

English, Lyn (Ed):

## **Mathematical and Analogical Reasoning of Young Learners.**

New Jersey: Lawrence Erlbaum & Associates, 2004, 224 pages, ISBN 0-8058-4945-9.

Bharath Sriraman, The University of Montana

### **Polya revisited: The cognitive links between mathematical and analogical reasoning**

#### **Introduction**

Reasoning by analogy is to mathematics and science as metaphor is to literature and philosophy. The preceding sentence has the logical structure of a typical GRE test item, namely  $A:B :: C:D$  but in a metaphorical sense hopefully illustrates to the reader the propensity of human beings to analogize. It however begs the question as to why we should bother with this strange albeit useful human trait. Simply put, this book explores the relationship between everyday analogical reasoning and mathematical reasoning in ten comprehensive chapters. It is tempting to ask what, if any relevance does the study of analogical reasoning have for mathematics education? In order to answer this question, let us examine the use of analogies in mathematics. Consider the following rather evocative quote (Corfield, 2003) by the French mathematician André Weil.

“As every mathematician knows, nothing is more fruitful than these obscure analogies, these indistinct reflections of one theory into another, these furtive caresses, these inexplicable disagreements; also nothing gives the researcher greater pleasure”...

#### **The use of analogies in mathematics**

Examples of the powerful uses of analogies in mathematics abound. Polya's (1954) classic treatise reveals the critical role that analogies play in the discovery of or (for the non-platonists) the creation of solutions to troubling problems as well as leading to new mathematics. In my opinion, the most beautiful illustration used by Polya (1954) is Bernoulli's solution to the troublesome Brachistochrone problem by constructing the appropriate analogy with the path of light in the atmosphere. Polya's (1954) rich collection of examples show that analogies can unify seemingly disparate ideas across disciplines. Hilbert (1900) in his famous address at the International Congress in Paris spoke of mathematics as a whole. He said that although mathematics was expanding into specialized branches one could discern the similarity in the logical devices used, the relationship of the ideas in mathematics as a whole and the numerous

analogies in its different departments. Barrow (1992) brings us back to the literary and philosophical realm by posing the metaphysical question “What is mathematics?” His answer

Mathematics is also seen by many as an analogy. But, it is implicitly assumed to be the analogy that never breaks down. Our experience of the world has failed to reveal any physical phenomenon that cannot be described mathematically. That is not to say there are not things for which a description is wholly inappropriate or pointless. Rather, there has yet to be found any system in Nature so unusual that it cannot be fitted into one of the straitjackets that mathematics provides. This state of affairs leads us to the overwhelming question: Is mathematics just an analogy or is it the real stuff of which the physical realities are but particular reflections? (Barrow, 1992, pp.21-22)

The popular press' coverage of the solution to Fermat's Last Theorem illustrated the intricate use of analogies to convey to the general public the essence of Andrew Weil's work. In the NOVA (1997) video “The Proof” the use of verbal analogies and metaphors were invaluable (and ingeniously used) to convey to the general public notions of modular functions, elliptic equations, Galois representations and other abstract mathematical ideas. These are but a handful of examples that illustrate the value of as well as the propensity to reason by analogy in mathematics and communicate via analogy about mathematics. Yet, the phenomenon of analogical reasoning has received scant attention in mathematics education research. Problem solving research in the 1980's partly focussed on analogical behaviors engaged in by expert and novice problem solvers. This research revealed that expert problem solvers in mathematics and science engaged “in metaphorical processes as they constructed problem representations, they looked for analogies between the problem at hand and other familiar situations” (Silver, 1987, p.45). In the last 15 years a majority of mathematics education books have focused on the social and cultural issues related to mathematics learning/teaching. Yes, the social and cultural dimensions are important and relevant but convey only part of the spectrum of mathematical learning.

#### **What is the book about?**

The cognitive dimension of mathematical learning completes this spectrum and is equally important but has received less attention. This book fills a niche in the current literature. It takes us back to the very roots of learning by investigating foundational questions on the nature of, the evolution of, and the complexity of reasoning in young children. After reading the first chapter one couldn't help but recall Dubinsky's (1999) review of an earlier book edited by the same author (English, 1997) which examined the use analogies and metaphors in a global and theoretical sense. Aptly enough it was easy to recollect this review because of an analogy used by Dubinsky (1999) in which the book was compared to a sandwich! Dubinsky (1999) said, “As with many sandwiches the bread is more interesting than the

innards.”(p.555). Therefore the reviewer was skeptical about the innards of this book.

The innards extensively report the results of a 3- year longitudinal and cross-cultural study of the analogical and mathematical reasoning patterns of young children (4-7 year olds) in Australia and the United States. Cross-cultural themes are partially addressed from the standpoint of the variance or the invariance of the findings across these two cultures. The innards are as interesting as the bread in this case!

Earlier in this review some examples were used to demonstrate the value of analogies both in everyday life and in mathematics. Clearly the use of analogies is part of a mathematician’s repertoire. As argued earlier, this suggests examining the development of mathematical reasoning and its connections to analogical reasoning in young children is an important and under-investigated area with tremendous implications for both the teaching and learning of mathematics. The question then is (1) how do we study this phenomenon? (2) What is the relationship between everyday analogical reasoning and mathematical reasoning? (3) Is reasoning by analogy in mathematics a natural phenomenon in children or is it something that is learned in a formal setting? Before we embark on a pursuit of these answers in the book, a working definition of “analogy” is called for.

Cognitive scientists define an analogy as a mapping from a base to a target (Gentner, 1983). For instance in the analogy, “feathers are to birds as fins are to fish” [feathers: birds:: fins: fish] the base is “feathers : bird” and the target is “fins : fish”. The base elements are mapped into the target elements such that the relations in the base and target correspond. The relation in this example is not completely transparent but something we will come back to. On the other hand in the analogy den:tiger :: burrow:rabbit , the relation is obviously “shelter of”. By analogy (pun-intended), in mathematics ideally two systems are analogous only if there are clearly definable relations of the respective parts. One can no longer focus on just any superficial type of similarity, rather one must get down into the structure. The first example could result in either a superficial type of relation such as “appearance of” or it could be taken down to the structural level where “exoskeletons of” could be considered. In general analogy is defined as the ability to reason with relations, to detect patterns, identify recurrences given superficial variations and abstract from these patterns (Gentner, Holyoak & Kokinov, 2001). One notices immediately that this general definition lends itself to the specific context of mathematics. Formulating the statement of a theorem requires the abstraction of structural relationships from a class of varying examples within which a particular pattern re-occurs.

The book essentially seeks to understand the relationship between mathematical reasoning and children’s “natural” tendency to create analogies. The research reported in the book includes a careful exposition of and an analysis of the measures constructed to study the growth of analogical reasoning. In the second chapter Alexander and Buehl’s attention to ensuring construct validity and item reliability in the instruments provides great overall credibility to the findings. The items increased in

difficulty over the course of the three years starting with plastic attribute blocks of varying colors and shapes progressing to pictorial and geometric analogy tasks in the second year, followed by geometric and verbal analogy tasks in the third year (via attribution and sequencing problems). In addition, in the third year the children were asked to construct two analogy items. In the third chapter Buehl and Alexander report the statistical results of the study. Overall, the researchers found that the development of mathematical and analogical reasoning in young children followed a similar developmental path, with some individual variation but insignificant cross-cultural variation. The insignificant cross-cultural variation is attributed to the similarity in the socio-economic demographics in the Australian and U.S schools. The improving trends seen in the quantitative data, which suggests that young children do reason analogically and the reasoning does improve with maturation, is consistent with the massive literature on children’s mathematical reasoning. However the general literature does not consist of longitudinal studies that document how this reasoning ability changes and improves over an extended time period, and even more importantly how these changes relate to the classroom instruction. This issue is tackled in depth in chapters 4 and 5.

These chapters focus on narratives of classroom discourse and an analysis of discourse patterns that aid the development of mathematical and analogical reasoning. The discourse vignettes provides the reader a glimpse of the ability of children to reason in non-contrived contexts. Chapter 5 consists of three student case studies in which Deal and Hardy paint a descriptive picture to complement the quantitative data reported earlier. It is alarming to note that although analogical reasoning was never explicitly taught the children improved significantly in this facility, whereas their mathematical reasoning did not show any significant or dramatic gains in spite of the explicit instruction. This finding although alarming is consistent with Sfard’s (1994) claim that since language comes a priori to mathematics, one has a natural facility to manipulate language to create analogies and metaphors. However since language is used as a vehicle to construct new mathematics, it is natural to expect that children have difficulty with “manipulating” mathematics in spite of explicit instruction.

Chapters 6 and 7 are devoted to investigating the beliefs of the teachers about the development of mathematical and analogical reasoning of their students. These two chapters are like the final pieces of a jigsaw puzzle that enables the reader to construct a complete picture of the study. Our understanding of teaching from teachers’ perspectives complements our growth of understanding of learning from learners’ perspectives, which in turn, enriches the idea of schooling as the negotiation of norms, practices and meanings (Cobb, 1988). In Chapter 6, White, Deal & Deniz provide a substantial theoretical overview for the lay person on teacher knowledge, beliefs and practices and report on teachers’ conceptions of mathematical and analogical reasoning. The next chapter contains three teacher case studies constructed from interview data. The teachers in the three case studies are

classified as “theoretical”, “experiential” and “intuitive” on the basis of their beliefs and pedagogical practices. In this reviewer’s opinion this triadic classification is very reminiscent of Alba Thomson’s seminal work in the field of teacher beliefs (e.g.,Thompson,1984). Another interesting angle to pursue with the interview data in future work is to explore the relationship between the teachers’ philosophy of mathematics and their beliefs about teaching and learning in lieu of the prior work of Lerman (1990).

Commentaries by Goswami, and Hatano and Sakakibara, in chapters 8 and 9 respectively critique the results reported in the preceding chapters. Hatano and Sakakibara explore Cobb’s (1988) ideas and discuss how mathematical knowledge is acquired via social interaction. The rich data generated by exploring the question of the role of classroom discourse in the development of analogical and mathematical reasoning is analyzed with the much larger framework of educational psychology. The commentators observe that spontaneous analogies employed by children in everyday language (in natural settings) are by and large absent when children employ the language of mathematics (engage in mathematical reasoning). This finding is attributed to limitations in a child’s mathematical knowledge as well as to the norms and practices transmitted to students in an instructional setting.

Goswami’s commentary from the cognitive science perspective is piercing and exposes the strengths and weaknesses of the results in the previous chapters. The reviewer concurs with Goswami’s conclusion that the main hypothesis of the study, namely, there exists a reciprocal relationship between pattern recognition and application done analogically with the ability to recognize numerical patterns and create abstractions and generalizations when reasoning mathematically, is validated by the study. Although this research is a first step in better understanding the analogical tendencies in children’s mathematical reasoning, the direct implications of these findings for teaching are tentative. To put this rather bluntly from the practitioner’s viewpoint, is it pedagogically appropriate to use analogies when teaching mathematics? Goswami is cautious about answering this question and points out the paradoxes that arise when using base-10 blocks as a concrete analogy of the base-10 system as many children have difficulties with discerning the structure of the source analog. “Is it possible to use analogies to teach mathematics when the best analogies (concrete representations for base-10 numbers) themselves require interpretation and understanding?” (Goswami, 2004, p.184).

### Concluding Points

An important point conveyed by the editor in the concluding chapter is that teachers must understand the analogies themselves and know how to use them effectively (and also know which analogies are appropriate and which aren’t when it come to their use). They have to make the relationships explicit for the child. Teacher knowledge is crucial here. The editor also points

out that the analogical paradoxes arising in mathematics are quite different from those arising in a discipline such as the life sciences where spontaneous analogies work well because children have a much larger a priori linguistic base, whereas in mathematics children’s pre-existing knowledge base is limited. This necessitates that both practitioners and researchers are sensitive to the major role that the knowledge base plays in the use of analogies for mathematics learning. The examples of the use of analogies in professional mathematics at the start of this review suggests that a better understanding of children’s natural tendency to use analogies should be harnessed and properly nurtured within the context of mathematics. The paradoxes that arise from the haphazard use of concrete materials suggest we re-examine the prescriptive instructional practices that have become the norm in elementary schools. The book’s attempt to create a research base on children’s analogical reasoning and its relationship to mathematical reasoning is a giant first step in the right direction. It is certainly an eye-opener for anybody interested in complexities of the teaching and learning of mathematics.

### References

- Barrow, J.D. (1992) *Pi in the Sky: Counting, Thinking, and Being*. Little Brown and Company.
- Cobb, P. (1988). The tension between theories of learning and theories of instruction in mathematics education. *Educational Psychologist*, 23, 87- 104.
- Corfield, D. (2003). *Towards a Philosophy of Real Mathematics*. Cambridge University Press.
- Dubinsky, E. (1999). Book review of Mathematical reasoning (Edited by Lyn English). *Notices of the American Mathematical Society*, pp. 555-559.
- English, L.D. (Ed.). (1997). *Mathematical Reasoning: Analogies, Metaphors and Images*. Lawrence Erlbaum & Associates.
- Gentner, D. (1983). Structure mapping: A theoretical framework for analogy. *Cognitive Science*, 7, 155-170.
- Gentner, D., Holyoak, K.J., & Kokinov, B.N. (Eds.). (2001). *The analogical mind: Perspective from cognitive science*. MIT Press.
- Goswami, U. (2004). Commentary: Analogical reasoning and mathematical development. In L.D. English (Ed.), *Mathematical and Analogical Reasoning of Young Learners* (pp.169-186). Lawrence Erlbaum & Associates.
- Hilbert, D. (1900). Mathematische Probleme: Vortrag, gehalten auf dem internationalen Mathematiker-Congress zu Paris 1900. *Gött. Nachr.* 253-297.
- Lerman, S. (1990). Alternative perspectives of the nature of mathematics and mathematics teaching to instructional practice. *Educational Studies in Mathematics*, 15, 105-127.
- Nova (1997). The Proof. 60 minute PBS video.
- Polya, G. (1954). *Mathematics and Plausible Reasoning: Induction and Analogy in Mathematics* (Vol.1) Princeton University Press.
- Sfard, A. (1994). Reification as the birth of metaphor. For the Learning of Mathematics, 14,1,44-55.
- Silver, E. (1987). Foundations of cognitive theory and research for mathematics problem solving. In A. Schoenfeld (Ed.) *Cognitive Science and Mathematics Education* (pp.33-60). Lawrence Erlbaum & Associates.
- Thompson, A. G. (1984). The relationship of teachers conceptions of mathematics and mathematics teaching to instructional practice. *Educational Studies in Mathematics*,

15, 105-127.

---

**Author**

Bharath Sriraman.  
Dept of Mathematical Sciences;  
Editor, The Montana Mathematics Enthusiast  
<http://www.montanamath.org/TMME>  
The University of Montana,  
Missoula, MT 59812,  
USA.  
Email: sriramanb@mso.umt.edu