

## Teaching mathematical modelling through project work

- Experiences from an in-service course for upper secondary teachers<sup>1</sup>

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**Abstract:** *The paper presents and analyses experiences from developing and running an in-service course in project work and mathematical modelling for mathematics teachers in the Danish gymnasium, e.g. upper secondary level, grade 10-12. The course objective is to support the teachers to develop, try out in their own classes, evaluate and report a project based problem oriented course in mathematical modelling. The in-service course runs over one semester and includes three seminars of 3, 1 and 2 days. Experiences show that the course objectives in general are fulfilled and that the course projects are reported in manners suitable for internet publication for colleagues. The reports and the related discussions reveal interesting dilemmas concerning the teaching of mathematical modelling and how to cope with these through "setting the scene" for the students modelling projects and through dialogues supporting and challenging the students during their work. This is illustrated and analysed on the basis of two course projects.*

### Introduction: Modelling and project work in the mathematics curriculum

The direct occasion to develop an in-service course on how to teach mathematical modelling through problem-oriented project work for upper secondary teachers was the new thorough reform in the Danish gymnasium (high-school, grade 10-12) which has been effective since August 2005. The introduction of modelling into the mathematics curriculum is not a new thing. Since 1988 modelling has been one of three aspects of mathematics that the students became

acquainted with.<sup>2</sup> But these aspects were not mandatory parts of the testing and in practise they were if not totally absent then inadequately treated.

The objectives of the mathematics teaching in the new gymnasium are explicated through four categories: goals, criteria for evaluation, mathematical goals, and content – which are further divided into core content and supplementary content. Aspects of the modelling process are explicitly mentioned in all of these categories. It is required that the students gain insights into how mathematics can contribute to understand, formulate, and treat problems within different subject areas; that they can apply models describing simple relationships in nature, and recognize their potentials and limitations; that they can reflect and argue about idealisations and the range of existing models, and are able to move between the theoretical and practical sides of mathematics in connection with modelling and problem-solving. Finally it is stressed that they should be able to use various mathematical concepts like the derivative and differential equations in connection with modelling.

In short, modelling has been given a more prominent position in the new mathematics curriculum. There is a stronger emphasis on the students' work with the entire modelling cycle. The intentions are not only that the students get familiar with mathematical concepts and theories that *can* be used in modelling but that they also are aware of the different phases in the modelling process and that they are competent in engaging in critical discussions and reflections about models and their possible applications.<sup>3</sup>

The reform is not only explicit about content it also makes demands on teaching methods. For mathematics it is stressed that a considerable part of the teaching should be organised as projects where the students in smaller teams or groups work independently with mathematical as well as interdisciplinary problems and prepare a report. On a more general level the gymnasium should support the transition from pupils to high school students and for some of them further on

<sup>1</sup> The development of the in-service course has been supported by the European Social Fond.

<sup>2</sup> The other two aspects were the history of mathematics and the inner nature of mathematics. See UVM (1988).

<sup>3</sup> See the guidelines for mathematics in the gymnasium UVM (2005).

to university students, and one way of doing that is to make sure that the students are placed in situations where they must make decisions, carry out delimitations and specify definitions.

In conclusion the new mathematics curriculum challenges the teachers in a two-folded manner. They have to work with the full mathematical modelling cycle in their teaching, and – even more challenging – it is required that they bring mathematics into play in projects in cooperation with other subjects. Hence the mathematics teachers need to be able to see, and discuss with teachers from other subjects, the potential mathematical content in form of modelling in different types of projects. In addition the teachers need to be able to organise and administrate project work in their own mathematics teaching.

Our position is that mathematical modelling – building, analysing, criticising – is learned by doing it. In this context problem-oriented project work in modelling presents itself as a suitable didactical choice,<sup>4</sup> and it takes care of some of the central challenges in the new reform.

As teachers at IMFUFA, Roskilde University we both have more than 10 years of experience with teaching mathematical modelling in its full sense and with problem-oriented project work as a teaching and learning method (Niss 2002). During the past 8 years we have developed and taught – some times together – a first year undergraduate course in mathematical modelling (Blomhøj et al. 2001, 2004). In the development of the in-service course in hand, we have drawn heavily on these experiences. Even though the institutional context in the gymnasium is very different from that of a project organised university our experiences from supervising different types of students' modelling project – not least the small projects that are the backbone of our modelling course – has proven to be very helpful in the development and teaching of the in-service course.

The in-service course is at present running for the third time with 13 participants and 32 teachers have already completed the course. In the paper we present the general organisation

and content of the course, as well as our understanding of the modelling process and problem oriented project work as we present it at the course. Two examples of modelling projects developed at the course are used to illustrate important dilemmas connected to the teaching of mathematical modelling through project work.

### **Developing an in-service course in modelling projects**

There are two problems connected with the developing of an in-service course addressing the issues stated above. First, since modelling has not seriously been a part of the mathematics education in the gymnasium earlier there is no real tradition for teaching this subject. In general the mathematics teachers in the Danish gymnasium have a master degree in mathematics and one other subject, normally within the sciences but not rarely in the social sciences or humanities. Modelling as such is not included in the standard curriculum in the education of mathematics teachers, so most teachers have not received teaching in mathematical modelling in their education. Modelling is different in nature than internal mathematics, concepts and theories so traditional teaching methods in mathematics cannot just be transferred to modelling, so many teachers are left in a new territory with no tested tools for upper secondary teaching.

Second, problem oriented project work is not included in their pedagogical training. So if the intentions for mathematics in the reform are to be taken seriously there is a need for in-service courses that introduce these subjects.

However, it is well known that knowledge about the mathematical content and the pedagogical methods does not automatically course the intended changes of a new curriculum in the practice of mathematics teaching. In a system like the Danish one, where the teachers have a high degree of autonomy concerning choice of teaching materials and the didactical planning of their teaching, changes in practice necessarily presuppose that the teachers have personalized the ideas behind the curriculum – at least to some degree. For this to happen the teachers need to have some personal experiences showing to them that the new curriculum

<sup>4</sup> In a section below we will explain thoroughly what we understand by problem-oriented project work.

elements actually can function in their own teaching and for the benefit of their students' learning. For an in-service course to be helpful in this process it needs to focus on the teachers' development of their own practice, and it needs to create the opportunity and support for the teachers to try out new elements in their teaching.

Accordingly we have developed an in-service course which main objective is to support teachers to develop, try out in their own classes, evaluate and report a project based and problem oriented course in mathematical modelling. The teachers' experimental practice with mathematical modelling and project work is the core element of the in-service course. By formalizing the experimental teaching as a key element the in-service course takes on some of the teachers' responsibility and hereby opens for their professional development. In order to support co-operation among teachers from the same school and thereby hopefully maintain long lasting effects we encourage and give priority to participants from the same school, and 2/3 do in fact come in pairs or larger groups from the same school.

The course is offered on marked terms and the teachers have to apply for the course to their headmaster. The course covers 5 ECTS points in the European university system and the participants will receive a certificate, but in general it will have no formal significance except maybe in future local negotiations of the teachers' salary. At the present state of affairs a mathematics teacher in the gymnasium can expect to get an in-service course approximately every five year.

### **The organisational structure of the course**

The course starts with a 3-days seminar where the participants are introduced to mathematical modelling and problem oriented project work. During the seminar the teachers are working in groups of 2 or 3 developing project organised courses on mathematical modelling covering approximate 10 normal lessons and equal two normal written homework assignments (altogether 5 hours of extra homework for the

students).<sup>5</sup> The teachers have in advance indicated in which class and on which level they want to carry through their experimental teaching. The groups are formed accordingly and teachers from the same school are in the same group.

In the first phase of the development of the projects the groups are asked to emphasize the following four issues: (1) Intentions for own development. What is it in particular that they want to try out and experiment with as a teacher in their project organised modelling course? (2) What are the main intentions for the students' learning in the course? (3) How to set the scene for the students project work? (4) How to evaluate the students' learning through observations and/or product evaluations. The groups are encouraged to limit and focus their answers to these questions as much as possible.

The groups' first proposals are presented and discussed and then further developed. Two weeks after the first seminar the preliminary descriptions and student materials for the modelling projects are distributed to all participants. Within a period of two months the groups finish the detailed planning and each teacher try out his or her experimental course in at least one class. In some cases it is possible for teachers working at the same school to observe part of the course in the colleague's class. This has proven to be very supportive for the teachers' reflections and for their further co-operation with improvements of the course or development of new courses.

After the period with experimental teaching a 1-day seminar is held with the aim of supporting the teachers in reporting their courses and their related reflections in a form that could be helpful for colleagues that would like to try out their courses or similar modelling projects. A first version of the reports is distributed to all participants a month later. These are presented and discussed extensively at a last 2-days seminar after which the groups receive written feed back from the course organisers including suggestions for improvements before publication on the internet, if the teachers so want. A final

<sup>5</sup> These – not so ambitious – frame conditions are to ensure that the experimental courses can be carried through even within the old curriculum under which the teachers are actually teaching at the moment.

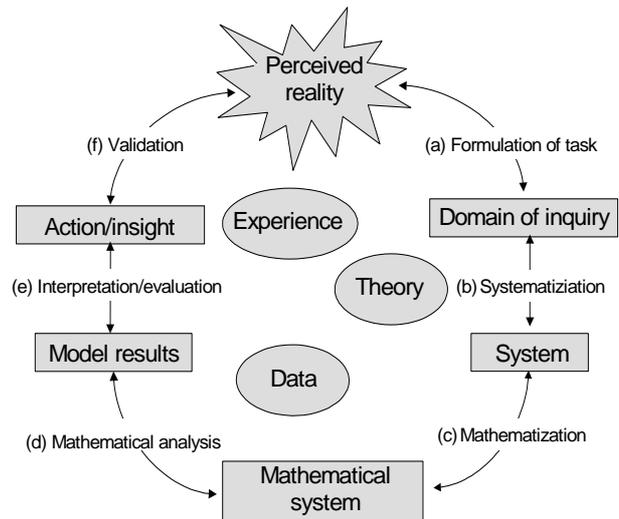
version of the reports is sent to the organisers approximately one month after the last seminar.<sup>6</sup>

### Introducing mathematical modelling

While working through different modelling examples at the in-service course we introduce the graphical model of the modelling cycle shown in figure 1. This is a complex model of the modelling cycle that can be used for different purposes, see (Blomhøj, 2004) and (Blomhøj & Jensen, 2006). The modelling cycle is consistent with the one presented in Blum & Leiss (in press) and discussed in Borromeo Ferri (2006) in the sense that “domain of inquiry” and “system” can be interpreted as “situation model” and “real model”, but this is not the place for a detailed discussion about the general modelling cycle. At the in-service course the modelling cycle is used to capture and define the different elements in mathematical modelling competency<sup>7</sup>, which we in accordance with the Danish KOM-project define as:

*Mathematical modelling competency means being able to autonomously and insightfully carry out all aspects of a mathematical modelling process in a certain context.*<sup>8</sup>

Based on the modelling cycle mathematical modelling competency can be specified to include the abilities to carry through the following sub-processes: (a) Formulation of a task (more or less explicit) that guides one to identify the characteristics of a perceived reality (real situation). In order to do so, one needs to be able to see one self in the situation or in other words to construct a mental model of the situation (a situation model). (b) Selection of the relevant objects, relation etc. from the resulting domain of inquiry and idealisation of these in order to make a possible mathematical representation.



**Figure 1:** A model of the modelling cycle consisting of six sub-processes (a)-(f) each related to a sub-competence in mathematical modelling competency (Blomhøj & Jensen, 2006). The three ellipses indicate that the epistemological basis for the different sub-processes can be of very different nature.

These processes, that may be based on theoretical ground, experiences or purely on ad hoc assumptions lead to a system (or a real model) that – in principal – can be described mathematically. (c) Representing the objects and relations in the system mathematically in a coherent way. (d) Use mathematical methods to achieve mathematical results and insights. (e) Interpretations of these results and insights in relation to the initiating domain of inquiry. (f) Evaluation of the validity of the model by comparison with experiences, observed or predicted data or with theoretical knowledge and reflections of the entire modelling process. (Blomhøj & Jensen, 2003)

It is important to notice that even though it in principal is possible analytically to identify and describe these six sub-processes in all mathematical modelling processes it does not imply that these processes can be learned separately. On the contrary, the pedagogical idea behind identifying mathematical modelling competency as a specific competency is exactly to highlight the holistic aspect of modelling. Modelling competency is developed through the practice of modelling. However, in order to challenge and support students in this learning process it is important to know the different sub-processes and related pitfalls in modelling.

<sup>6</sup> Reports (in Danish) from the two previously held courses can be found at <http://mmf.ruc.dk/MAT/matefteruddannelse.htm>

<sup>7</sup> See also the paper by Katja Maaß (2006) in this issue of ZDM.

<sup>8</sup> The Danish reference is Niss & Jensen (2002). Part of the project is translated into English in Niss & Jensen (2005).

In our interpretation the ultimate learning objective regarding mathematical modelling in the gymnasium is to support the students' development of mathematical modelling competency. That means that the students sufficiently often should be challenged to work with the entire modelling cycle. But of course the learning process has to start somewhere and therefore it is really important for the teachers to have an idea about how to notice, describe and support progress in students' modelling competency.

According to the KOM-project, progress in a competency, e.g. mathematical modelling competency, can be described along three dimensions: (1) Technical level, according to which kind of mathematics the students are using and to how flexible they are in their use of mathematics. (2) Radius of action, according to the domain of situations in which the students can perform modelling activities. (3) Degree of coverage, according to which part of the modelling process the students are working with and the level of their reflections. (Blomhøj & Jensen, 2006)

The modelling cycle and the three dimensions of progress are used at the in-service course as an analytical tool for analysing different tasks for modelling projects, ways of setting the scene for the students' modelling activities and possible ways of challenging the students during their project work. In particular we discuss with the teachers how to find and develop tasks that are suitable as point of departure for modelling projects. The following list of criteria for good modelling tasks is presented and discussed at the course.

It is a quality if the task

- can be recognized and understood by the students. That is to say that the students should be able to make sense of the task and to give their own formulation of the problem or in other words build a situation model of the task.
- gives an appropriate challenge for the students to work independently of the teacher's support with (some parts) of the modelling cycle.
- is authentic or includes authentic data. Meaning that the problem is relevant in some real situations outside the school

context and hereby is open for the use of the students' experiences and for collecting information or data from other sources.

- opens for interesting modelling results. Showing that mathematical modelling can add meaning to the situation and provide new insights into the problem.
- opens for critique of the model and/or the way that the model results is or can be used.
- leads to activities that are in some sense representative for a particular type of modelling situations.
- challenge the students appropriately to work with concepts and methods that are relevant for their mathematical learning.

The teachers in general find the modelling cycle and the definition of modelling competency relevant for their own understanding and useful as a tool for didactical planning. But they are very hesitating to teach these general ideas about mathematical modelling to their students – and with good reasons we think. Some of the teachers include in their courses a simplified description of the modelling cycle consisting of four steps: Starting from reality → “real” model → mathematical model → mathematical treatment/solution → interpretation in the “real” situation with the possibility of going through the cycle several times. The model may be presented to the students at the beginning of the project but the general discussion is typically placed at the end of the project period and seen as a way to help the students reflect about what they have been doing in their modelling project.

### Introducing problem oriented project work

We define a *problem oriented project work* as a didactical situation where a group of students are working together on a problem during a longer period of time.<sup>9</sup> In this context a problem is a question or task, which is challenging for the students and to which they do not know a direct method for solving the problem. Hence, what constitutes a good problem for a project work is

<sup>9</sup> The time period should be long enough for the students to personalise the problem, to find information from other sources, to use each other in discussions about the problem, not focus too early and too much on the end-product. In the context of the in-service course for pragmatic reasons we define a project period to be at least two weeks.

very much depending on the students' competences and previous experiences and of course also on the conditions given for the project.

In a problem oriented project work the problem should be the "guiding star" for the work. Every act taken in the project should ideally be justified by its contribution to the solution of the problem. Didactically this is very important because it gives the opportunity for the students to take over – part of – the responsibility for directing the project. And this is crucial when the learning goal is the development of high level competences such as mathematical modelling competency. A very important element in modelling competency is to consciously steer the process through the modelling cycle and to reflect about the different steps. This is not something that can be learned by working with a series of questions related to different modelling situations but presupposes a certain degree of autonomy in the students' work. In order to develop modelling competency the students need to experience situations where they are responsible for – most of – the decisions involved in the process of modelling. From a pedagogical theoretical point of view problem oriented project work is therefore an obvious choice of didactical method for developing modelling competency.

However, such an approach immediately gives rise to the dilemma of how to teach directed autonomy (Blomhøj & Jensen, 2006). On the one hand we want the students to take the responsibility for their modelling work but on the other hand we also want them to make "good decisions" in order for them to be able to get through the modelling cycle using their mathematical knowledge. In other words we need the students' work to be effective for their development of modelling competency but if we try to ensure the effectiveness by didactical interventions there is a risk that we are missing the point. This dilemma and ways to handle it are discussed intensively at the in-service course where we argue that the dilemma can be handled – not solved – by balancing two forms for didactical interventions, namely by "setting the scene" for the students' project work and through dialogues with the groups of students during the process.

By setting a specific scene for the students' project work in form of the formulation of the task for the project work, the conditions for the working process in form of milestones and overall time constraints and requirements for the end-product, e.g. a report directed to a specific target group fulfilling certain requirements for documentation, it is possible to direct the project work and still leave room for the students to take important decisions and hereby take over the responsibility for the project. However, in order for this to happen, the "scene setting" needs to be sufficiently open. Therefore, in the design the teachers have to deliberately leave important decisions to the students. This has as a consequence that some groups may get into trouble in their projects and that some of the learning potential in the modelling task are not realized by the students. However, as a teacher one can try to cope with these didactical problems by getting into dialogues with the groups during the project work<sup>10</sup>.

As part of the design the teachers are also challenged to foresee possible ways of interacting with different groups of students through dialogues. They are even asked to construct what they find could be realistic dialogues with groups of students that need support or challenges in relation to some specific part of their modelling work. Thus, as a teacher one can prepare one self also for this type of interaction with the students. In the following paragraphs we present two examples from the in-service course that illustrate how the two types of didactical interventions were balanced differently with clear didactical implications.

### **The asthma drug project**

This project was planned and designed by two teachers who worked together in the same school. The project was carried out in a class of first year science students (i.e. 10<sup>th</sup> graders) during these students' first semester in an ordinary public Danish gymnasium.

The overall modelling task was – on the basis of data for the concentration of an asthma drug in the blood at different times – to suggest a plan

<sup>10</sup> "Scene setting" as a didactical tool in mathematics education is discussed in Skovsmose (1994).

for administering doses of the drug in such a way that the concentration would never exceed 20 mg/l and never get below 5 mg/l. The task was given by a fictitious doctor who needed such a mathematical well-documented plan. It was important that the doctor was able to understand the argumentation for *why* the suggested plan would fulfill the medical requirements.

The teachers had no earlier experiences with using problem oriented projects as a teaching method in mathematics on the contrary they saw their teaching method as rather traditional primarily dictated by the national finals. The intentions they formulated for their own development as teachers as well as for the learning output of their students need to be understood with respect to their “normal” teaching practice.

The teachers characterized in their report their usual mathematics teaching as divided into two parts. The first part is completely governed by the written final exam.<sup>11</sup> In order to be prepared students must have developed particular routines for solving the set of standardized exercises that could appear on the final. If this is not accomplished the students will run out of time during the final. This (huge) part of the teaching is directed narrowly towards the written final. Normally the exercises can only be solved in one way but some of the problems can be viewed differently and they require an individual interpretation by the student. Most of the problems are theoretical mathematics exercises. Modelling exercises do appear but in such cases the problem is to determine whether a relation between two variables is linear or exponential. There is always only one correct answer. There are no debates about the interpretation of the model, its applicability etc.

The second part build on the first part in that here the focus is on *why* the procedures for solving the exercises trained in the first part actually work, why the mathematical theorems are true. This goes on at a higher level of abstraction and where the students in the first

part were active in searching for solution procedures they passively reproduce proofs from the textbook in this second part. Only at very rare occasions are they able to produce a proof by themselves.

Viewed against this teaching practice the following description of what the teachers’ intentions were for the students’ learning output of the asthma modelling project is put into perspective as a radical change both in teaching practice, in what kind of problems the students are faced with, and the requirements for the students’ working process.

The teachers had the following intentions with the project:

The students should:

- work during a longer time span where they themselves should organize and manage their work within the project.
- work with a more authentic, complex and non-delimited problem, where the students do not know a standard procedure.
- learn how to formulate a (modelling-) problem.
- work with a *problem* and not a standard exercise so the modelling, the mathematisation, and the interpretation of the results also become a part of the project.
- understand the notion of a mathematical model.
- use known mathematical concepts like graphs and given equations for functions in concrete situations.
- develop a basic understanding of the nature of the exponential function.
- develop their mathematical communication skills to formulate mathematical problems to non-mathematicians.
- use electronic equipment and software through out the project.

To fully appreciate the challenges the students were confronted with in this project it is important to be aware of another agenda the teachers had which also broke fundamentally with their traditional teaching method. Besides working with a *problem* – in our sense of the concept – of a type the students had not seen before and for which they did not know a standard procedure they also did not know the

<sup>11</sup> A national 5-hour written exam consisting of a system of tasks for one hour without any auxiliary materials and a 4-hour assignment with cas-calculators and other materials.

underlying mathematical concepts. The exponential function was unknown to the students. The teachers wanted the students to “discover” new mathematics during their work with the problem. The idea was to challenge the students to develop themselves the “new” mathematical theory of exponential decline/growth, to discover by themselves the properties of the exponential function and realise its analytical expression.

This is a very ambitious project and the challenge for the teachers was to “set the scene” in such a way that these intentions could be fulfilled. They did that by dividing the whole project period into four parts – or phases – as they called it.

In the first phase the teachers introduced the students to the different aspects of the project, i.e. the asthma drug problem, “net studies” which is the school’s software program for organizing team work, the social contract for the responsibility of the individual student with respect to his or her group, a mini course on how to use Excel, and finally a brief introduction to the different elements of the modelling process. The teachers used three lessons of 45 minutes each on this introduction. The introduction was accompanied by the following handouts prepared by the teachers

1. A general description of the problem and set of data
2. The “hidden” teacher-agenda
3. The different phases of project work
4. The different phases of the modelling process
5. Excel-exercises, the social contract, p-notebook (logbook and project management).

If we compare the content of this first phase with the teachers intentions for the students’ learning process and the kind of insights they wanted the students to achieve we can see that during this introduction the teachers provided the students with the necessary initial knowledge about the modelling process, the self-managing of a longer project in teams, and the computer software. In relation to the modelling process the introduction helped the students to construct a situation model of the problem and gave them an idea about the modelling cycle.

Phase 2 took care of the problem formulation part of the modelling competency. The teachers were very much aware that the ability to formulate problems that can be answered by mathematical modelling is a skill in itself that needs its own attention. In order to train the students in that particular part of the modelling process they set aside two lessons for problem-formulation and –decision during which they engaged in discussions with the students about what kind of questions mathematics can answer.

Before the students were set free to work on the actual project the teachers put in a third phase consisting of six 45-minutes lessons where the students in groups worked with four mathematical exercises formulated by the teachers. During these exercises the students were guided through some of the mathematical difficulties and steps that they would need for solving the problem during the project work. The students were supposed to hand in their answers to these exercises. This was a way through which the teachers could “fertilize” the ground so the students would not be left in a total vacuum when they were set free to work on the actual project.

After these three phases the students then worked on the project independent in teams using ten 45-minutes lessons at the school. The requirement was that each team prepared a report addressed to the doctor, which meant that it should be formulated in ordinary language. The report should give an account of how the concentration of the drug in the blood declines with time, how a plan for repeated administering of a fixed dose at fixed time intervals could be designed in such a way that the concentration – first after a couple of doses, and then immediately through a start dose – would lie between 5 and 15 mg/l. The students were asked to reflect on whether this “dose-time”-plan could be used on another patient. All the claims made in the report should be validated mathematically in an appendix, so all results could be controlled.

The grading of the project was based on (1) the language in the report, is it written for a non-mathematician, is it understandable, is the problem and the solution presented in a transparent way? (2) are graphs used to illustrate the problem and its solutions? (3) are all calculations, mathematical considerations and

assumptions clearly formulated and stated in the appendix, is it possible to understand how the results were found? (4) the level of advancement on which the problem is solved, (5) are the group of students consciously aware of the modelling process? Together these five elements cover the original intentions the teachers had for the students' learning.

This very careful and in a way very controlled design of the three phases leading up to the project work (phase 4) was crucial for the success the students experienced during their work on the project. It prepared the students in such a way that they were able to continue on their own. And the design made it possible for the teachers to support the students during their work without draining the situation for challenges and hereby undermine the intentions for the students' learning.

At this point there are two things to which we want to call attention. Firstly, the role of the teacher during phase 4, that is during the actual problem solving phase, and secondly, the significance of the "scene setting" for the realisation of the "discover the mathematics yourself" agenda.

According to the first point, one of the teachers' intentions was to create and try out a teaching situation where the students should work independently on a complex non-standard problem. This was accomplished through the three first phases during which the students were properly prepared to work on their own. During phase 4 the teachers chose to play the role as consultants that could be called on for help, but they did not structure what went on during phase 4. They were present but did not intervene in the students' work neither the organising nor the strategy chosen to solve the problem.

With regard to the second point concerning the students' discovering of the mathematics on their own the teachers on the one hand used the data shown below were they had changed the original data in such a way that the half-life period was shown very clearly. On the other hand they supported and encouraged the students' own investigations during dialogues with the groups when they were called upon for consulting assistance. In these dialogues the teachers had before hand decided how far they

would go in their guidance of the students toward the properties and analytical expression of the exponential function.

Time hours	Concentration mg/L
0	10,0
2	7,0
4	5,0
6	3,5
8	2,5
10	1,9
12	1,3
14	0,9
16	0,6
18	0,5

Table 1: The data given to the students showing the decline of the drug concentration in the blood.

The following dialogue gives a nice illustration of this type of supportive dialogues where the teacher deliberately leaves a specific mathematical challenge for the students to work with.

- S1: We can see that it decreases all the time, but it decreases less and less.  
 T: Which one?  
 S1: Eehh ... that must be the concentration of the drug  
 T: Yes. Is there a pattern?  
 S2: Yes. It decreases by one half every fourth hour.  
 T : Yes. So it doesn't decrease with an absolute amount, but what is it then that is fixed?  
 S1: That with which it decreases ... that is the magnitude with which we multiply.  
 S3: That must be the percentage ...  
 T: Does it also decrease with a fixed percentage if we only consider intervals of two hours?  
 S3: It decreases from 10 to 7, which means there will be 70% left.  
 T: Yes, and when we want to find 70% with what factor do we then multiply?  
 S1: We multiply by 0.7  
 T: Is it true that it continues to decrease to 70% when we go on?  
 S2: If we multiply 7 with 0.7 we get 4.9 and that does not fit because it should be 5.

- S3: It fits perfectly. It is almost the same! It is just a truncation error.
- T: If it also decreases with a fix percentage every hour, how many percent would that be?
- S1: It must be 15%.
- T: Try it out and se if it fits. With what factor should we multiply to get one hour ahead .... (The teacher leaves)<sup>12</sup>

This dialogue shows that the modelling problem in whole and the project organised work motivated the students to examine mathematical questions regarding the properties of exponential functions. For all of the groups such mathematical investigations eventually resulted in the formulation of the analytical expression of an exponential function describing the data. Half of the groups figured out that the correct rate was the fourth root of a  $\frac{1}{2}$ , which equals 0,84, while the second half operated with 0,85 (1-15%).

The students' reports show that all the groups were able to develop a sensible plan for administrating the drug doses for the patient. Some even managed to compare and present to the doctor different feasible plans according to convenience regarding sleep and work patterns.

### The swimming pool project

This project was also designed by two teachers working together at the same high school. It was carried out in a class of second year students (grade 11) who had chosen mathematics at the highest level. Some of the students also had physics at the highest level and the rest of them had chemist at the highest level. The teachers characterised the class as very strong in mathematics and with a lot of drive.

The teachers wanted to experiment with a problem oriented project teaching approach to remedy a frustrating dilemma they had observed in the class. On the one hand the teachers found the students to be very strong in mathematics in the context of normal class teaching. But at the same time they also have had quite a few

experiences of the following type in the class (quotes from the teachers' report):

"It is well-known that questions like: "Where do we have the highest rate of change?" can overwhelm even the most capable students"

"If a student accidentally is asked to come up with a simple, increasing function the result will in most cases be a paralysing silence. That  $f(x) = ax + b$  for  $a$  positive would be useable 'is just too much'."

The teachers' objectives for the students' learning were centred on modelling and the use of IT-tools to solve differential equations numerically using a special programme called Fpro. In addition the teachers saw the modelling activities as a mean for enhancing the students' perception of  $f$  prime as a rate of change and their general understanding of the concept of a function. Through modelling activities the students could be challenged to create functions that behave in a certain predetermined way.

Finally it was also part of the teachers' intention to develop the students' modelling competency especially regarding the processes of mathematisation, mathematical analysis and interpretation of the results, e.g. process (c)-(f) in figure 1.

The project had its source from an article in the Danish newspaper called Jyllands-Posten. There was a picture of children playing in a garden with an inflatable pool, see figure 2. Under the picture one could read that if the water was above 25 degree Celsius every single bacterium would turn into two in 20 minutes. Together with the following questions this story formed the point of departure for the project: *What are the consequences of this? How many bacteria are we talking about per hour, per day? Is that reasonable?*

The curriculum on differentiation had been taught in class before the project was invoked but not in a modelling context. The plan was to use only 7 lessons on the project because the teachers found it difficult to cover the curriculum if more time was spend on the project. Before the first lesson the students were given a set of handouts consisting of the

<sup>12</sup> The dialogue is based on the observing teacher's notes and translated from the final report of the Asthma drug project by the authors.



Figure 2: The picture in the newspaper article with the headline “For Children” and the sub text: “After three days the children’s best play toy had turned into a bacterial bomb....Heated by the sun to over 25 degree and every single little bacterium turns in to two every 20 minutes.”

newspaper article with the picture and an introduction to the project.

The criteria for the end product were quite loose. The only requirements were well-functioning Fpro-programmes that solve the different models numerically and a text document with the model results and descriptive comments.<sup>13</sup>

In contrast to the asthma project this project was much more causally designed. The only “scene setting” elements were the introduction and the requirements of the end product. It is questionable whether these very unspecific requirements for the end product can function as a scene for the students’ project work. The

<sup>13</sup> In a parenthesis to the requirement of descriptive comments it says: (“Did you “invent” the model yourself? Do the involved parameters have an interpretation? What assumptions have you made? Do the predictions of the model coincide with data?)

introduction on the other hand was – as the following quotation shows – very detailed:

“Let us dive into the soup, where we in some detail will build a simple model. We imagine a – for bacteria – nurturing soup. Let  $N$  denote the number of bacteria, where  $N$  of course depends on the time,  $t$ . For small  $N$  the quantity of nourishment can be taken to be unlimited. We want to know how the number of bacteria develops with time. The very, very, very central magnitude is the *rate of change*  $N'$ . In causal mathematical language  $N'$  denotes *the change in number of bacteria per unit time*. If we measure the time in minutes and if  $N' = 10$  this means that the number of bacteria increases with 10 every minute. ...

The most simple model of the bacteria population is to suppose the rate of change is proportional to the actual number of bacteria, that is  $N' = k \cdot N$ , where  $k$  is a constant.”

After the introduction the students were “set free” to work on their own setting up models for the following scenarios given by the teachers:

1. The amount of food is not unlimited. Suppose a certain amount of food is added per unit of time. How will the amount of bacteria develop over time?
2. Suppose a purification device remove a certain fraction of the bacteria per unit of time. Model!
3. Some bacteria liberate poisonous substances. Think for example of a wine balloon ... the poison inhibits the reproduction of the yeast cells. One can also say that the alcohol kills a certain fraction of the yeast cells – a fraction which of course increases with the amount of alcohol in the balloon. Model!
4. Many poison substances are unstable. Suppose now that a certain fraction of the poison disappears per unit of time. Model!

If we compare with the experiences expressed by the teachers, the thorough treatment of the simple model and the challenges the students are faced with in the project there is a didactical contradiction. The difficulties the students have in the classroom when asked about the behaviour of the rate of change or to construct a function that can describe an increasing phenomenon suggest that the task of modelling the simple situation would have been a challenge to them – a challenge that actually would have had the potential of fulfilling the initial goals of the teachers'. Instead the teachers took the students by the hand and led them through crucial steps in the modelling cycle. If the students cannot by themselves model a phenomenon of growth using the simplest possible linear model and if they cannot model the simple scenario it is hard to see how they should be able to do any thing on their own with the four extended scenarios.

This was also reflected in the course of the project and at some point in the process the teachers felt the need to hand out a “key” to the different scenarios:

1. Limited amount of food: The relative rate of change decline with seize of the population, that is:  $N' = (a - b \cdot N) \cdot N$

2. With purification:  $N' = (b - a \cdot N) \cdot N - c \cdot N$
3. The wine balloon: There are two essential magnitudes, the number of micro  $N$  and the content of poison  $g$  in the system. If a single organism produces poison at a constant rate we have:  $g' = c \cdot N$ . A fair model for the number of organisms will then be:  $N' = a \cdot N - b \cdot g \cdot N$ . The parameter  $b$  symbolise the “poisonness” of the poison. We get a system of two differential equations:  
 $g' = c \cdot N$  and  $N' = a \cdot N - b \cdot g \cdot N$
4. If the poison is degradable, the model has to be modified. A suggestion could be:  
 $N' = a \cdot N - b \cdot g \cdot N$  and  $g' = c \cdot N - d \cdot g$ .  
If, finally the source of food is limited, one can imagine the following model:  
 $N' = (a - e \cdot N) \cdot N - b \cdot g \cdot N$  and  $g' = c \cdot N - d \cdot g$

In our interpretation, the teachers' ambition for the technical level of the students' modelling activities was much too high and this prevented the students to actually engage in the modelling process. Accordingly the degree of coverage in the students' work regarding the modelling cycle became quite limited in the project.

The importance of “setting the scene” becomes very clear in this project. The swimming pool story is an excellent problem and the project has the potentials for fulfilling all of the initial goals for the students' learning. A more careful planning on the teachers' side where they stick to the original idea from the newspaper and maybe just the scenario with the purification could have formed a nice project with an appropriate challenge for the students to develop their modelling competency.

The students were set free to work on open problems but the scene was not set in such a way that the groups of students could work independently, on the contrary they were left in an empty space (3/4 of the students found the project either totally frustrating or rather difficult), and this caused the teachers to provide a full solution. Evaluated against the initial goals the project was a failure, but – interesting enough – the conclusion drawn by the teachers – based on observations during the project and a questionnaire answered by the students – was, that the whole experience was rather positive.

### Dilemmas teaching mathematical modelling

The experiences from the in-service course in form of the teachers' reports on their experimental modelling projects, the presentations given at the last seminar and the discussions with the teachers during the course show that in general it is possible for the teachers to plan, carry through, evaluate and report about problem oriented modelling projects in the gymnasium. In the vast majority of the experimental courses it appears that the students have actually done a project directed by a problem and have engaged seriously in mathematical modelling activities. Supported also by teachers' written evaluation of the in-service course we tent to conclude that the course is quite effective in relation to the objectives given.

Analysing the reports from the perspective of teachers' professional development it has become very clear that the most successful projects are those where the teachers have been very explicit and focused about their intentions for the students' learning as in the asthma drug project. In these cases the students' work has been evaluated against specific intentions and hereby provided the teachers with the possibility to reflect very specifically on how to improve the project. Accordingly, in our future in-service courses we would like to be even more supportive for the teachers in their formulation of explicit intentions for their students' learning.

Another important common characteristic in the experiment projects developed at the in-service course is the interplay between the overall scene setting for the projects and the possibilities for the teachers to get into dialogues challenging and supporting the students during their work without taking over the students' work.

As discussed the differences between the asthma drug project and the swimming pool project illustrate the importance of the scene setting. Also in the other projects, the teachers' preparatory discussions about how to set the scene for the students' modelling activity prove to be crucial. These discussions let the teachers to decide about which information the students should be given in the written material and where to leave decisions to the students and to open for the students' own collection of

information and data.<sup>14</sup> In some projects it also gave raise to discussions among the teachers about how to challenge the students through dialogues in possible situations that might occur during the students' modelling work. In general the experiences from the in-service course underline the crucial importance of the introductory phase of a modelling project, where the scene is set in order for the students to be able to engage in serious modelling activities.

Of course, the really interesting thing to know about the in-service course is if and how the teachers would teach project work and mathematical modelling in the future. We only have a few incidentally personal reactions from teachers telling us that the in-service course have made a difference for their way of teaching - not only in mathematical modelling. If possible we would very much like to do a survey on this question in a few years time.

From a research point of view the in-service course already gives basis for describing and analysing the following three interrelated dilemmas of teaching mathematical modelling.

- (1) The understanding of mathematical modelling competency from a holistic point of view or as a set of sub-competencies.

In order for the teachers to support the students' development of modelling competency effectively they need to be aware of the different sub-processes involved in mathematical modelling and about how to support and challenge the students to work with these. The teachers tend to focus on the inner part of the modelling process, e.g. process (d)-(f) in the modelling cycle. These processes are closest to the mathematical content that teachers are used to teach and they expect with good right that the students need much support in connection with these processes, especially that of mathematisation. However, for the students to develop modelling competency they need to work with the entire process - not always, but

<sup>14</sup> Kaiser & Schwarz (2006) in this issue also point to the fact – without using the term scene setting – that the framing and degree of openness of the modelling tasks are crucial for the students' motivation and the way their modelling work can develop.

sufficiently often, and they need to develop along the road a meta-cognition of what it means to practice mathematical modelling. This dilemma of balancing between a holistic and a reductionistic approach became evident in many of the projects developed at the in-service course<sup>15</sup>.

- (2) Seeing mathematical modelling as an educational goal in its own right or as a mean for motivating and supporting the students' learning of mathematics.

As described, in our interpretation the new curriculum includes the development of the students' mathematical modelling competency as a goal in its own right. On the other hand mathematical modelling in the gymnasium should also be seen as a didactical mean of teaching mathematics that place the relation between real life and mathematics into the centre of teaching and learning mathematics, and this is relevant at all levels. Modelling activities may motivate the learning process and help the learner to establish cognitive roots for the construction of important mathematical concepts. However, these two perspectives can not be pursued simultaneously but need to be balanced in the practice of teaching mathematical modelling. The case of the swimming pool project shows very clearly that the teachers' intention of presenting interesting and advanced mathematical ideas to the students was preventing the students from engaging in relevant modelling activities. In the asthma project it seems as if there was a successful shift in the project from the modelling perspective to a mathematics perspective concerning the nature of an exponential development and back again to the modelling perspective.

- (3) The dilemma of teaching directed autonomy.

In order to develop modelling competency in its full entirety the students – at some stage – need to take over the responsibility of all parts of a modelling process. Especially in relation to the

initial parts of the modelling process, e.g. process (a) and (b) in figure 1. This constitutes a didactical dilemma. The students need to be responsible for most of the decisions but the decisions also need to be “the right ones” (or at least feasible ones) in order for the students to be able to carry through the modelling process. This is the dilemma of teaching directed autonomy (Blomhøj & Jensen, 2006). The institutional context has strong influence of the teachers' possibilities for handling this dilemma. And so do the teachers own beliefs about learning, mathematical modelling and its role in mathematics teaching as discussed by Kaiser & Maaß (2006).

The conditions for most of the experimental projects developed at the in-service course did not allow the students to use very many hours on the initial parts of the modelling process. From a psychological point of view it is also important that the students do not get too frustrated in the beginning of the project. Problem oriented project work is a way to handle this pedagogical dilemma, but of course it still leaves the teacher with the important non-trivial task of finding, developing, and negotiating relevant problems for the students to work with. Such experiences from teaching project work in mathematics – supported by the in-service course – may eventually facilitate the integration of mathematical modelling in the practice of mathematics teaching.

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<sup>15</sup> This dilemma is discussed extensively in Maaß (2006) in this issue, both from a theoretical and an empirical point of view.

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