Commentary on "The articulation of symbol and mediation in mathematics education" by Moreno-Armella and Sriraman[1]

Gerald A. Goldin (USA)

Abstract: In this paper the author critiques and comments on the global ideas presented by Moreno-Armella & Sriraman (2005) on the development of representational systems drawn from the prehistory and history of mathematics, their discussion of tools and technology as mediators of mathematical action and cognition and the claim that we regard present-day computational media as mediators and mathematics itself as constituting "symbolic technology."

ZDM Classification: C30, C50

This is a truly ambitious paper, evoking a series of great and sweeping ideas – the need for global theories, aspects of the human brain, implicit *vs.* explicit cognition, the role of human intentionality, and the seminal thinking of Charles S. Peirce (1998) on the nature and evolution of symbolic representation. After mentioning some cogent examples of the development of representational systems drawn from the prehistory and history of mathematics, Moreno-Armella and Sriraman turn to a discussion of tools and technology as mediators of mathematical action and cognition. With some problem solving and reasoning situations in mind, they suggest we regard present-day computational media as such mediators; indeed, they see mathematics itself as constituting "symbolic technology." The discussion is all taken to be part of a necessary "pre-theory'' of mathematics education, pointing the way to the eventual unifying framework the authors favour.

It is not easy to comment at such a broad level of generality. Despite a few quibbles (e.g., I would not call mathematics "*strictly* symbolic"), I would basically agree with the essential importance of all these ideas, while arguing for the inclusion of a few more (see below). There are some useful points of contact with the "models and modelling" perspective on mathematics education, advanced in the recent volume edited by Lesh and Doerr (2003). Let me highlight here one compound idea, articulated in the article, that I believe to be especially deserving of attention; namely, *"Mathematical symbols co-evolve with their mathematical referents and the induced semiotic objectivity [allows] them to be taken as shared in a community of practice."*

In connection with this notion of co-evolution, we may focus on either of two distinct sorts of "referents". One possibility is to consider referents that are themselves encoded in a previously-invented, well-developed system of symbolic representation. Then, in discussing the co-evolution, we may give central attention to how the prior symbol-system undergoes change and extension as the new system develops. A second possibility is to consider referents that are encoded imagistically – e.g., visually, spatially, kinaesthetically – or, more broadly speaking, conceptually. Here we may discuss the co-evolution without, or apart from, any prior mathematical/symbolic system of representation, giving central attention to conceptual change as we look at the co-evolution. In practice, we have both kinds of referents in many mathematical situations. Moreover, notational or conceptual changes in the system of referents may happen not only in response to the reciprocal influence of the new system, but also in response to other influences.

In earlier discussions of the development of new representational systems modelled on pre-existing systems, I focused on three stages – (1) a "semiotic" stage, in which symbols in the new system are first assigned meanings with reference to configurations in the earlier system (for example, the introduction of the letter 'x' to stand, in various contexts, for a particular real number whose value is unknown); (2) a "structural development" stage, in which relations in the new system are built up modelled on properties of the earlier system (for example, the development of notation and syntax for algebraic expressions and equations using letters, modelled on standard arithmetic notation for numbers); and (3) an "autonomous" stage, where the new system becomes "detached" from the necessity of its earlier meanings, acquiring new interpretations in new situations and functioning independently of the earlier system (for example, 'x' comes to be regarded as a variable having a range of possible values, rather than as standing for a particular number; and algebraic expressions come to be manipulated as objects in their own right). Such an analysis may be applied to the historical development of representational systems, or to their cognitive development within the individual learner (Goldin, 1998; Goldin & Kaput, 1996).

But the description of such stages tacitly regards the prior representational system as held fixed while the new system evolves. It neglects the "co-evolution" that Moreno-Armella and Sriraman so rightly highlight in the present article. Thus it can be, at best, only a first approximation to a good theoretical framework. In the example of a developing algebraic symbol-system, both the formal symbolic system encoding numbers and the underlying imagistic representations of "number" undergo changes consequential to the algebraic system – both historically, and cognitively in the individual. To cite just one example, we have the introduction of the "imaginary number" *i* as a solution of the quadratic

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equation $x^2 + 1 = 0$, so that i is the square root of -1. Then earlier-developed properties of "numbers" generalize, so that $-i$ is identified as a distinct "imaginary number" satisfying the same equation, while $3 + 5i$ becomes a "number" of a new sort, a "complex number". Complex numbers then come to be represented as points in a "complex plane" within which the "real number line" is embedded. And now the concept of number, as well as the set of possible domains of variables in algebraic expressions, has been enlarged.

The second part of this important idea has to do with "semiotic objectivity" permitting the shared interpretation of mathematical symbols in a "community of practice." This alludes to the social and cultural dimension of symbolic cognition. But the sharing of mathematical symbols and underlying concepts is not just a *result* of the process of co-evolution – it is an essential part of the mechanism through which co-evolution occurs. In an immediate sense, we have a small-scale "community of practice" in every mathematics classroom, where meanings of mathematical symbols are negotiated and established, and representational systems co-evolve. Here the norms and expectations of students' out-of-school neighbourhoods, families, language groups and cultures may mesh with – or clash with – the norms and expectations of school cultures. To successfully teach conceptually challenging mathematics in America's inner-city classrooms, often situated in low-income, predominantly minority communities, such factors need to be explored for the untapped resources they can offer. In some classrooms, the students may come to "speak mathematics" with each other, and come to "own" the symbolic meanings they have ascribed to symbols, in a process having many analogies to the learning of a natural language. In other classrooms, the students may carry out symbol-manipulation procedures quite detached from any shared understanding or "semiotic objectivity," remaining to varying degrees "apart from" or "outside" the semiotics. The "maps" from students' developing internal cognitive representational systems and their referents to the idealized external (shared) system and its referents are not only imperfect, but are likely to be culturally dependent.

In the context of these observations, I find myself comfortable with the notion of mathematics as "symbolic technology;" with tools and technology as mediators of action and cognition, particularly in classrooms. This is, in a sense, a traditional "cognitive science" perspective (Davis, 1984). But I think there could also be some value in shifting our perspective, to see mathematical action and cognition as mediating symbolic (co-)evolution. For example, it is suggested that we study further the learning of group theory and proof in computational environments. However, I would conjecture that the effectiveness of such learning is highly dependent on which fundamental mathematical entities are represented in the computational medium, and how they are represented. Do they build from students' prior bases of experience, or not? As mathematical actions and cognitions occur and develop, mediated by technology, so should these actions and cognitions eventually result in *modifying* the technology – e.g., reprogramming the computer toward representations that are conceptually more salient. Thus it is important that the *locus of executive control* in our technological mathematical environment remain with the human learner.

The goal of a global theoretical framework for mathematics education is certainly ambitious, though I am not entirely sure of my interpretation of the authors' intent. My own research direction has mainly focused on achieving a unifying framework for describing individuals' mathematical learning and problem solving, a far more modest objective that is nevertheless daunting. One feature of a global framework might be the ability to work with a variety of different units and/or levels of analysis. Possibilities here include the study of particular mathematical concepts, lessons, or lesson sequences; the individual learner's cognition and affect during mathematical problem solving and learning; pathways of conceptual change and learners' conceptual development over time; small-group interactions during mathematical learning; individual teacher behaviours; school classes over the course of a unit or semester; students' and teachers' structures of mathematical beliefs (Leder, Pehkonen, and Törner, 2002); school and community cultures; policies pertaining to mathematics education; as well as numerous aspects of social, cultural, societal, historical, epistemological, or abstract mathematical dimensions of education. With such a lofty, all-embracing, and necessarily complex edifice in mind, even the "building blocks" discussed in the present article may not suffice. At least, I would like to suggest two further emphases, alluded to tacitly in the present article, that I think that are aligned with the authors' focus on symbol and meaning in mathematics and essential to creating a "global framework" as proposed.

The first of these is the study of *affect* as a system of internal representation, and its particular role in relation to symbolic cognition. This includes the *symbolic function* of emotional feelings, shared affect, and the development of affective structures around mathematics. Among the meanings encoded by emotions (and co-evolving) may be information pertaining to the mathematical problem being solved, the mathematical concept being learned, or the relation of the student himself or herself to the mathematics. Of special interest are recurring sequences of emotional feelings, or *affective pathways* that may occur, contributing to the construction of global affect – affective structures such as mathematical integrity, mathematical self-identity, and the capacity for mathematical intimacy (DeBellis and Goldin, 2006, in press). The referents of emotional feelings are highly ambiguous and context-dependent. In particular, *meta-affect* (i.e., affect about affect, affect about cognition about affect, affective monitoring of cognition and affect) may profoundly transform emotional feelings in relation to mathematics.

The second idea needing further discussion is the role of *ambiguity* in mathematical cognition. This includes ambiguity within a representational system (e.g., in the construction of symbol-configurations from primitive signs, or in their structural relation to each other), and ambiguity in the referential relationships that may exist between or across systems. Resolution of ambiguity may make reference to the representational system itself, to contextual information outside the system to which the ambiguous symbol-configuration belongs. "Mathematical ability" sometimes translates into skill in resolving ambiguities from contexts, or skill in interpreting the tacit assumptions of teachers, textbook authors, examination writers, or the "school mathematics culture."

To sum up, Moreno-Armella and Sriraman have suggested many broadly-formulated but essential notions, some more well-known than others, in their "pre-theory" exploration. I have discussed but a small subset of these, and suggested the importance of further, broadly-formulated ideas. However, my view is that the really difficult task lies ahead. It is to go beyond this very general level, to create a detailed, specific, and practically useful characterization of the processes of mathematical learning and development – a characterization that takes realistic account of the mathematical, psychological, and sociocultural complexities that intersect in the educational domain.

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Author

Gerald A. Goldin
Professor of Mathematics, Physics, and Education
SERC Bldg. Rm. 239, Busch Campus
Rutgers University
118 Frelinghuysen Road
Piscataway, NJ 08854
USA
E-mail: gagoldin@dimacs.rutgers.edu