Connecting Arguments to Actions – Dynamic Geometry as Means for the Attainment of Higher van Hiele Levels

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Abstract: New technology requires as well as supports the necessity to raise the level of geometric thinking. Freudenthals view of van Hiele's theory corroborates a dynamic multi-level curriculum that offers material support for higher levels. For levels higher than 2, the dynamic locus capability of Dynamic Geometry software is crucial, e.g. in the study of loci of orthocentres and incentres. Discrepancies between their algebraic and geometric descriptions can motivate a deeper involvement with basic curve theory on the side of the teacher, who thereby can predict in which cases the students may succeed in restructuring the construction to overcome the discordance.

Kurzreferat: Die Erhöhung des geometrischen Denkniveaus wird von Neuen Technologien sowohl erfordert als auch unterstützt. Freudenthals Sicht der van-Hiele-Theorie bestärkt ein stufenübergreifendes, dynamisches Curriculum mit materialbasiertem Zugang zu höheren Niveaus. Ab Stufe 3 ist die dynamische Ortslinienfunktion der Dynamischen Geometrie-Software essentiell, z.B. um Ortslinien der Schnittpunkte von Höhen bzw. Winkelhalbierenden zu untersuchen. Diskrepanzen von algebraischer und geometrischer Beschreibung können Lehrende zur Auseinandersetzung mit elementarer Kurventheorie motivieren – womit sie vorhersagen können, in welchen Fällen es Schülern gelingen kann, durch Restrukturieren der Konstruktion die Diskrepanzen zu überwinden

ZDM-Classifikation: C30, C70, E40, G10, G70, U70

1 Background

1.1 The van Hiele levels

Certainly a central aim of geometry teaching is to raise the level of students' geometric thinking. The model of van Hiele (1986) provides a concise description of students' status and progress. It has been adopted as rationale for the arrangement of tasks and topics in subsequent grades by the NCTM standards (NCTM 2000), aiming at advancement to higher levels: "Systematic reasoning is a defining feature of mathematics. Exploring, justifying, and using mathematical conjectures are common to all content areas and, with different levels of rigor, all grade levels." The usual interpretation of the levels reads as follows: "Level 0 (Visualization): Students recognize figures by appearance alone, often by comparing them to a known prototype. The properties of a figure are not perceived. Level 1 (Analysis): Students see figures as collections of properties. They can recognize and name properties of geometric figures, but they do not see relationships between these properties. When describing an object, a student operating at this level might list all the properties the student knows, but not discern which properties are necessary and which are sufficient to describe the object. Level 2 (Abstraction): Students perceive relationships between properties and between figures. At this level, students can create meaningful definitions and give informal arguments to justify their reasoning. Logical implications and class inclusions, such as squares being a type of rectangle, are understood. Level 3 (Deduction): Students can construct proofs, understand the role of axioms and definitions, and know the meaning of necessary and sufficient conditions. Level 4 (Rigor): Students at this level understand the formal aspects of deduction, such as establishing and comparing mathematical systems." (Mason 1998, p.4, with the levels' numbering adjusted to van Hiele's counting)

1.2 The role of dynamic geometry

Nowadays dynamic geometry software (DGS) is widely recognized as a tool of visualization that may further students progress. But presently, most of the material proposed for the classroom seems to be concerned with levels 0 (Visual), 1 (Analysis) and 2 (Argumentation). It is well-known that dynamic manipulations help in the transition from the first to the second van Hiele level (Fuys, Geddes & Tischler, 1988). Namely, it is recommended to let students analyse dynamic figures in order to discover their characteristic properties and relate them. Furthermore, Battista (1998) has developed a sequence of activities with the Shape Maker microworld that he claims to "encourage students to pass through the first three van Hiele levels." Level 3 (Deduction) is commonly identified with proof. Its use in classroom is a somewhat controversial issue – the survey of Battista & Clements (1996) concludes that "ironically, the most effective path to engendering meaningful use of proof in secondary school is to avoid formal proof for much of students' work. By focusing instead on justifying ideas we can lead students to appreciate the need for formal proof." But it is questionable that this can be supported well by dynamic visualizations: "Since such [level 1 or2] students do not doubt the validity of their empirical observations, they tend to experience it [deductive reasoning] as meaningless, or 'proving the obvious'." (de Villiers (1999), p.12) Hence there are fewer proposals for level 3 (e.g. loc. cit., p.20) and virtually none for higher levels.

1.3 New goals for geometry teaching

Nevertheless there is a need to further develop these levels – and to utilize DGS for this: "Dynamic geometry environments can (and should) completely transform the teaching and learning of mathematics. Dynamic geometry turns mathematics into a laboratory science rather than the game of mental gymnastic, dominated by computation and symbolic manipulation … As a laboratory science, mathematics becomes an investigation of interesting phenomena, and the role of the mathematics student becomes that of the scientist: observing, recording, manipulating, predicting, conjecturing and testing, and developing theory as explanations for the phenomena." (Olive, in press). This clearly suggests to diminish the role of *separate* deductive activities and to aim at higher levels by "developing theory as explanations for the phenomena", as proposed by Olive. This view is shared by working mathematicians like Whiteley (2000), who stresses the need to bridge the gap between traditional geometry courses and the increasingly sophisticated and widespread applications of geometry in science and daily life, as well as the key role of DGS: “This overlap of learning tools and research tools is very strong and should be made visible in the class room. The student is not going through a phase with an educational toy but is learning a key modern tool of the trade for geometers”

1.4 Freudenthal’s alternative view of the levels

We developed teaching materials that address level 3 (and higher) in a slightly different way than traditionally, but fit to Freudenthal’s account of van Hiele levels: “Good geometry instruction can mean much – learning to organize a subject matter and learning what is organizing, learning to conceptualize and what is conceptualizing, learning to define and what is a definition. It means leading pupils to understand why some organization, some concept, some definition is better than another. Traditional instruction is different… All concepts, definitions and deductions are preconceived by the teacher. “(Freudenthal’s 1973, p.418) In contrast to this, Freudenthal viewed progressive *mathematization* as the main goal of school mathematics. For this ongoing task, he provided a framework by recursively defined levels: The activity of the lower level, that is the **organizing activity** by the means of *this* level, becomes an *object of analysis* on the **higher** level. The box below illustrates this in the case of geometry.

0. Visual: Recognize *objects* as geometric shapes.
1. Analysis: Distinguish *shapes* by their geometric properties.
2. Argumentation: Argue by relations between *properties*.
3. Deduction: Systematize *arguments* to form deductive proofs.
4. Rigor: Analyse *deductive systems*.

Progression through these levels will not occur all by itself, but needs to be triggered by giving the students suitable tasks that really afford the building of new concepts.

1.5 Attainment of van Hiele levels

This is not too high, at least with the activities provided for the levels’ usual interpretation: Healy & Hoyles (2000) investigated in a nation-wide survey the capability of high-attaining British 10th graders to recognize and construct correct proofs. Given the marginal role of these activities in the curriculum, their findings are not surprising: “the majority of students do not incorporate deductive reasoning in their constructed proofs and very few even attempt to construct a formal argument.” (op. cit.) But contrary to common beliefs that proving is more accessible in geometry than in algebra, the attainment there is even lower: for instance, only 19% could give a complete proof of the familiar observation that the angles in a quadrilateral sum up to 360°, whereas 52% had some basis, but could not perform any deductions and 24% could not even find any basis for the proof! The research group of Reiss has replicated and refined these investigations, observing e.g. in Reiss & Thomas (2000) that though many of the students they surveyed (German 13th graders) had enough declarative knowledge to solve the posed proof problems, they lacked the methodological knowledge to organize the relevant facts into a cogent argument. So we can conclude that the majority of students has not mastered the transition from level 1 to level 2 – and in fact, it can be readily observed that this applies as well to a significant portion of teacher students, especially for primary and lower level secondary schools. By mathematics lectures, usually aiming at level 3 or 4, these students are neither furthered in their understanding of proof nor prepared for its application resp. adaptation in their prospective service, as becomes apparent from the following episode:

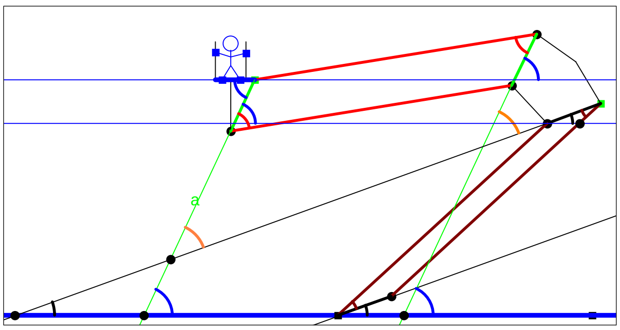
1.6 Problems in teacher education

Two middle school teacher students had to develop a grade 8 lesson on a mechanism for parallel transport by parallelogram linkages. It was surprisingly difficult for them to transform a series of arguments for the right functioning of the mechanism from a jumble of angle theorem applications as in fig. 1 to a neat 5-step-sequence that utilizes the *notion* of parallelogram as given below – even though they were certainly well accustomed to it and had themselves been concerned about their messy sketches! One reason for this difficulty is probably that these students were most likely educated about the matter in rather formal level 3 style at high school – whereas their university geometry courses focussed on an axiomatic level 4 treatment of (non-) Euclidean geometry, probably not mentioning parallelograms at all. Consequently, they felt the need, but were not able to reshape their original proof in a more feasible way – though this is certainly one of the “restructuring activities” in Freudenthal’s description of level 4.

Experiences like this one suggest

- to link the treatment of higher levels more closely to (teacher) students’ actual *geometric experiences*,
- to guide their progression by *material* that is capable of *visualizing* the *concepts of the subsequent level*.

Figure 1: Parallel transporter mechanism in a hoist



2 A material-based approach for the problematic step from level 1 to level 2

2.1 Representation problems in the teaching of proof

An ongoing problem in the teaching of proof as well as in preparational activities is the following: A proof is a sequence of arguments to derive the purported hypothesis from the precondition. In the practice of elementary geometry, however, these arguments are recognized, illustrated and verified by means of suitable drawings. But in these drawings, the hypothesis will always hold true as soon as one arranges the premises to be satisfied – by the very theorem one is going to prove! Especially in passing from van Hiele level 1 to level 2, this is a nearly insurmountable hurdle: Mastering level 1 means the ability to recognize and state true statements about a geometric configuration – now one has to distinguish some pivotal facts that hold in a configuration (the preconditions) from several inessential ones and has draw conclusions from them in order to derive a particular fact, singled out in advance (the hypothesis) – with due care not to make use of any facts that follow only afterwards from this one! But these distinctions are drawn easily (if at all) only by a master's mind – and often, only after a successful completion of the proof. Especially the identification of accidental circumstances is not at all straightforward, since it may well involve a thorough restructuring of both the theorem and the proof – as is brilliantly exposed in the masterpiece of Lakatos (1976).

2.2 Material support

The study of mechanisms can help to make this distinction clear: in their construction, the clear-cut purpose of a mechanism is achieved by imposing suitable conditions on the parts – but usually, not directly, thus invoking the very elementary geometric knowledge the student is to acquire. For instance, by manipulation of the linkage in fig. 1 it becomes clear to the students that

- equally coloured bars have the same length – this is a property that *can be directly controlled*: thus it is a *precondition.*
- equally coloured angles have the same size – this a property that *can **not** be directly controlled*: thus it is a *hypothesis.*

If this distinction is established, one may proceed to formulate a dynamic version of the **Parallelogram theorem:** *A four-bar linkage with opposite bars of equal lengths has opposite angles of equal size.*

At this stage, the dynamics serves a twofold purpose:

1. To distinguish *contingent* from *necessary* facts: That the blue angles in fig. 1 are of size 30° is a *contingent* fact, because it *can* be changed via direct manipulation – that the red bars are of length 5cm is a *necessary* fact, since it can *not* be changed via direct manipulation
2. To distinguish *preconditions from hypotheses*: the former ones are true by virtue of the *utilized constructions* – the latter ones by virtue of *geometric theorems*.

In this way, the study of mechanisms allows to formulate conjectures on level 1 that give rise to an investigation on level 2 – namely: *why* is it that these very preconditions seemingly necessary entrain the observed hypotheses? It is important to understand that the awareness of this necessity itself can - and should! - already be obtained on level 1: By altering some of the prescriptions one can observe which of them are crucial to the truth of the theorem. For instance, just changing the lengths of the red bars is not harmful – as long, as they remain of the same size. We propose to really practise such a check, since all to often preconditions are neither remembered nor can they be utilized for a proof. ("Pythagoras? $a^2+b^2=c^2$!" But the knowledge of the underlying right-angle is sparse…)

Figure 2. Which mechanism ensures parallel transport?

a) b)

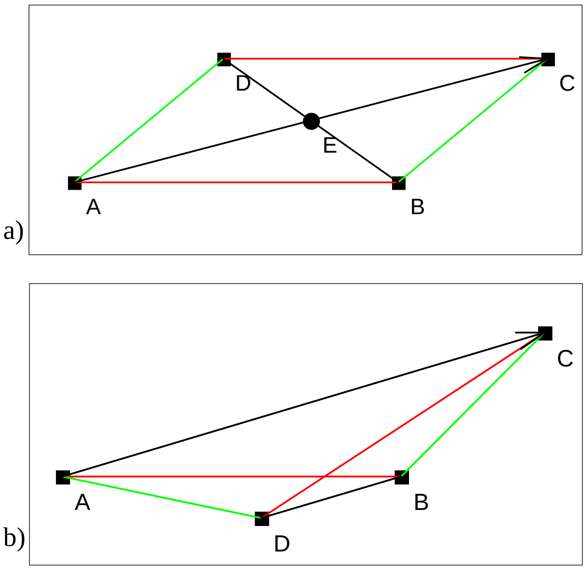
2.3 Experiencing elements of proof

To let students interiorize this check it may be helpful to show that a theorem will otherwise fail by means of memorable examples. Fig. 2a shows such an example: Two mechanisms are tested whether they yield parallel transport of the passengers (observe the teacher students' ingenuity in preparing real models for a hoist). Only the parallelogram mechanism accomplishes this aim (fig.2b)!

After the two distinctions above are established, one may now proceed to investigate the connection of preconditions and hypotheses. In this situation it should be clear that the students favourite technique for answering questions, namely by measurements, is of no use, since thereby one can address only contingent facts, but what is now of interest is rather the interconnection of necessary facts. Nevertheless this can be obtained by performing concrete actions – these must be only of a nature that does not rely on the contingent facts. Measurements necessarily remain on this level – and this is why they are a real barrier towards level 2: they can be readily utilized to check the theorem in every concrete instance but are unsuitable to produce evidence towards its truth in general! But students do possess other techniques - e.g. mirror reflections. And they can convert their ability to always produce the "right" reflection into a sequence of arguments, e.g.: "If the red and the green bars have the same length respectively, we can always find a mirror reflection that transforms one (red, green)-pair of bars into the other one. Namely if the given bars are joined in one end point, say A (they always are joined), we can join also the other end points, B and D, and bisect the segment

BD to get a point E, in which we can reflect the given bars to get the other ones.” (Fig. 3a)

Figure 3. Two configurations of a quadrilateral with opposite sides of equal length



2.4 The strength of material-based reasoning – correcting the parallelogram theorem

Observe, however, that contrary to wide-spread claims in the didactic literature the parallelogram theorem does *not* hold true in general – as one can readily check by a material realization of its construction! Indeed one easily obtains the configuration of fig. 3b by manipulating pairs of sticks in order to form a quadrilateral. It is a good exercise to utilize this mechanism to find out where the “proof” just given falls short – as well as the proof of this theorem in your favourite textbook…

This often-overlooked fact signifies *the strength of material-based reasoning*. Also, it may suit to step forward to level 3, on which one formalizes geometric reasoning to avoid such shortcomings. However, it is well known that the teaching experiences of this approach are not too promising. Also, it may well be doubted if formalization really enables students to avoid or even discover pitfalls not previously known to them.

Rather, we would advocate at this stage to stick to semantic reasoning, but to ensure its correctness by a suitable refinement of the precondition – e.g. if ones states the above theorem as a theorem on quadrilaterals one may introduce the precondition that they may not be crossed. This is a nice instance of a proof-generated notion in the sense of Lakatos (1976): indeed, such a precondition would of course be very artificial in either deductive or formal reasoning as long as its necessity is not demonstrated by virtue of an example. But if it is derived *in the course of the proof*, it becomes feasible. In this vein, deductive reasoning does not stick to the flavour of ‘proving the obvious’ but rather of ‘delineating the extension of truths’.

2.5 The didactical advantage of dynamics

The crucial role of dynamizing the geometric configuration at hand is not only that it enables one to discover a theorem – one may also investigate its scope as well as the validity of a proposed proof: By manipulating the contingent preconditions of the configuration one moves from one instance of the configuration to another one and is able to check whether the theorem and its proof do carry over – or not…

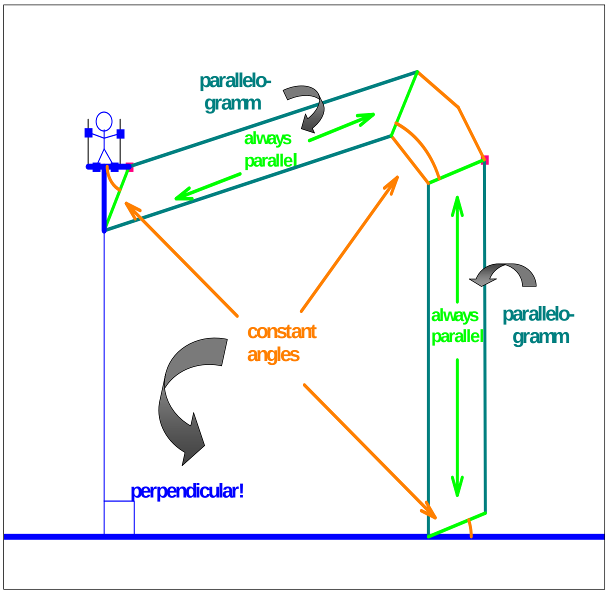
Until now it does not really matter which ones we utilize– as long as the chosen mediums allows one to perform manipulations in necessary generality. E.g., the only DGS that allows one to freely manipulate all vertices of the stipulated parallelogram is the German DGS “Euklid DynaGeo”. However, for the further steps onto higher levels, it becomes increasingly difficult to give material support for the envisaged reasoning. It will turn out that DGS is better suited for this aim since. Before turning to this issue, let us however solve the problem at hand:

2.6. The issue of the parallel transporter mechanism

Restructure the mathematical model as in fig. 4 to observe that

- the mechanism contains two *flexible* parallelograms (green) that – according to the parallelogram theorem! – keep the inclination between the attached parts *constant*,
- the two flexible parallograms are attached to three *fixed* parts (orange) that thus convey also a *constant* angle to the adjacent parts,
- the constant angles are such that combining them, that the platform and the ground are *always* kept *parallel*, regardless of the adjustment of the flexible parts.

Figure 4. Restructuring fig.1 for an argumentation



By means of the parallelogram theorem one can thus explain the correct functioning of the apparatus – and it seems plausible that the application to such a real device sheds some light on the nature of precondition and hypothesis in this theorem also.

The question why there are two parallelograms and not just one yields one step beyond the mere application of mathematical theorems: obviously this does not stem from the functioning of the mechanism alone since that could be achieved with just one parallelogram. Rather, one has to consider questions of stability as well as limitations of size: Both aims can be met clearly better by a machine consisting of two opposite-directed parallelograms.

3 The current and prospective role of DGS as material basis for higher levels

3.1 Material support for higher levels – by DGS

We have seen by the example in 2 that material may faciliate learning in a twofold way:
1. students are supported in the transition from level 1 to level 2 by making clear the distinction of the different epistemological status of cocomittant facts (premises, hypotheses,…) ,
2. teacher students are supported in the transition from level 2 to level 3 by rendering possible the uncovering of gaps in an argumentation to be made formal.

However, the increasingly theoretical character of notions and techniques on higher levels renders support by traditional material more and more difficult. In contrast to this, dynamic constructions are better suited as material base for higher levels since
1. dynamic constructions are subject to direct manipulation by the user,
2. dynamic constructions behave in a way that is directed by the theory to be learnt.

DGS is thus an appropriate material support for higher van Hiele levels. This aspect has been especially stressed by C. Laborde: „Cabri provides a 'real' model of the theoretical field of Euclidean geometry in which it is possible to handle, in physical sense, the theoretical objects which appear as diagrams on the screen. The behaviour of Cabri is based on geometrical knowledge: it offers feedback which can distinguish diagrams drawn in an empirical way from diagrams resulting from the use of geometrical primitives.“ (Laborde 1998, p. 184)

3.2 Current use of DGS on different levels

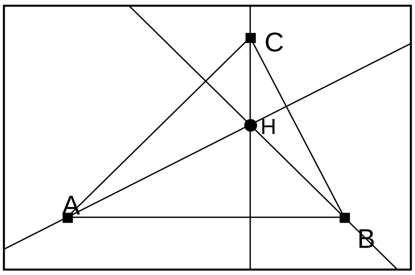
We introduce the role of DGS in the shift of view that we have in mind by an example that illustrates how different van Hiele levels may correspond to the use of the three main instruments incorporated in any DGS, namely **drag-mode**, **macros** and **loci**. E.g. a *dynamic* version of fig. 5 can be utilized

1) to visualize that the altitudes of a triangle *always* meet in one point by **dragging** one of the base points of the construction,

2) to explore the properties of the orthocentre *H* – when does it e.g. coincide with *C*? What if one iterates the construction, namely what is the orthocentre of *ABH* etc.? To this end, one may produce a **macro** to merge the construction steps into a single one that can be readily repeated,

3) to investigate the curve generated by *H* when *A* varies e.g. on a circle through *B* and *C*. This **locus** can be constructed and varied by DGS.

Figure 5. Triangle with orthocentre



Van Hiele levels of may be attributed to these DGS activities as follows: 1) is clearly a level 1 activity. 2) starts at the same level, but may be "upgraded" to level 2, when students use properties of the orthocentre to explain their observations. The same could of course be said of 1), but note that in this case the figure has to be *augmented* and *restructured*: one *embeds ABC* into a larger triangle *DEF* such that the altitudes of *ABC* become the side bisectors of *DEF* and *reinterprets H* as circumcentre of *DEF*. Usually these steps have to be provided by the teacher – whereas in 2) and 3) students can go for their own, sticking to what they already *see* and know. Thereby they can actually set out for the "level-raising process" of concept building envisaged by Freudenthal: "Most definitions are not preconceived, but the finishing touch of the organizing activity."(1973, p.417). How this process is supported on a material base by DGS, can be seen most clearly in 3): Remember Freudenthal's definition that the activity of the lower level, that is the **organizing activity** by the means of *this* level, becomes an *object of analysis* on the **higher** level -*before*, the orthocentre has been the *organizing activity* by the means of level 2 (find a construction in terms of the level 1 properties) – now it is an **object of analysis** on the higher level: its properties become tools for argumentations about its global behaviour which is *materialized* by the locus as simultaneous instantiation of a family of orthocentres. Note that in a dynamic environment, loci indeed have at least partial object status, being subject to the drag mode. So they can also become an object of analysis at level 3! Thus, we can utilize the *dynamic* tools for advancing to *several* higher levels, as depicted below:

> The drag mode is a key tool to advance from level 1 to level 2.
> Macros and loci suit to support the step from level 2 to level 3.
> Families of loci can be used to progress from level 3 to level 4.

This interpretation corresponds to the fact that teaching examples in the literature – addressing the step from level 1 to level 2 – draw mostly upon the drag-mode only. Also, it is clear that macros and loci are just the tools to materialize the "chunking" involved in the building of a new concept by reorganizing an ensemble of geometric constructions to form a new one – and this is precisely what Freudenthal describes as advancement to a new level: an "algorithmically constructive and creative definition…[that] models new object out of familiar ones."(1973, p.458). So, one may utilize a macro for kites to "instrument" (in the sense of Rabardel) the shift of attention from the exploration of "kitehood" on its own to its utilization in more complex activities like tiling the plane – which is a fruitful way of moving from level 2 to

level 3, as described by Mason: "Students perceive relationships between properties and between figures. At this level, students can create meaningful definitions and give informal arguments to justify their reasoning. Logical implications and class inclusions, such as squares being a type of rectangle, are understood." (Mason 1998, p.4)
In the same vein we now propose to utilize loci in 3) to step from investigating *the* orthocentre of *one* triangle to studying *an* orthocentre as member of a *family* of varying triangles. Such a generalization from properties of individuals to properties of the category they form certainly qualifies as a level 3 activity in the above description – and it is also "local ordering" in the sense of Freudenthal (1973) We are thus lead to the following strand of thought:

3.3 A dynamic multi-level curriculum based on the central idea of locus, as materially realized by DGS

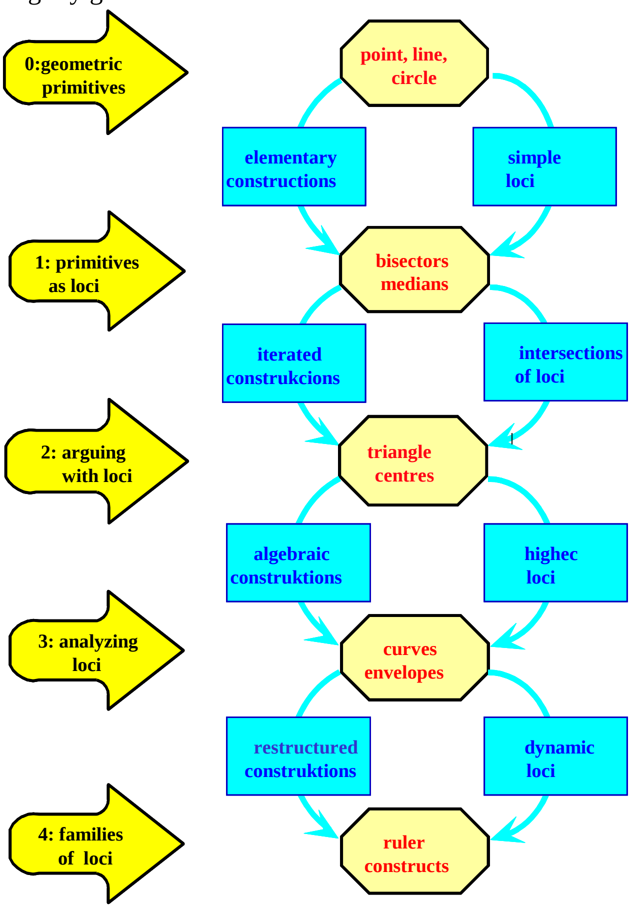
We therefore propose to adjust the curriculum in order to promote the treatment of such level-raising topics that can be approached by DGS-based activities in the same way as successfully established for progressing from level 1 to level 2. In the Freudenthalian spirit, we will not *deduce* the underlying conceptual change from an a priori restructuring of the levels but *establish it as a unifying concept* for our (necessarily brief) sketches of various activity series – this would be an "anti-didactic inversion"! But for the ease of the reader, we will nevertheless state this unifying concept in this paragraph. Note, however, that it really emerged from the concrete activities presented afterwards. To underline the feasibility of our material for (different kinds of) classrooms, we will

- sketch several sequences of such tasks that all start with standard tools and topics,
- let come into view from each one a series of increasingly sophisticated investigations,
- point out how DGS tools can be readily exploited in level-advancing problem solving processes.

But first **the main idea:** Viewing geometric primitives like straight lines ore circles as loci of points with a certain condition is an activity targeted at leading from level 1 to level 2. The content of this activity is closely connected to the insight required in this step for the following reason: The concept of locus right away parallels Freudenthal's view of the level transition: A locus is a set of point that fulfil a certain geometric condition. These points form a new geometric object, usually a curve: in elementary cases, it is a straight line or a circle, in more advanced cases, it is a conic section or an algebraic curve (algebraic as long as the defining condition amounts to a ruler-and-compass construction). In summary:

> Shifting the focus of attention from the **individual points** that satisfy *the defining property of the locus* to *the properties* of **the locus itself** is precisely the same as turning the **organizing activity** of *the current* level, into an *object of analysis* on the **next** level.

The same can be repeated on higher levels: The study of properties of loci leads us from level 2 to level 3. Going one step further, we can consider families of loci again as an object of our study (namely the drag-figure, compare Gawlick (2001) or Gawlick (2002)). This naturally leads to a new view of level 4 also: this level is devoted to the study of family of loci!
Now follows an **overview of the activities** sketched afterwards: The first step is described in A): To make use of such an approach in classroom, it has to be rooted in elementary and well-known concepts, from which new ones gradually emerge – connecting levels 1, 2 and 3 by the same topic: viewing the circle as a locus to solve distance problems. In B) we give an analoguous thread leading from level 2 via level 3 to level 4, namely from properties of the orthocentre to properties of its locus to properties of families of this locus. The more refined this example becomes the more apt is the need for an algebraic restructuring of the geometric situation – we argue that this triggers advancement to level 4. C) poses a problem that rouses the need for this step in the geometric domain. In considering the geometric conditions and implications of such restructurings one practices "global ordering" – level 4 in the sense of Freudenthal (1973). D) and E) explain how to pursue this further systematically for our approach: D) gives a geometric technique in order to overcome the problem that aroused in C): via ruler constructions the realm of elementary constructions is reorganized – i.e. "globally ordered" by feasibility for rulerization. E) gives an algebraic criterion whether this technique can be applied successfully to find a dynamic ruler construction for a given geometric locus. This superimposes a new order to the already globally ordered field – which means that we can interpret this step as leading to level 5.
The whole **multi-level process** can be summarized in this slightly generalized form:



4 Local ordering – how to guide students towards level 3 (and beyond)

A) The circle as a locus – in argumentations leading from level 1 to 3

At level 0, the circle is identified just by its shape. At level 1, students *recognize* its defining property, so to distinguish it from an oval they argue like "this is a circle, because it has constant perimeter". (At that stage, students are not aware of Reuleux triangles – it would be interesting to see whether they can convince them to argue by "constant radius" instead. However, thereby arises the extra difficulty to find the centre of the locus in question.) At level 2, they can *use* this property in solving tasks. We sketch a sequence of tasks to let evolve the notion of circle as equidistancy locus of *one* point and perpendicular bisector as equidistancy locus of *two* points:

a. "Anna lives 2km away from school. Where on the map could her home be ?"
b. "To whose tree do the fallen apples belong?" (Fig. 6)
c. "Where are the points with the same distance from A and B?" (Fig. 7)

In a., students use *one static* circle to find the possible locations of Anna's home - thereby one can root the new notion in a known use of a tool. The locus notion furthermore occurs only implicitly, since no further conclusion are drawn than that the searched point is one of all the circle's points – whereas in the task b. one already needs that this circle contains all points with a certain property (namely lying in a given distance from the tree's centre). Also, it becomes necessary, to consider *two dynamic* circles (resp. a manifold of static ones) – thereby the circles themselves gradually become themselves an object of study. In c., this goes one step further. The dynamic circles are now only tools that provide all points with a certain distance to A resp. B – and now it is necessary to take into account that the circles indeed contain all points with that property. Note the formal character of this argument, clearly aiming at level 3, if compared to the more casual, heuristic use of circles in b. Nevertheless there is a natural bridge between the two tasks – some apples may not belong to a unique tree in terms of proximity – namely those lying on an intersection point of two delimiting circles, or –equivalently, as developed by c. – on the perpendicular bisector of the corresponding tree centres A and B.

Figure 6. Dynamic circles as tool to delimit neighbourhoods

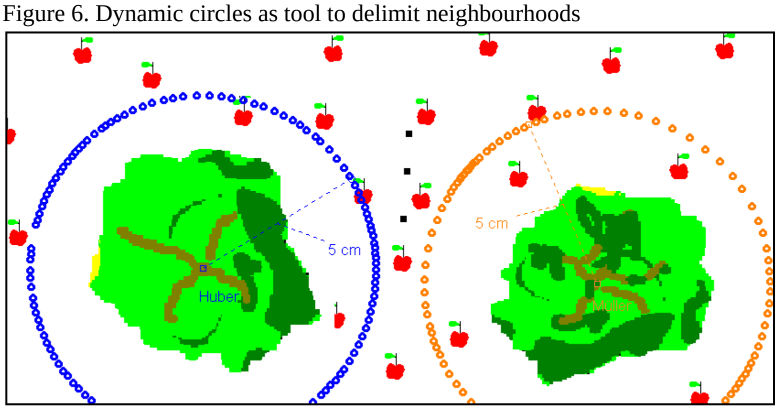
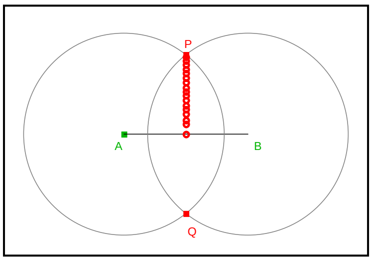
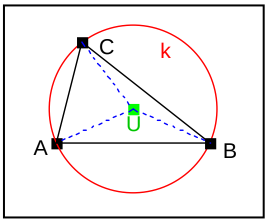


Figure 7. Dynamic circle as tool to determine equidistancy



At level 3 such arguments can be *sequenced* logically: "If U is equidistant to A and B as well as to B and C, then the same holds for A and C. Thus A, B and C lie on the same circle centred at U."(Fig. 8) One strives for an understanding of such proofs that no longer rests upon a visual interpretation. Gawlick (2002a) describes an empirical study on the successful implementation of this material.

Figure 8. Circumcentre as Locus of Equidistancy

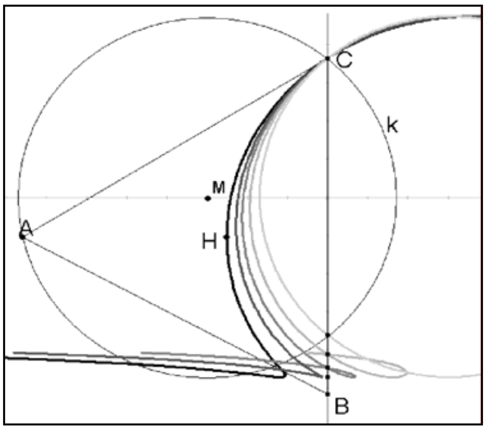


B) Higher locus investigations – from level 2 to 3 to 4

It is known that the locus *O* of the orthocentre *O* of a triangle *ABC* inscribed in a circle k is a circle, when *C* varies on *k* – namely *k* reflected by *AB*. Traditionally, one proves this via transformation geometry, but runs into trouble when trying to *justify rigorously* the visually obvious existence of the reflection. So this should *not* be reckoned as a well suited entrée from level 2 to level 3! But even worse: Hölzl (2001) reports empirical evidence that for many students the hypothesis itself does not seem to be worth noting. In order to elucidate its peculiarity, Hölzl has proposed to untie *B* from *k* and *then* explore the situation (fig. *O*). His classroom transcripts show that students are now motivated

- to explain why *O* is originally a circle (level 2 to 3),
- to survey the whole family and the change of its members' properties – e.g. by deriving and investigating a coordinate equation (level 3 to 4).

Figure *O*. Orthocentre loci with B varying on a line



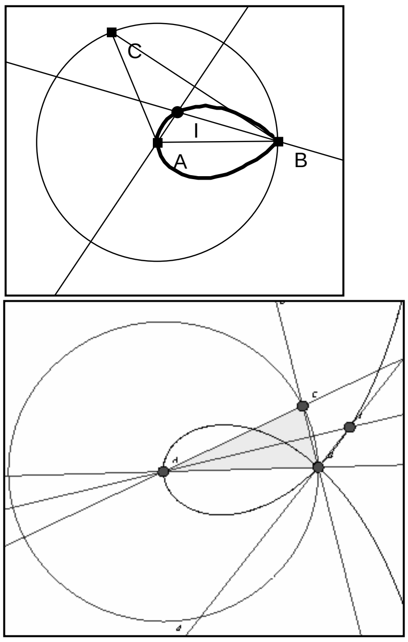
By systematically exploiting this approach of restructuring constructions algebraically and translating the results back to geometry, one can foster the development of the *Cartesian Correspondence* between curves and their equations. Interconnecting previous knowledge in algebra and geometry for utilization as tools, the students' level 3 knowledge now becomes the object of their investigation, which is precisely Freudenthal's description of advancement to level 4! Deinhardt (2000) exhibits this level advancement in the case of a primary teacher student.

C) The necessity to restructure a construction – a problem triggering the transgression of level 3

The locus of the incentre *I* of an isosceles triangle *ABC*, when *C* moves on a circle *k* through *B* centred at *A*. The straightforward way of constructing the locus I yields seemingly a curve with a cusp (Figure *I*a). But elementary trigonometry yields a parametric representation for *I* that can be readily transformed to a locus equation (Warneke 2001). This equation, however, turns out to describe a strophoid – and the strophoid is known to have a node as singular point! What's going on? Plotting reveals that *deterministic* DGS (Gawlick 2002) like "Cabri" or "Sketchpad" yield as locus only the part of the curve within *k*. So one is tempted to conclude that the *continuous* behaviour of "Cinderella" is favourable since it produces the complete locus (Fig. *I*b).

But one has to pay a price for this: namely to accept that then the incentre *I* has to move out of the triangle *ABC* in every second pass of *C* through *k*! *Both* drag modes are thus unsatisfactory, and because of the exclusion principle (Gawlick 2002) we cannot hope to combine them. Thus we are lead to rethink such constructions – this will yield a surprising remedy, as we are going to see…

Figure *I*. Two loci of the incentre - which one is correct?



a)

b)

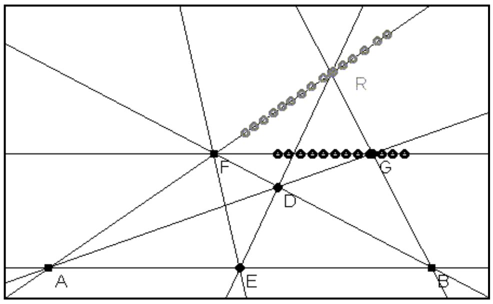
5 Global ordering – activities leading teachers (and teacher students) towards level 4 and 5

D) Rulerize dynamic constructions – a level 4 activity

The "defective" loci of *O* and *I* prompt the question: Is dynamic geometry "just like with ruler and compass"? To answer this, we have to consider *macros* and *loci* as new tools. It can be shown that the rich possibilities of these *dynamic* tools can be concisely described by the properties of a *static* ruler! In particular: the circle can be *dynamically* constructed with the ruler alone! (Gawlick 2003). If (teacher) students are lead to restructure their level 2 knowledge by investigating the power of the dynamic tools, this becomes a sequence of well-supported activities for level 3 activity.

Beforehand the following should be stressed by the instructor: that middle perpendiculars, angle bisectors and altitude can be constructed by ruler is surprising as they are based on metric properties like "perpendicular" or "halving". These of course cannot be represented by ruler alone – but it is possible to encapsulate them in the set of starting points: From $x_1 = (1,0)$, $x_2 = (2,0)$, $y_1 = (0,1)$ and $y_2 = (0,2)$ one can construct the coordinate axes with their origin *U*. These are thus straight lines on which one has two points and their midpoints. For such a straight line *AB*, however, the parallel through a given point *F* can be drawn, provided one has another point *R* on *AF* at one's disposal – its existence must be ascertained beforehand: If *E* is the given(!) centre of *A* and *B*, $D = ER \cap BF$ and $G = AD \cap BR$, thus $AB \parallel FG$, cf. fig. 9.

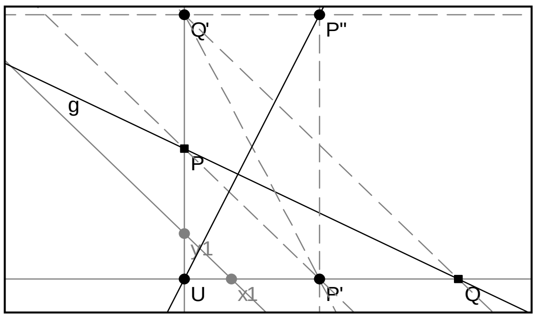
Figure 9. Ruler construction of a parallel



This fundamental property often seems like a miracle to students and appears not be easily discoverable on their own. But with its help they can *themselves* reduce most of the elementary geometric constructions to drawing parallels. Consequently, a parallel ruler is readily seen to suffice for most constructions of elementary geometry:

Perpendicular to a straight line *g* through *U*: *g* intersects the *x*-axis in *Q*. The parallel to x_1y_1 through *P* shall intersect with the *x*-axis in *P'*, the parallel through *Q* shall intersect with the *x*-axis in *PQ'*. *P'Q'* is the mirror image of *g* = *PQ* at the first bisector. Let the parallels to the coordinate axes through P' and Q'intersect in *P"*. Then *UP"* is perpendicular to *g*, see figure 10.

Figure 10. Ruler construction of a perpendicular



Perpendicular to *g* in an arbitrary point *R*: This perpendicular is obtained as parallel to the perpendicular just constructed through the point *R*.
Midpoint *M* = *MP* (*A*,*B*) of the points *A* and *B*:. By *restructuring* the construction of fig. 14, we get *M* = *E* from the harmonic properties of the complete quadrilateral (see e.g. Bieberbach 1952).
Mid perpendicular of the points *A* and *B*: this is of course the perpendicular on *AB* in *M*.
The medians and altitudes of triangles are obtained similarly. Consequently, the centre of the triangle's circumcircle, the point of gravity and the orthocentre can also be constructed by means of the ruler alone!

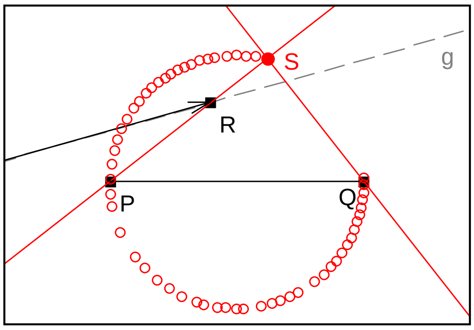
In doing this, students can conclude for their own that one cannot only reduce drawing parallels to dropping the perpendicular as usual, but also vice versa. For reasons of simplicity, it is suggestive to use a ruler which commonly (at least in German schools) serves to construct perpendiculars as well: the *angle-hook*. In summary: *Elementary geometry is geometry by angle-hook!*

NB: The power of the angle-hook does *not* exceed that of the common ruler, it only enhances the practical feasibility of a ruler constructing, the starting points of which contain the metrical data.

Many ruler constructions are classical (Pappus, Steiner). For practical purposes, however, these constructions were not feasible because of the necessary effort. This is where DGS provides its *one* decisive contribution: The "rulerized" constructions can be encapsulated in macros and thus for the first time comprehensively carried out and concatenated.

For constructing the circle, still *another* contribution of DGS is essential: *dynamics*. Once a "general" right angle above the segment PQ has been constructed (by angle-hook or macros!), one produces the entire circle as locus of its vertex S by moving any point R of the angle-hook on a straight line *g*: The circle thus proves to be a ***dynamic ruler construct***, as depicted in fig. 11.

Figure 11. Dynamic ruler construction of a circle



E) Reordering a global ordering – by a principle
It is well known in static geometry that the ruler-constructible points are just those which have coordinates rationally dependent on the coordinates of the starting points. Dynamizing this, one gets an analogous description of rational parametrizable curves. By combining two classical results of elementary algebraic geometry (Brieskorn & Körrer 1986) , namely the Plücker formula and the Lüroth theorem, we derive a handy criterion for ruler constructibility:

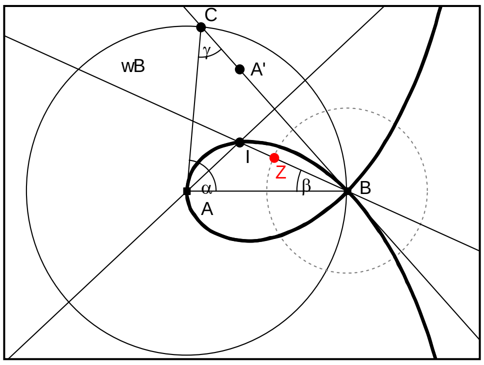
The ruler principle *Most elementary geometric constructions can be accomplished dynamically by ruler alone: A locus is a ruler curve iff its degree d and its number r of double points satisfy:* $r = (d-1)(d-2)/2$. *If so, a construction for its general point can be restructured to a ruler construction.*

We have to refrain from developing this result but believe that it is well within reach of a one semester-course in dynamic geometry, comparable in prerequisites to the usual ones on axiomatic geometry, but certainly more applicable and hence probably making more sense for teacher students. Whereas we do not claim that the principle itself should be established by the learners themselves, it is certainly crucial that they get accustomed to use it as a "meta-tool" for questions like: when is it possible to apply D) successfully to a given geometric construction? Usage of the Plücker formula requires only basic calculus and algebra. From practising it for several examples, they can gain the insight that this level 5 principle is a handy criterion to predict the outcome of the level 4 activity "rulerization".

We sketch this line of thought for the example of the strophoid generated by a moving triangle's incentre (Gawlick (2003a) has more examples!): Though from the classical geometric theory we could derive readily ruler constructions for altitude, perpendicular bisectors etc., the situation is different with angular bisectors. As a rule, these are not ruler constructible – but nevertheless this applies to the incentre! With regard to the angles themselves, note that they are constructible by ruler iff the angle's tangent is rational. Accordingly, not every constructible angle can be halved by ruler, e.g. the 45^o angle cannot. But the locus of *I* while varying *C* on *k* is a strophoid which is a ruler curve by the criterion above – this is strong evidence that one may also find a ruler construction of the incentre itself! To get the right idea, take into account that ruler constructions are necessarily deterministic, so one thereby *directly* obtains only the part of the strophoid within *k*. Thus in order to let a deterministic *I* leave the triangle, one has to *restructure* its construction!

This can be achieved as follows: Rotate the angular bisector *wB* by varying $\beta = \angle ABC/2$ instead of *C* – e.g. by moving the point *Z* on a circle around *B*. Since the bisector at *A* is ruler constructible as perpendicular bisector of *BC*, we obtain a ruler construction for *I* from one for *Z*. But we know from fig. 11 how to get this one! Consequently, the whole strophoid as well as the extra-triangular incentre can be produced by a deterministic DGS via a ruler construction (fig.12).

Figure 12. A ruler construction of the complete strophoid



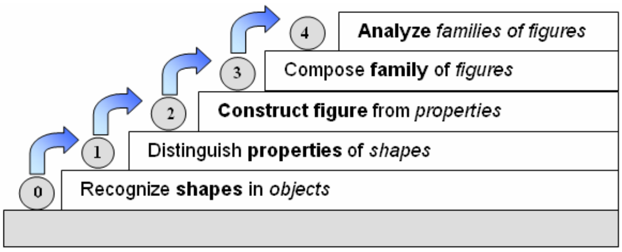
NB: The *only* rationale to attempt the "rulerization" of this curve lies in the ruler principle!
But there are a lot more interesting examples. Gawlick (2003a) applies the ruler to

- the loci of the orthocentre – these yield all types of quadrics and cubics,
- the conchoids of Nicomedes,
- the trisectrix of MacLaurin.

In contrast to this, it is far more difficult to find "classical" curves that are *not* rulerizable: Perhaps the most elementary example are the Cassinian ovals. In their construction the compass is indispensable.

5 Summary

The traditional topics associated to van Hiele levels are known to be less appropriate to sufficiently develop advanced level thinking. To that end, we recurred to Freudenthal's broader view of the levels and proposed alternative activities for level 2, 3 and 4 that in our opinion suit better the goals of geometry education, because they utilize DGS to *explore geometric phenomena at **all** levels.* This cannot be accomplished via static visualization. The following figure concisely depicts our reinterpretation of levels:



The value of our *dynamic* approach is thus twofold:
1. it can be continued to higher levels and prepared on lower levels (see A)), so students get accustomed to the tools as well as to a "discoverer's" habit of mind.
2. on *all* levels, it provides a material base for the sequential *phases of learning* in van Hiele's description of progression from one level to another: namely they can *explore* the topic in a phase of directed orientation via DGS and then *build the new concepts* for themselves, drawing upon their *previous knowledge.*

Anticipating a detailed account of classroom experiences, we assure the reader that though this approach levers the epistemological barriers between the levels not all at once, students clearly are better supported and more motivated to tackle them within the proposed framework.

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