Mathematical modelling as bridge between school and university

Gabriele Kaiser, University of Hamburg, Germany
Björn Schwarz, University of Hamburg, Germany

Abstract: *The paper reports on university seminars of mathematical modelling at school, that were held together by the departments of mathematics and mathematics education and some schools in Hamburg. Prospective teachers together with students in upper secondary level carried out modelling examples either in ordinary lessons or special afternoon groups. They tackled authentic problems proposed by applied mathematicians working in industry. In this paper we consider three of these examples in more detail and describe students' attempts at solving the problem.*
After a description of the theoretical framework of modelling in schools and the framework and the structure of the course diverse modelling attempts by students to solve two modelling examples are presented. Finally an evaluation of the series of seminars is presented.

ZDM-Classification: C70, M10

In the following sections, we describe modelling examples and experiences from a series of university seminars on modelling in schools conducted at the University of Hamburg (in co-operation with Claus Peter Ortlieb and Jens Struckmeier from the Department of Mathematics).

1 Theoretical framework for modelling in mathematics education

The relevance of promoting applications and mathematical modelling in schools is widely accepted. For example the PISA study emphasises as goal of mathematics education to develop within students the capacity to use mathematics in their present and future lives. It means that students should understand the relevance of mathematics in everyday life, in our environment and for sciences.
This perception of the objectives of mathematics teaching has impact on the structuring of mathematics lessons. It is insufficient to simply impart competencies for applying mathematics only within the framework of school curriculum. Instead more mathematics teaching should deal with examples from which

- students understand the relevance of mathematics in everyday life, in our environment and for the sciences,
- students acquire competencies that enable them to solve real mathematics problems including problems in everyday life, in our environment and in the sciences.

This demand for new ways of structuring mathematics teaching meets the goals for more reality-oriented mathematics education as postulated in many didactical positions since the middle or the end of the 20th century. It has been agreed that mathematics teaching should not be reduced to just reality based examples but that these should play a central role in education (for an overview see Kaiser-Meßmer 1986, Blum 1996, Maaß 2004, Kaiser & Maaß 2006). As consequence it became clear that besides the application of standard mathematical procedures (such as applying well-known algorithms) in real world context and real world contexts serving as illustrations of mathematical concepts (e.g. usage of debts for the introduction of negative numbers), modelling problems as reality based contextual examples are increasingly important. In addition, it has been discussed that students should acquire competencies which enable them to solve real mathematical problems from different extra-mathematical areas. Within the actual discussion it is stressed that it is not sufficient at all to deal with modelling examples in lessons but that the stimulation of modelling competencies through self-initiative is of central importance (see Maaß 2004).

Since several years, within the mathematics didactical discussion, the following ideal-typical description of modelling processes has become widely accepted. A real world situation is the process' starting point. Then the situation is idealised, i.e. simplified or structured in order to get a real world model. Then this real world model is mathematized, i.e. translated into mathematics so that it leads to a mathematical model of the original situation. Mathematical considerations during the mathematical model produce mathematical results, which must be reinterpreted into the real situation. The adequacy of the results must be checked, i.e. validated. In the case of an unsatisfactory problem solution, which happens quite frequently in practice, this process must be iterated.

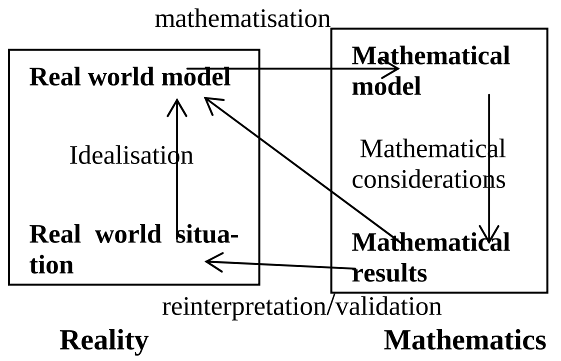

Fig. 1: Didactical modelling process (from Kaiser 1995, p. 68 and Blum 1996, p. 18)

In applied mathematics, typically one does not distinguish a real-world model from a mathematical model, but the transition from a real life situation into a mathematical problem is regarded as the core of modelling. Here again, the starting point is a real-world problem or a phenomenon to be explained. Then, a mathematical problem is developed from it, in a sense as an effigy of reality. This problem has to be solved by mathematical methods, and the mathematical solution is then interpreted concerning its reality-related meaning and its relevance for the real-world problem is verified (Ortlieb 2004, p. 23).

The most striking difference between both descriptions is their focal point for which the description coming from applied mathematics, emphasises especially the normativity and subjectivity of the transition from a real phenomenon towards a mathematical problem and therefore takes this phase as modelling.

The competencies or abilities needed for this kind of modelling process are still the topic of current controversial debate. The current discussion differentiates between modelling competencies and modelling abilities. Modelling competencies include, in contrast to modelling abilities, not only the ability but also the willingness to work out problems with mathematical aspects taken from reality through mathematical modelling. Efforts to develop detailed and concrete descriptions are under development. For instance, Maaß (2004, p. 35f), in her elaborate empirical study, gives a list of modelling competencies. In lieu of this study and based on own research results, we consider the following competencies as necessary:

- Competence to solve at least partly a real world based problem containing mathematics through a mathematical description (mathematical model) developed individually by one's own;
- Competence to reflect on the modelling process by activating meta-knowledge about modelling processes;
- Insight into the connections between mathematics and reality;
- Insight into the perception of mathematics as process and not merely as a product;
- Insight into the subjectivity of mathematical modelling, i.e. the dependence of modelling processes on the aims and the available mathematical tools and students competences;
- Social competences such as the ability to work in a group as well as to communicate about and via mathematics.

This list is far from being complete since more extensive empirical studies are needed to receive well-founded knowledge about modelling competencies (for a more detailed description of modelling competencies and their evaluation see Kaiser, 2006).

In practical modelling work the following guidelines related to these competencies have been considered as helpful for carrying out modelling examples:

- Formulate the real question or problem at the beginning as precisely as possible, clarify, which issues are relevant and which are irrelevant;
- Clarify the information needed to proceed: Is the information complete? Might the given information be useless or even misleading?
- After these two steps it makes sense to reflect upon the mathematical question to be treated and to formulate it precisely;
- Simplify the problem radically at the beginning, enlarge the model gradually if necessary;
- Check the mathematical solution found, whether it solves the real world problem; if not modify the model;

- Examine the model, whether it fulfils the criteria of admissibility, correctness and suitability, discuss the limitations of the model and assess it.

In order to promote a modelling-based understanding of mathematics and to develop competencies for carrying out modelling processes at school, it seems to be necessary to teach such competencies to prospective teachers during the course of their studies. Future teachers have to become familiar with modelling examples, because if not, later the barriers for the integration of such examples into lessons will be too high (see for example Kaiser & Maaß 2006). In the following we will report about a university course with future teachers through which these students could acquire competencies for implementing modelling processes in their prospective teaching and through which their students could acquire competencies for carrying out modelling processes.

2 Framework and structure of the seminar

The project "Mathematical Modelling in School" was established in 2000 within the framework of the initiative "Mathematics at the Interface between School and University" financed by the Volkswagen Foundation and conducted by the Department of Mathematics in co-operation with the Didactics of Mathematics at the Department of Education at the University of Hamburg. This university course project with future teachers for upper secondary level teaching carried out every year with only one exception since year 2000, aims to establish a conjunction between university and school as well as between mathematics and didactics of mathematics. Student groups from upper secondary level (age 16-18) supervised by the prospective teachers are the focus of the course. Each group works independently on one modelling example within the regular lessons or in separate after school working groups.

The main objective of the course is to change the academic curriculum of the Department of Mathematics and of the Didactics of Mathematics, so that in future mathematical modelling and associated teaching experiences will play a central role. Through this project, the prospective teachers will be enabled to implement modelling processes in mathematics teaching in their future professional work.

It was hoped that the participating students would acquire competencies to enable them to carry out modelling examples independently, i.e. the ability to extract mathematical questions from the given problem fields and to develop autonomously the solutions of real world problems. It is not the purpose of this project to provide a comprehensive overview about relevant fields of application of mathematics. Furthermore, it is hoped that students will be enabled to work purposefully on their own in open problem situations and will experience the feelings of uncertainty and insecurity, which are characteristics of real applications of mathematics in everyday life and sciences. An overarching goal is that students' experiences with mathematics and their mathematical world views or mathematical beliefs are broadened.

Each course extends over a period of two semesters with the following structure (in each cycle various modifications occurred; for details see Kaiser et al. 2004). After a short introduction into questions of teaching modelling, in a start-up lecture an authentic real life problem is presented by an applied mathematician. That is the problem which will be dealt with during more or less three months within the framework of school lessons. First results will be presented by students at the end of the winter semester. During the summer semester a further real world modelling problem is worked on. Since modelling processes are carried out twice, both, the students and the attending future teachers, can review their experiences from the first run. Again the students present their results of the second run at the end of the summer semester. For both presentations all participants of the project come together and the results are presented by the students whether in short lectures or on stalls with posters. Figure 2 shows an overview of the presentation of the stalls by the students.

Fig. 2: Overview of a presentation

Simultaneously a university course is carried out where the students' solution attempts, the problems and experiences of the future teachers are discussed.

Among others, until now the following modelling problems were treated:

- Mathematical methods within risk management
- Mathematics in private health insurance
- Mathematical and methodical problems of fishing sciences
- Optimal position of rescue helicopters in South Tyrol
- Radio-therapy planning for cancer patients
- Identification of fingerprints
- Pricing for internet booking of flights
- Price calculation of an internet café
- Traffic flow during the Soccer World Championship in 2006 in Hamburg
- Construction of an optimised time table of school

Main criteria for the selection of the modelling problems are their accessibility for students, i.e. the real world context of the examples should be understandable for the students without too much additional work and possible modelling approaches should be within the mathematical horizon of the students. In addition the availability of experts willing to present real modelling examples to us, which they have once tackled or met in their work determined the selection of the problems. Other criteria such as their representativeness for different types of models and their applications were not considered in the selection process.

Supplementary activities include excursions to companies for which mathematical models are of importance in order to demonstrate a broader variety of modelling examples. To give students an adequate imagination of the extensive applications of mathematical models, a series of lectures conducted by applied mathematicians is offered in which mathematical models from various fields of profession are presented at a level matching the students' knowledge.

In the following paragraph three modelling examples will be described in detail in order to show the wide variety of solutions developed by the students.

3 Description of three modelling examples

In the following, three modelling problems will be presented more thoroughly. They have been practiced this way within the framework of the first and the fourth run of the modelling project at the Gymnasium Tonndorf, Gymnasium Grootmoor, Albert-Schweitzer-Gymnasium in Hamburg and Gymnasium Harksheide in Schleswig-Holstein. For this reason, this part of the contribution focuses on those attempts of solutions developed by the students that really came up during the project.

3.1. Pricing of Air Berlin

The first problem, how the low price airline Air Berlin makes its pricing was presented by one of the participating applied mathematician to the students and the prospective teachers as well. Air Berlin sells its flights predominantly by an online booking system via internet, and the prices for the various destinations are not fixed. For each flight the prices are indicated separately and change very often which arises the question how Air Berlin determines its prices. The participating students were asked either to develop an adequate description based on the prices announced on the internet or to develop an own price system.

The problem proved to be challenging for the participating student groups because on the one hand the price algorithms of Air Berlin were not known by them and, on the other hand, various attempts are possible. Thus, each group developed a different attempt. However, in the following part, we will restrict our description to the attempt of the Gymnasium Harksheide in Norderstedt because this attempt has to be regarded as the most mature one. The course is an advanced mathematics course, in year 12 (age 17-18) with 11 participating students.

At the beginning, the students collected data from the internet and observed the changes of prices. Doing that, they noticed that the changes in price obviously happened more or less arbitrarily. Some of the observed flight prices did not change over a longer period of time before then suddenly the prices doubled. Other prices did not change at all until the end of the observation period, while for some other flights additional charges, such as the kerosene surcharge, augmented. The students identified the following factors that influence the development of prices:

- date/departure time of the flight
- date of booking (referring to the flight-date)
- number of free seats
- place of departure and destination
- time of departure, meaning the day and the time of the day
- number of connected bookings
- flight number
- market situation

After that, the group divided itself into two sub-groups: The first group worked on the amount of prices in relation to the number of remaining free seats for certain flights. The second group dealt with the amount of prices in correlation to the time of booking. For flights inside Germany a model about the rise in prices in relation to time left until the departure could be developed, based on a step-wise augmentation. For international flights no such model could be reconstructed. Likewise, a correlation of prices to the number of remaining free seats also could not be reconstructed from the data. Because of these big problems they faced with the development of a model to determine the flight prices, the students of this group decided to develop a model by their own.

For this, the group split again into two sub-groups and they followed two different attempts: The first group decided to describe the development of flight prices by means of an exponential function, due to experimental considerations with real values. Thus, as model for the development of prices this group developed the function $f(x) = e^{cx} + b$ with b as initial price and c as description of price behaviour (meaning a steep or slow rise). For the determination of the parameter c, the students, assisted by the prospective teachers, referred to the mean value theorem of integral calculus, and for determining the price behaviour, they developed the following model: $F(t) = \int_0^t (e^{cx} + b - 1)dx = a\ t$, in which a represents the average price of a flight. The factor -1 was apparently introduced by the students in order to adapt their description to the data.

The second sub-group agreed on a description of price trends starting 30 days before departure by means of the exponential function $f(t) = basic\ price\ a^{30-t}$, (where t is the number of days until flight time), with the assumption that the basic prices vary between 25€ and 125€ and that no flight shall cost more than 300€. As growth factor *a*, based on the collected data, the group experimented with factor 1.024 and 1.03. Then, this attempt was modified further and generalised by taking the amount of remaining free seats into consideration by inserting *m*(10-s) as additive factor with *s* as number of remaining free seats and *m* as not yet fixed lump sum. The basic price was fixed depending on flight distance for which they referred partly to the real flight distances of Air Berlin and further assumptions with $\frac{z}{16} = basic\ \ price$, where z is the distance of the flight in km.

Both attempts were carried out simultaneously with two modifications:

- Consideration of dependence on season, modelled by means of a cosine function: $0.25\ \cos\left(\frac{2\pi c}{100}\right) + 1.25$, where c is a factor depending on the time of flight which was determined by analysing older data
- Rounding up to the following decade

The different meaning of c as factor depending on the time of flight in contrast to the meaning of c as description of price behaviour as it was used at the beginning was not realised by the students and did not cause major difficulties in the further development of the project.

The first sub-group produced as model for the development of prices:

$$f(c,t) = (0.25\ \cos\left(\frac{2\pi c}{100}\right) + 1.25)\int_0^t \left(e^{cx} + b - 1\right)dx = a\ t$$

The second sub-group produced as model for the development of prices:

$$f(c,t,s) = \left(0.25\ \cos\left(\frac{2\pi c}{100}\right) + 1.25\right)\left(\frac{z}{16}\ a^{30-t} + m\ (10 - s)\right)$$

The students did not use the notation with several variables and it is unclear whether they realised the problems resulting of their model containing more than one variable or not.

Then, both models were compared with each other concerning the price behaviour from which the following became obvious: model 1 regards the time period from the beginning when the prices were offered first until the time of departure with an exponential rise in prices, while model 2 regards a constant price with a fir-tree-like increase over the last 30 days. Because in model 1 factor *c* determines the steepness of the increase of the prices, both attempts led to quite similar results which are well adaptable to reality.

The students who had worked on this problem presented their results not only in a short lecture in front of the other participating classes but as well on a stall with a poster and a computer presentation.

Both, lecture and stall were of remarkable high quality; figure 3 and 4 show their stall and poster.

Fig. 3: Stall of the students of the Gymnasium Harksheide

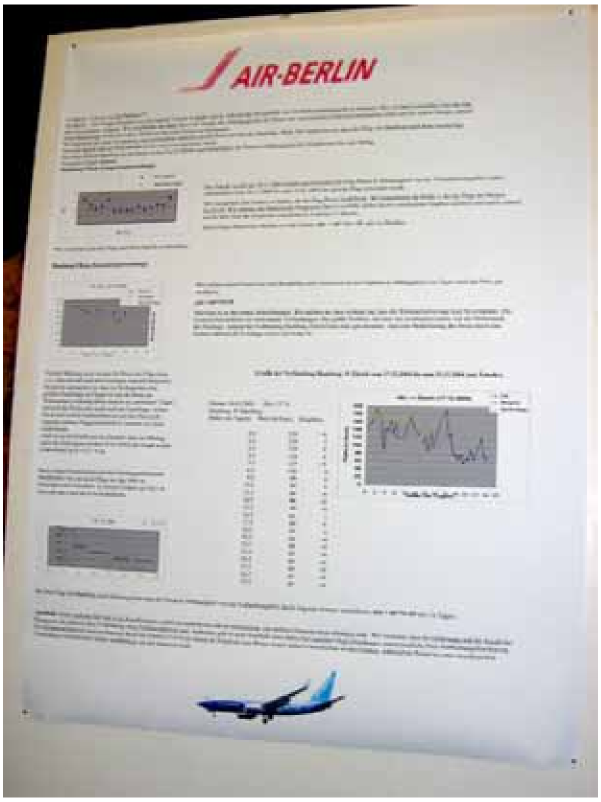

Fig. 4: Poster produced by the students about the Air-Berlin-problem

3.2. Pricing of an internet café

A problem similar to the one described above was set to the students in the second part of fourth run of the project. The question then dealt with the price calculation of an internet café. Although the problem had some parallels to the Air-Berlin-problem different aspects have to be taken into consideration and so the solutions showed clear differences to the ones presented above. We will restrict ourselves to short remarks on one approach, which shows remarkable differences to the approach described above.

The question of the problem ‘Pricing of an internet café’ was how to calculate - reasonably and with the aim of profit - the prices of an internet café. The solution developed by a group of students at year 11 from the Albert-Schweitzer-Gymnasium is characterised by far-going and to some degree strongly simplifying assumptions. So on the one hand the model’s proximity to reality seems to be doubtful, but on the other hand this simplification enables students to develop a complete and concrete solution. The group distinguished three factors that are influencing the price: the costs to be borne by the owner, competition and demand. The second and third factor the students recognised only after being asked further by the future teachers. After some research in the internet, the costs were estimated by a total amount. Furthermore, the consideration of competition was renounced because it would have made the problem too difficult. Starting from some basic economic knowledge, which - as a crucial deviation from the normal proceeding of the project - the future teachers introduced during a specific learning unit, the relation between price and demand has been modelled based on two modelling attempts:

- At a price of 0 € per hour internet access 10,000 hours are demanded.
- With an increase of price of 0.10 € per hour of internet access, demand decreases about 5 %.

Based on these assumptions the so-called price-response-function was developed:

$N(x) = 10,000 \times 0.95^{10x}$,

with x indicating the price for one hour internet access in €/h and N(x) the number of demanded hours at a price of x €/h

The sales-volume was calculated based on the number of demanded hours which was then multiplied with the price for one hour internet access. With S(x) as sales-volume at a price x one receives the so-called sales-volume-function:

$S(x) = x \times N(x) \Leftrightarrow S(x) = x \times 10,000 \times 0.95^{10x}$

Based on the above described context, the optimal price was to be determined as the point at which the sales volume function reaches its maximum. This was carried out almost alone by the students, only the specific derivation of an exponential function had to be found out with the assistance of the future teachers. The optimal price in the described meaning is received after some calculation and becomes $x = \frac{-1}{10 \ \ln(0.95)}$, approximately $x \approx 1.95$ €.

In the continuation of the project the students modified their assumptions and tested which influence the modifications have upon the result. Thus, a modified number of demanded hours at a price of 0 € does not influence the result, while in contrary to that a modified number of demand with an increase of price per hour clearly changes the optimal price.

These two very different approaches show the variety of possible approaches and their pitfalls:

- very general models based on complex assumptions might be too ambitious for the students' mathematical knowledge and might lead in addition to very general descriptions, which describe the modelled situation, but do not provide the students with concrete answers;
- very specific assumptions might be more adequate for the students' mathematical knowledge and can lead them to tangible results, which might be far away from reality.

There seems to be no general solution for this problem, especially because realistic models of not too complicated real world situations lead quite often and easily to functions with several variables, which are usually not treated at upper secondary level in German mathematics teaching and are often above the horizons of average students.

3.3. Risk management

Starting point of a third problem was the application of mathematical methods in risk management. This problem was realised at the Gymnasium Tonndorf and Gymnasium Grootmoor. The question was how to determine the risks of professional activities in the financial sector and how one might hedge against it. Examples are the risks of so-called international transfer risks, which mean the problem of possible losses caused by the different exchange rates of currencies. Another example is the risks with stock trading because stock values naturally are never stable.

At the beginning of the project, the participating students were told some basic facts about risk hedging in currency and stock trade by using options by a professional banker from an important German bank. Options are used to sell ('put-option') or to buy ('call-option') shares at a specific moment ('European option') or within a particular time period ('American option'). The presentation dealt in addition with a modelling example which mathematically was mainly based on geometrical binomial processes. However, when the project started, the students decided not to comprehend or to develop that model further but to deal with questions chosen by themselves, especially from the domain of stock price modelling. This means a significant deviation from the presentation's topic, because options offer different possible actions than direct stock trading.

At the Gymnasium Tonndorf the question was put into concrete terms by the problem of predicting the development of stock quotations. There, a working group planned to develop a deterministic forecast formula (an ambitious aim which if it was realisable would mean the breakdown of today's market structures). For this, at first, the students calculated the arithmetic average value of the stock in question from the quotations of the last 38 days. Then, from this value and the lowest quotation of the last three months the arithmetic average value was calculated again which then was used as forecast value of the stock value after n days. Written as formula this means:

$$P_n = \frac{A_{38} + L_{3M}}{2}$$

P_n: forecast value after n days; A_{38}: arithmetic average value of stock prices of the last 38 days and L_{3M}: lowest quotation of the last three months.

It was a problem that on the right side of the equation the value of n was not used so that it remained unclear why this formula should predict a stock's quotation after n days. Furthermore, it was not explained at all why the period of exactly 38 days has been chosen for the average value. One might assume that the group came onto the number of 38 days because the gliding 38 days average generally is used in financial practice. Later, through a modification, an alternative formula was offered in which the highest quotation was received instead of the lowest quotation value of three months. As criterion for the formula to be applied served the calculation whether the sum of all market values of the last three months that are higher than the 38-average value is higher as the sum of market values below the average value. As further modification, this criterion was modified in such a way that not the absolute market value but the sum of the absolute values of the deviations from the mean value has to be summed up. This strategy indicates that the conception of variance has been considered. At the end of this procedure, the group tested its formula with the real development of share prices.

A second group at the Gymnasium Tonndorf started out from the assumption that share prices develop by chance in contrast to the approach described before. For this, they defined an average daily price fluctuation S by constructing a mean value from the price fluctuations of the last three months; for this, probably, the absolute values of price fluctuations were looked at in order to avoid mutual settlements of price fluctuations. Then, by using a start share price they simulated the development of a share price over n days that way, that they added or subtracted the value S to or from the last share price in order to get the next share price n times. The decision whether S should be added or subtracted was made by using a randomizer. Thus, the corresponding formula is:

$$P_n = P_0 \pm \underbrace{S \pm \ldots \pm S}_{n\ times}$$

with P_n: forecast value after n days and P_0: current stock price.

Also this group then modified its approach. For this they started out from the idea that the general development of the share price over n days before the simulation starts should be considered too. For this they compared the 38 days average value of the day the simulation started with the 38 days average value n days before the simulation started. In case the average value n days before was higher, one assumes a generally falling tendency and values the losses higher; for each loss one subtracts 1.15 S from the last share price, and for each increase of share price one only adds 0.85 S. In case the average value n days before is lower than when the simulation starts, one turns the weighting factors over so that the increases of stock prices are valued higher. If the average values differ only less than 50 Cent from each other, no weighting factors are used. The values for the weighting factors are received through trying and plausibility checks with real shares. A problem in doing so was the constant value of 50 Cent as benchmark for comparing the average values. Of course, 50 Cent with an average value of 100 € is less important than with a value of only 2 €. Therefore, the presented concept of weighting was changed that way that it was compared whether the average value at the beginning of the simulation differs more than 2% from the average value at n days before the simulation starts.

At the Gymnasium Grootmoor not the development of share price but the risks arising from it became the focus of interest. At the beginning, the participants constructed a mind map that showed the possible risks, in which they displayed a high number of influential factors, such as market risks or psychological risks as rumours.

In the following, one group tried to find a more precise definition of the ambiguously understandable concept of risk and to quantify it. The result from the first trial was the following definition of „risk“: *“Risk is the absolute value of the deflection of the share price development line to be expected at most”*. In this definition not the absolute but the relative alteration was meant. The problem with this definition was that the maximum deflection can be given directly: Upwards the share prices are unrestricted while downwards they can fall to complete worthlessness. For this reason, the definition was revised and the concept of mean average value was integrated into the definition: „*The risk is the average of the absolute values of the of share price development curve within one day within a defined time period. We want to call this value the relative width of variation, shortly R-value*.” Following, the concept was tried with real data for which the students experienced problems with obtaining those data. At this point the future teachers intruded once and they gave the hint that the mean average value might produce problems because extraordinary changes of share prices as it was the case with the Black Friday in 1929 might interfere the mean average value. Nevertheless, the students determined the R-value with examples and made the hypothesis that the more data are used the better the results produced by the R-value will be. From the following modelling they got further modifications; one of them was to weight the data variously in order, to weight newer data more than older ones because it was assumed that newer data are more relevant for actual risk forecasts. For this the weighted average was used:

$$R_w = \frac{\sum_{i=1}^{n} g(i) \cdot a_i}{\sum_{i=1}^{n} g(i)}$$

R_w: weighted R-value, a_i: absolute value of the price change of day i; g(i): weighting factor of the absolute value of price change of day i; (i means the index of the share price in question, the data get a serial number, for which the oldest share price gets the value i=1).

The weighting factors are subjected to three conditions:

- They should be describable by a function which depends on the data’s time distance to the actual time.
- For short periods to be predicted newer data should be weighted more, for longer periods the weighting factors should approach 1, which means the mean average value should correspond to the arithmetical average.
- The weighting factors should decrease with the number of available data which means that with only a few available data, those data should be weighted quite equally.

As a result, the students developed the following equation to calculate the weighting factor:

$$g(i) = 2^{\frac{i \cdot n}{l^2}}$$

n: number of data; l: length of the period to be predicted.

One can see that the formula meets the above described conditions; the square in the denominator of the exponent arose from the assumption that the length of the forecast time period is more important than the number of data.

Additionally, the selection of the function to determine the weighting factor demonstrates exemplarily the process-like character of modelling in schools: base 2 instead of the originally intended e-function was used because the students did not find the e-function in the excel software they applied.

For comparison purposes, the group calculated the unweighted and the weighted R-values for two companies, the Deutsche Bank und the Deutsche Telekom and received the following values:

Deutsche Bank: weighted R-value: 1.66%, unweighted R-value: 1.54%

Telekom: weighted R- value: 2.53%, unweighted R- value: 2.49%

Due to the fact that the weighted and unweighted factors were located relatively near to each other the students assumed that the share prices changed only a little during the time period in question because the equal and weak weighting of the older data yielded quite similar results. Furthermore, the group concluded that due to the higher R-value of the Telekom it is a greater risk to invest in this share than in one of the Deutsche Bank.

Another group at the Gymnasium Grootmoor planned to develop a model which should include the coincidental development of share prices. The problem was that the group did not have much knowledge about stochastics. For this reason, the participants decided to interrupt the modelling process - which was contrary to the usual procedure of this project - in order to have conventional lessons about some aspects of stochastics. Especially the binomial distribution was introduced by the future teachers. This must be kept in mind because the model developed afterwards reflects only fragmentarily the group's state of knowledge. The realised model bases on the assumption that the share price increases or decreases only by a stable percentage on each day; in other words, it does a so-called 'random walk'. The assumption of those constantly changing values is of course a problem because in reality share prices increase or decrease by inconstant percentages. This was also recognised by the group. The probability distribution for the number of days the prices increase or decrease is received by the binominal distribution. In order to determine the probability of increase or decrease of share prices, the relative frequencies of share price increase or decrease within a fixed time period were calculated and served as approximate estimated value for the underlying probabilities. Because of the given probability distribution and the known elementary probabilities expected values for the number of days on which the share prices rises could be computed. This description enables to develop forecasts of the stock price development. Besides that, probability statements were possible, for instance in which time interval the price will be after n days at which probability.

4 Experiences with the modelling courses

4.1. General observations

If reflecting retrospectively the students' way of working with various problems, it can be stated that due to the problems' openness, a feeling of excessive demand was dominating the atmosphere at the beginning. But, later on, at the middle of the modelling work, creative processes arose just out of this openness and led to various very different ways of dealing with the problems. Then, most participants attended the modelling process with great enthusiasm.

Especially at the beginning of a modelling process, it is necessary for nearly all kinds of problems to collect the needed information about the concerning extra-mathematical area, for instance about shareholding. Only then it is possible and makes sense to deal with the problem in the meaning of mathematical modelling as described in paragraph 1. Only when being equipped with the needed knowledge, a reality based model can be developed and mathematical solutions can be reinterpreted and validated. To deal with non-mathematical aspects in mathematics lessons was a completely new experience for the students. Although getting knowledge about the specific non-mathematical area is a basic condition for executing the modelling circle and therefore forms part of applied mathematics, at the beginning students regarded this part of the project as an area totally independent of mathematics. However, by the following work with the problem, the students understood more and more that to work in areas outside mathematics in order to get the needed knowledge means an essential component of mathematical modelling.

Furthermore, it can be stated that for instance at the time, when the students worked on the stock trade-problem within the framework of the project (decade 2000-2001), this was a highly up-to-date topic. That time, the stock exchange was booming so that this topic existed and was quite popular in the students' everyday life – a fact which has decisively pushed the students' motivation. Thus, students got the feeling to work on a mathematical problem which really has a practical purpose. However, quite soon it became quite clear that the students' high-pitched aims and expectations to develop a model to predict share prices was less realistic because such a model would lead to a collapse of all stock markets.

Altogether, from our observations in the project, we can state that the motivation of the students often increased when the problems were part of their own life experiences or if a practical purpose could be identified directly. Also if the students were asked to develop problems autonomously, motivation increased. A fact which became clear from various ways of posing a problem: Students from the already mentioned Gymnasium Harksheide suggested as modelling problems the optimisation of the flow of traffic during the Soccer World Championship in 2006 and the construction of an optimised time table of a school. The motivation for the second suggestion originated in the students' own experience with many interruptions of their daily school life by free periods. Finally, at the end of the course, the students realised self-critically that their expectations had been far too high. Despite this setback, most of the students felt that they have got central insights into the modelling approach. Retrospectively, it should be reconsidered whether it might have been better to get the students off from their suggestion, because the optimisation of the time table is a highly complex problem with a non-linear optimisation for which up to the moment no satisfying solution algorithms exists. The same happened with other given problems for which no precast solutions exist nor may be expected to be found within the project, a fact which caused less difficulties. In the case of the time table of the schools, the expectations of the students were very high because the problem concerned them directly.

Besides that, it could be observed that the students themselves claimed a very high quality to their own solutions. Thus, many students separated clearly between their insights they got into mathematical modelling and the achieved result. While the first aspect, the insight they got into modelling they regarded to a great part very satisfactory, the second aspect, the solutions they found, even very good solutions – considering the conditions under which the project takes place - were evaluated very critically by themselves. A high level of expectation for finding a complete solution of the problem could be observed, but this often could not be hoped within the framework of the project. Against this background, particularly the presentations at the end of each modelling phase, must be underlined. On the one side, for many participants it was a very special experience to present their results in front of a big audience, and, on the other side, the students got a positive feed-back, often from the experts of the specific area, so that sometimes it was only then that they realised the high quality of their performance.

Further experiences of a more general character will be described in the following chapter.

4.2. Results of the evaluation of the modelling courses

The three modelling courses carried out from 2001 to 2004 were evaluated intensively. 180 students from 10 schools in Hamburg and its surrounding and 32 prospective teachers participated in the project. At the beginning and the end of the second and third run of the project all participants were questioned, whereas during the first run – due to organisational reasons –the questioning could be done only in the middle of the run. 138 students and 22 prospective teachers answered an open questionnaire which contained 11 questions on the domain beliefs about mathematics and mathematics teaching (based on the typology developed by Grigutsch 1996), about application of mathematics in everyday life and in the sciences; additionally students were asked what they want to study what kind of profession they would like to choose. The second questionnaire included additional questions about the evaluation of the modelling examples they had worked on. For the prospective teachers the questionnaire contained additional questions about the university course. The evaluation of the questionnaires was conducted in accordance to the method of "thematic encoding" by Flick (1999) in which the codes were thematically deduced from the research questions as well as received through empirical open encoding. Due to a lack of space, the results are not represented in detail here (but an unpublished report is available from the authors), so that in this contribution we concentrate more on some central results which are exemplified by direct verbal statements by students.

The following central results were achieved:

Result 1:

It has been shown that complex and high standard modelling examples are feasible in schools. A large number of the students involved participated actively up to the end of the course despite the long time period needed for the examples and the complexity of the examples. Most of the students expressed satisfaction with the results and with the course.

This is exemplified in the following statements of the students taken from the evaluation, in which they assess the modelling examples positively:

> That "this project ... makes lessons more varied and more interesting".

> "Yes, because of this, relation to reality has been demonstrated, lessons are more flexible

> to so that teamwork is also promoted."
>
> "I am happy since in my opinion we achieved much."

Result 2:

The results of the evaluation make clear that complex modelling examples are not reserved for highly talented and high performing students. On the contrary they can be carried out by average students in ordinary schools. The students' solutions achieved a remarkably high standard, if we consider the wide spread of abilities and achievements that are usual in average German classes.

Result 3:

A change in the mathematical beliefs of the students and the prospective teachers concerning mathematics and mathematics teaching could be detected:

Before the project had started, static views about mathematics were dominant for most of the students and prospective teachers. In the following, some exemplary statements are given:

> "Mathematics is constant which means what one learns today, will still be valid tomorrow". Mathematics means "dealing with numbers", "a subject where one must calculate a lot."

After the project was finished, opinions that showed a more application oriented view came up such as

> "Mathematics is the basis of many aspects of life and many professions".
>
> Mathematics is relevant, "because in my opinion mathematics is essential in our everyday life, since mathematics holds many things together and makes many things possible for us."

Particularly in connection with descriptions of what good mathematics teaching should look like, the demand to include modelling examples into ordinary mathematics teaching was emphasised.

> Students wished, that mathematics lessons have "real world contexts" and should be "near to everyday life" and "deal with real problems of everyday life".

Result 4:

The prospective teachers assessed the seminar even more positively than the students.

> It was positively noted that "own teaching experiences were enhanced" and that "modelling in school practice" was learnt.

It is demanded that

> "practical training for prospective teachers should generally be more integrated into university study".

Result 5:

Altogether, the evaluation shows a positive judgement of the modelling course, but clearly differentiated judgements about single examples are given. The modelling examples are generally judged as being "very near to reality", allowing interdisciplinary insights, discovering danger e.g. in the fields of biometry. Students were critical of some examples, particularly the one concerned with the identification of fingerprints because of the considerable usage of computers, especially the usage of MATLAB. Students said that "problems with the programmers" occurred.

Result 6:

The teamwork practised within the modelling course was evaluated positively. The students described that through teamwork they could better sustain the uncertainty which is characteristic of modelling activities. In addition a greater selection of solution attempts was achieved. The students said:

> "Working in groups is super. The other participant can give you hints about things you don't notice. By sharing the work one achieves the result more quickly."

The learning of new ways of working, such as teamwork, is praised as a positive effect of the project.

Result 7:

Although one aim of the project had been to shift the students' intentions for their further university study towards mathematics only a small shift could be detected. Nevertheless a tendency towards favouring technical studies could be observed. This might be due to the fact that the lectures given by the applied mathematicians together with the modelling examples studied had shown the great applicability of mathematics in various fields, which might have directed the interest of the students towards these topic areas.

The following conclusions can be drawn from the experiences described above. There is a consensus that while on the one hand mathematics teaching changed positively through modelling examples, on the other hand the lessons became much more challenging and time-consuming.

Furthermore, it became clear that especially the learning and teaching practice at university changed positively through this kind of seminar.

In summary, the positive reactions from all participants demonstrate that through these kinds of seminars a change of mathematics teaching towards a stronger consideration of modelling and real world context is possible. Particularly by involving prospective teachers, changes can be expected.

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Authors

Gabriele Kaiser, Professor
Faculty of Education
University of Hamburg
Von-Melle-Park 8
20146 Hamburg
Germany
Email: **gabriele.kaiser@uni-hamburg.de**

Björn Schwarz
Faculty of Education
University of Hamburg
Von-Melle-Park 8
20146 Hamburg
Germany
Email: **schwarz@erzwiss.uni-hamburg.de**