to supporting networks of tools and colleagues. This is why job interviewers in future-oriented professions consistently emphasize the fact that the people who are in highest demand are not necessarily those who are skillful at scoring well on standardized tests. Instead, they tend to be people who are able to: (a) make sense of complex systems, (b) work within teams of diverse specialists, (c) adapt rapidly to a variety of rapidly evolving conceptual tools, (d) work on multi-staged projects that require planning and collaboration among many levels and types of participants, and (e) develop sharable and re-useable conceptual tools that usually need to draw on a variety of disciplines – and textbook topic areas (Lesh, Hamilton & Kaput, 2006, in press). So, the mathematical abilities that set these people apart from their peers often have more to do with expression (e.g., interpretation, description, explanation, communication, argumentation, and construction) more than computation or deduction; and, they have as much to do with imposing structure on experience as they do with deriving or extracting meaning from information which is presumed to already be given in a mathematically meaningful form.

- New kinds of products and design processes: Today, when some kind of mathematical thinking is needed to solve real problems, the products that need to be produced often involve much more than short answers to pre-mathematized questions. For example, they often involve developing conceptual tools (or other types of complex artifacts) which are designed for some specific decision maker and for some specific decision-making purpose – but which seldom are worthwhile to develop unless they go beyond being powerful for a specific purpose to being sharable with others and re-useable beyond the immediate situations in which they were first needed. Consequently, solution processes often involve sequences of iterative development→testing→ revising cycles in which a variety of different ways of thinking about givens, goals, and possible solution steps are iteratively expressed, tested, and revised (e.g., integrated, differentiated, or reorganized) or rejected. That is, the development cycles often involve a great deal more than simply progressing from pre-mathematized givens to goals when the path is not obvious. Instead, the heart of the problem often consists of conceptualizing givens and goals in productive ways.

Another reason for re-examining assumptions about what it means to understand and use mathematics is because, in research on concept development, mathematics educators have now developed relatively sophisticated mini-theories describing the development of whole number concepts, rational numbers concepts, early algebra concepts, and a wide range concepts related to other topic areas. Yet, these mini-theories often yield quite different descriptions of development – in spite of the fact that, in the minds of students, these concepts do not develop in isolation from one another, nor independently. For example, at the same time that whole number concepts are developing, early understandings also are beginning to develop about measurement, geometry, fractions, and even foundation-level ideas related to algebra and calculus. Furthermore, in recent years, researchers have produced a great deal of evidence showing that learning is far more piecemeal, situated (Lave & Wenger, 1991), socially mediated (Wenger, 1998), and multi-dimensional (Lesh & Yoon, 2004) than most currently popular theories have us to believe. For example, in ethnographic comparisons of experts and novices in a variety of fields, results have consistently shown that expert knowledge is organized around experience at least as much as it is organized around the kinds of abstractions emphasized in schools. Furthermore, in an age when problem solvers have nearly continuous access to things like computer-based search engines, spell checkers, data processors, and multi-media tools for communication and collaboration, it is obsolete to think of intellectual capabilities (e.g., information storage, retrieval, or processing) as if they resided exclusively within the minds of isolated individuals. The whole point of using conceptual tools is that they enable people to off-load information and functions that once needed to be carried on in the mind. But, the nature of these tools is that, when they are introduced into a situation, the situations themselves tend to be transformed in a variety of fundamental ways (Lesh, 2006, in press).

Research initiatives in classrooms with younger students in experiments involving the simulations of complex systems have tremendous potential for changing present conceptions of mathematical modeling. Our whole point is that if we wish to expose students to simulations of complex systems and research their understanding of these systems, we need to develop these simulations based on what the existing research on concept development has to say (and not simply what people in industry are telling us). In other words, one just cannot transfer a modeling scenario from industry into the classroom without giving any consideration into how children learn in the first place. If we do this, we are simply repeating the mistakes from the past (e.g., New math). The inference for modeling in schools is when children engage in these simulations, they will naturally begin to engage in elements of a modeling cycle.

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