Patterns – a fundamental idea of mathematical thinking and learning

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Abstract: Taking advantage of patterns is typical of our everyday experience as well as our mathematical thinking and learning. For example a working day or a morning at school displays a certain structure, which can be described in terms of patterns. On the one hand regular structures give us the feeling of permanence and enable us to make predictions. On the other hand they also provide a chance to be creative and to vary common procedures. School students usually encounter patterns in math classes either as number patterns or geometric patterns. There are also patterns that teachers can find in analyzing the errors students make during their calculations (error patterns) as well as patterns that are inherent to mathematical problems. One could even go so far as to say that identifying and describing patterns is elementary for mathematics (cf. Devlin 2003). Practising good interacting with patterns supports not only the active learning of mathematics but also a deeper understanding of the world in general. Patterns can be explored, identified, extended, reproduced, compared, varied, represented, described and created. This paper provides some examples of pattern utilization and detailed analyses thereof. These ideas serve as "hooks" to encourage the good use of patterns to facilitate active learning processes in mathematics.

Kurzreferat: Der Umgang mit Mustern prägt sowohl unseren Alltag wie auch das mathematischen Denken und Lernen. Ein Arbeitstag, ein Schulvormittag weist eine bestimmte Struktur auf, die sich in Form eines Musters beschreiben lässt. Regelmäßige Strukturen geben uns Sicherheit und ermöglichen es, Vorhersagen zu treffen, gleichzeitig bieten sie aber auch die Möglichkeit, kreativ zu werden und Abläufe zu variieren. Im Mathematikunterricht begegnet den Schülerinnen und Schülern Muster z.B. in Form von Zahlenmustern, geometrischen Mustern, Fehlermuster und Aufgabenmuster. Man kann sogar soweit gehen und das Identifizieren und Beschreiben von Mustern als elementar für die Mathematik zu bezeichnen (vgl. Devlin 2003). Damit unterstützt die Schulung des Umgangs mit Mustern nicht nur ein aktives Mathematiktreiben, sondern hilft auch, sich die Welt zu erschließen. Wie kann nun die Begegnung mit Mustern beschrieben werden? Muster werden erforscht, entdeckt, fortgesetzt, nachgezeichnet, verglichen, variiert, repräsentiert, beschrieben und entwickelt. Beispiele und eine genauere Charakterisierung werden hier näher vorgestellt und geben Anknüpfungspunkte, wie der Umgang mit Mustern und damit das aktive Mathematiktreiben angeregt werden kann.

ZDM-Classification: B10, D30

1. Introduction

Using patterns determines several relevant aspects of our lives. We strive to discern regularity, because recurrent structures simplify daily life. Recognized patterns produce a sense of security which allows us to perform our daily work and be open to new challenges. At the same time disorder motivates us look for a pattern. To discover patterns in plots, in courses, in pictures, in pieces of music, etc. gives us the feeling of mastering our experience.

The analysis of patterns and the description of their regularities and properties is one of the aims of mathematics, which Alan H. Schoenfeld (1992, p. 334) characterizes as "… a living subject which seeks to understand patterns that permeate both the world around us and the mind within us." Keith Devlin goes as far as to describe mathematics as the science of patterns: "It was only within the last twenty years or so that a definition of mathematics emerged on which most mathematicians now agree: mathematics is the science of patterns." (Devlin 2003, p. 3)

The world-wide discussion of educational standards – e.g. the NCTM-Standards of 1989 and 2000 (NCTM 1989; 2000) – draws math teachers' attention to the importance of patterns for learning and teaching mathematics.

The NCTM-Standards of 1989 demanded in the thirteenth standard for grade K-4 the study of patterns and relationships with the goal that students should be able to

- "recognize, describe, extend, and create a wide variety of patterns;
- represent and describe mathematical relationships;
- explore the use of variables and open sentences to express relationships." (NCTM 1989, p. 80)

In grades 5-8 the exploration of patterns was expanded. The emphasis shifts to an exploration of functions (see NCTM 1989, p. 98). The description and the representation of relationships with tables, graphs and rules were important. The aim was to use different representations in the process of problem solving.

At this point it is important to note that in the development of reports on NCTM-Standards from 1989 to 2000 there is an interesting discrepancy concerning patterns. Whereas in the report of 1989 patterns were an item in the list of standards, they only appear in the report of 2000 as a subcomponent of the algebra standard. This may be the effect of reduction of thirteen standards to ten.

The aims of the NCTM-2000-Algebra-Standard are worded as follows:

"Instructional programs from pre-Kindergarten through grade 12 should enable all students to –

- Understand patterns, relations, and function
- Represent and analyze mathematical situations and structures using algebraic symbols
- Use mathematical models to represent and understand quantitative relationships
- Analyze change in various contexts". (NCTM 2000, p. 394)

Thus, the universal character of patterns, which is present in all fields of mathematics, is no longer perceived as such in the standards. As the same time the algebra standards perspective became broader and richer. John A. van de Walle (2004, p. 417) characterizes appropriately this algebra standard with the following words:

"When the authors of Principles and Standards for School Mathematics chose Algebra as one of the five content standards to span all grades, pre-K-12, they were not thinking about the typical algebra of the eighth grade or high school. Rather, the algebra described in the Principles and Standards is about the development of algebraic thinking and algebraic concepts. The focus is on patterns, functions, and the ability to analyze situations with the help of symbols".

In the German educational standards for mathematics for the primary and middle schools – known as KMK-Standards[1] (KMK 2003; 2004) – patterns have a similar position of importance. The German KMK-standards for mathematics follow the concept of fundamental ideas. These are core ideas for mathematics and everyday life, which can be assimilated at different levels of abstraction in primary school and after (cf. Heymann 1996). One of the fundamental ideas in the KMK-standards for primary school is „pattern and structure". Students have to develop competence in recognising, describing and representing regularities (KMK 2004, p. 13). According to the Germany KMK-Standards for primary school, this means that pupils should be able

- "to understand and to use sequences of number representations (e.g. the hundred table)
- to recognize, to describe and to extend regularities in geometrical and arithmetical patterns (e.g. in number sequences or sequences of tasks)
- to develop, to systematically vary and to describe arithmetical and geometrical patterns"[2] (KMK 2004, p. 13)

In the KMK-standards for the primary school the fundamental idea of "pattern and structure" includes functional relations in real word situations. This usage prepares the way for comprehending functional relations, which are then subject to deeper study in secondary school. "Pattern and structures" are not named explicitly in the mathematics KMK-standards for middle school: only the description of the fundamental idea of "functional connection" refers to patterns.

The analysis of the standards shows that the usage of patterns is limited to geometrical and arithmetical patterns. The proposed usage of patterns concentrates on how to extend, to repeat or to vary sequences of numbers or sequences of geometrical figures. In my view this type of task does not make use of the wide range of possibilities for using patterns. The variety of abilities, which can be developed by working with patterns are cut short without a reason. At school students normally don't get enough "free space" from routine mathematical problems to experience "real math". One concept for getting pupils acquainted with "real mathematics" is that of „mathematical substantial learning environment" (Wittmann 2001; Steinbring 2005).

"A substantial learning environment, an SLE, … is a teaching/learning unit with the following properties: (1) It represents central objectives, contents and principles of teaching mathematics at a certain level. (2) It is related to significant mathematical contents, processes and procedures beyond this level, and is a rich source of mathematical activities. (3) It is flexible and can be adapted to the special conditions of a classroom. (4) It integrates mathematical, psychological and pedagogical aspects of teaching mathematics, and so it forms a rich field for empirical research." (Wittmann 2001, p. 2/3 in Steinbring 2005, p. 89)

The following section attempts to describe diverse activities using patterns which enable students to become active learners of mathematics.

2. Interacting with Patterns

Exploring, identifying, extending, reproducing, comparing, representing and describing are characteristic operations with patterns (cf. van de Walle 2004, p. 417 ff.; Radatz et al. 1998, p. 151). The order chosen is not necessarily the order in which interaction with patterns takes place. However, in the majority of problems relating to patterns several of the above listed operations can be identified. Clearly, interacting with patterns depends on the context in which the patterns are being used and on the tasks that motivate their use and the motivation of their use.

How can these characteristic operations be described? In order *to explore patterns*, these have to be presented in a given context. Then the task is to analyze the structure of patterns. Identifying and verifying patterns continually confirms our conjectures and makes them clearer and more precise. During the exploration it is important that the base elements or units of the phenomenon are found. Identifying the base element imparts a sense of security to the observer. Often the base element is not well defined (cf. fig. 1). In the example it might be the gray triangle, the square (black dashed) or the rectangle (gray dashed).

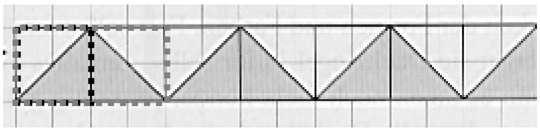

Fig. 1: A series of geometry figures (Radatz et al., p. 151)

To identify patterns is at first an intuitive process and relates to the given structure as a whole. Regularities are recognized which are often not intended. To detect a pattern means to distinguish between random phenomena and structured ones. Only in structured phenomena can patterns be identified. This is impossible in random phenomena. Which cognitive strategies will be used is described by Inge Schwank in her paper "Introduction to functional and predicative thinking" (Schwank 2003).

The task of *extending* a certain pattern formation is more than merely being aware of it. The identification of the characteristics in the pattern formation – its base element and its rules – is essential for extending a given pattern sequence. The typical rules are dependent on the context. Particularly geometry provides tools for extending geometrical patterns (fig. 1), i.e. in the form of congruence maps. Extending a pattern sequence is guided by conjectures arising from a first impression, which may be rather informal. In the case of number sequences, for instance, it may happen that the given sequence exhibits hints of possible rules for determining the next term, but that the sequence is far too short to establish an unambiguous further step. In addition to these factors, trial and error characterize the action of sequence extension.

[1] KMK stands for Kultusministerkonferenz (a standing conference co-ordination the work of all secretaries of education in the German Federal States)

[2] translated by the author

Reproducing and comparing patterns demands a very precise and sequential analysis. But this can lead to a loss of overview of the whole setting and undermine the discovery of general rules. Another aspect is that detecting and adding new patterns is context-dependent. The choice of the specific pattern unit determines the range of the analyzed pattern. When additional information is provided, for instance when the chosen sequence is extended, the pattern in question appears in a new light which may have consequences for its interpretation. Reproducing and comparing patterns is an important condition for detecting regularities and structure.

To represent and to describe patterns is to determine the rules that characterize the pattern structure by means of appropriate descriptive features (cf. Stern et al. 2003). On the basis of such descriptions it is possible to reconstruct patterns. The operation of pattern description enables the observer to establish a certain distance between him- or herself and the pattern under investigation. Adopting the perspective of an objective observer (Steinweg 2000, p. 8) creates a condition which makes pattern identification useful for a fruitful mathematical activity. The different tools for description depend upon the specific task and exhibit different levels of abstraction, which become apparent in Pascal's procedure (cf. the following example performed by Pascal).

3. An Example from Primary School

In this section we analyze a task which has been used by several authors (cf. Lorenz 1997, p. 58 f.; Steinweg 1999, p. 67, 2000; Steinbring 2005, p. 96 ff.):

The following sequence of rectangular numbers (fig.2) is presented as an array of dot patterns

(1) How would you continue?

(2) Find the 15th rectangular number

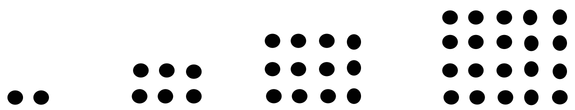

Fig. 2: The sequence of rectangular numbers given in the task

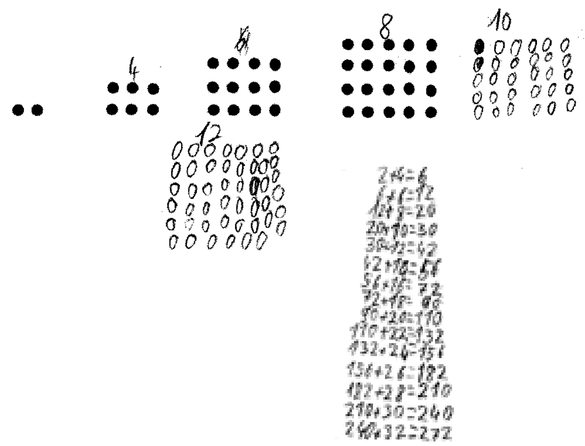

Fig. 3: Pascal's solution (Pascal is a third grade primary school pupil.)

Pascal's solution shows that by adding numbers of dots to the given sequence he gains clarity on the regularities ruling the dot patterns. The picture (fig. 3) does not show in detail how Pascal found the rules to continue. This record of Pascal's work is seen by him as a document with which he communicates with his teacher. Since he knew in advance that his sheet would be collected after the exercise, this kind of record represented for him a "public documentation" of his solution (Fetzer 2003; Ruf & Gallin 1998). In this setting the process of discovery is not documented (in contrast to the epistemological analysis of Heinz Steinbring 2005, p. 147 ff.)

Let us now analyze the task. What are the salient attributes of the given dot patterns? (cf. fig. 2)

(a) The basic configuration is a growing rectangle.

(b) The end result at each step is a rectangle with an additional column and an additional row compared with the preceding one.

Pascal draws the fifth and sixth rectangle by extending the sequence of dot patterns. His drawing indicates that Pascal recognizes both attributes. Furthermore he protocols all changes between pattern units by keeping track of the number of added dots. Thus he generates numerical differences making up the sequence. It may be that reading the second question has influenced his answer of the first. Obviously Pascal works on two levels, the design level and the arithmetic level. The question "What is the 15th rectangular number?" evokes a new strategy, different from the one evoked by "How would you continue?". I conjecture that question (2) leads Pascal to use a strictly arithmetical procedure, so that he now selects a different type of representation, which is more appropriate for answering question (2). The number of additional dots required to make a dot pattern "grow" into the next one (according to attribute (b)) was indicated by Pascal on top of each rectangular pattern (see Fig. 3). At the beginning there are two dots. For the next rectangular number four dots have to be added, obtaining 6 dots, which become the starting point for the subsequent step, etc.

This example shows the application of the operations "extending", "representing" and "describing" of patterns concretely.

4. Another perspective: Patterns for modeling

Working with geometrical and arithmetical patterns have been the main issues considered so far. Detecting and describing patterns is not limited to working with explicit external representations of pattern arrangements, however. Working with implicit reasoning on mathematical problems also involves pattern identification. Finding an adequate mathematical model for a real life situation demands "understanding, specifying, structuring" (cf. PISA-Konsortium Deutschland 2004, p. 48/49), which means that regularities and patterns of the situation have to be explored. This is the case we want to illustrate with the following example. The task was set as an assignment to a group of teacher students at the university.

Task: 20 people participate at a party and greet each other. How many handshakes take place?

We now analyze two solutions provided by two female students (Anke and Sabine):

I . Anke
She writes:

„I started with a small number of guests, namely 5.
I represented 4 of these guests in a row visually and let the fifth one shake hands with them 4 times. Then I put him aside and let person Nr. 3 shake hands three times with the rest. In the end I had counted 4 + 3 + 2 + 1 = 10 handshakes. I repeated the procedure with different numbers of guests and it was always possible to detect the underlying structure. Finally I considered 20 guests and obtained 19 + 18 + 17 + ... + 2 + 1 = 190 hand shakes“ [3]

This student restricts herself to her concrete situation, identifying a pattern applicable to it. She describes how she begins by treating the concrete situation of 5 guests and also how she translates her imagined situation into an arithmetical procedure (namely by adding the numbers of shaken hands). Her next step is a verification of her model consisting of repeated trials for several different numbers of guests. After assuring herself that her procedure works, she goes on to transfer it to the question posed in the task. Observe that she does not go as far as to choose another representation for her verification (e.g. drawing a graph).

II. Sabine:
She writes:

„Here we have an unordered, random sample without repetitions! (It does not matter who shakes hands for the first time with whom and if the 20th person has shaken hands with the 19th person, then the 19th person does not shake hands with the 20th person again).

Therefore: $E = \binom{n}{s}$

I check this with an example with five guests:
The fifth person shakes hands with the following guests:
4 3 2 1
The fourth person shakes hands with following guests:
3 2 1
The third person shakes hands with following guests:
2 1
The second person shakes hands with the following guest:
1

Sum: $20 = \binom{5}{2}$

Therefore: $E = \binom{20}{2} = 190$ “[4]

Sabine, the second student, describes the operation pattern in the task mathematically and decides that it is a combinatorial situation. I conjecture that she had combinatorics classes at school or university. Sabine probably recalls features of the combinatorial cases (“with order and repetition, with order without repetition”, etc.) and claims to have identified the case treated in the task. This is the way she justifies her chosen formula, as being the number of “handshakes”, which she identifies with “unordered pairs”, among twenty guests. Nevertheless, her verification for the case of 5 guests exhibits an interesting confusion. She writes down the ordinal numbers corresponding to the guests the fifth person shakes hands with, and goes down in this form to the second person. Then she ends up treating these ordinal numbers as cardinal numbers and adds them up. This seems to be the way she obtains the “20” as the sum of handshakes for 5 guests. This result confirms her remembered formula, but she fails to realize her error in calculating $\binom{5}{2}$, which equals 10. After her verification she uses her formula to solve the proposed task.

These two examples describe students with a task whose context – a party – stems form everyday life, while the question – number of handshakes – is artificial and has a mathematical motivation. Most of the PISA tasks have these characteristics. The two chosen solutions exhibit different approaches to pattern recognition and their mathematical treatment. The first one, Anke, concentrates on her concrete situation, recognizes a pattern for each the first 5 guests (in compact mathematical notation, this would be expressed by: each nth person shakes (n - 1) hands) and translates the result of her naïve heuristic into a simple mathematical form. The second one, Sabine, is at the other extreme of a continuum of possible solutions. She recognizes a pattern and associates it with a mathematical pattern from a discipline she has studied and deduces that she can use a well-known formula. Both aim at verifying the validity of their procedure but both fail in that they produce no mathematical argument or proof. Here recognizing patterns in analysing errors helps the math educator to understand how students think. This diagnostic utilization of pattern detection in students' solutions enhances the possibilities of learning from mistakes.

5. Conclusion: patterns as a fundamental idea

As we have seen, a systematic use of patterns can help in developing metacognitive principles. “Metacogniton deals […] with the knowledge and the control of one's own cognitive system.”[5] (Brown 1984, p. 61; cf. Schoenfeld 1987 too). Metacognitive abilities are important for the development of mathematical competence. Also the diagnostic competence of teachers may be supported by a solid knowledge of patterns (and further patterns of error).

In the different catalogues of fundamental ideas (cf. Heymann 1996) patterns are usually not explicitly listed. Yet, it is a popular view among mathematics educators that the use and understanding of patterns is central to mathematical activities (Steinweg 2001, Steinbring 2005, Wittmann 2003, Wittmann 2005, Vogel & Wessolowski 2005). The reason might be that patterns are seen more as

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a process component of mathematical methodology. This paper is intended to show that patterns are not just process components but also content components of mathematical thinking and learning.

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References

Brown, A.L. (1984): Metakognition, Handlungskontrolle, Stelbststeuerung und andere, noch geheimnisvollere Mechanismen - In: F.E. Weinert & R.H. Kluwe (Hrsg.), Metakognition, Motivation und Lernen. Stuttgart: Kohlhammer. pp.60-109
Devlin, K. (2003): Mathematics: The Science of Patterns. - Owl Books: New York
Fetzer, M. (2003): Verschriftlichungsprozesse im Mathematikunterricht der Grundschule aus interaktionstheoretischer Sticht. - In: Journal für Mathematik-Didaktik, 24 (H 3/4), pp. 172-189
Heymann, H.W. (1996): Allgemeinbildung und Mathematik.- Weinheim: Beltz
KMK – Kultusministerkonferenz (2003): Bildungsstandards im Fach Mathematik für den Mittleren Bildungsabschluss. Beschluss vom 04.12.2003. - (http://www.kmk.org/schul/Bildungsstandards/bildungsstandards-neu.htm, 08-08-05)
KMK – Kultusministerkonferenz (2004): Bildungsstandards im Fach Mathematik für den Primarstufenbereich (Jahrgangsstufe 4). Beschluss vom 15.10.2004. - (http://www.kmk.org/|-schul/Bildungsstandards/bildungsstandards-neu.htm, 08-08-05)
Lorenz, J.H. (1997): Kinder entdecken die Mathematik. - Braunschweig: Westermann
NCTM – National Council of Teachers of Mathematics (1989): Curriculum and Evaluation Standards for School Mathematics. - Reston, Virginia, USA
NCTM – National Council of Teachers of Mathematics (2000): Principles and Standards for School Mathematics. - Reston, Virginia, USA
PISA-Konsortium Deutschland (Hrsg.) (2004): PISA 2003. - Münster: Waxmann
Radatz, H., Schipper, W., Dröge, R. & Ebeling, A. (1998): Handbuch für den Mathematikunterricht. 2. Schuljahr. - Hannover: Schroedel
Ruf, U. & Gallin, P. (1998): Dialogisches Lernen in Sprache und Mathematik. Spuren legen – Spuren lesen. Unterricht mit Kernideen und Reisetagebüchern. - Seelze-Velber: Kallmeyer.
Schoenfeld, A. H. (1987): What's All the Fuss about Metacognition? – In: A.H. Schoenfled (Ed.), Cognitive Science and Mathematics education. Hillsdale, NJ: Erlbaum. pp.189-215
Schoenfeld, A.H. (1992): Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. - In: D.A. Grouws (Ed.), Handbook of research on teaching and learning. Old Tappan, NJ: Macmillan. pp. 334-370
Schwank, I. (2003): Einführung in funktionales und prädikatives Denken. In I. Schwank: ZDM-Themenheft: Zur Kognitiven Mathematik, 35 (H 3), pp. 70-78
Steinbring, H. (2005): The Construction of New Mathematical Knowledge in Classroom Interaction. An Epistemological Perspective. - New York: Springer
Steinweg, A.S. (1999): „Children's Understanding of Number Patterns“– In: Proceedings of the BSRLM Day Conference in Leeds 1998, pp. 65-70
Steinweg, A.S. (2000): Mit Zahlen spielen. – In: Die Grundschulzeitschrift, 14 (H. 133), pp. 6-10
Steinweg, A.S. (2001): Zur Entwicklung des Zahlenmusterverständnisses bei Kindern: Epistemologisch-pädagogische Grundlegung. Münster: LIT
Stern, E., Aprea, C. & Ebner, H. (2003): Improving cross-content transfer in text processing by means of active graphical representation. - In: Leraning and Instruction, 13, pp. 191-203
Van de Walle, John A. (2004): Elementary and middle school mathematics. Teaching developmentally. 5th ed. - Boston: Pearson Education
Vogel, R. & Wessolowski, S. (2005): Muster und Strukturen. Eine Leitidee für den Mathematikunterricht. – In: J. Engel, R. Vogel & S. Wessolowski (Hrsg.), Strukturieren – Modellieren – Kommunizieren. Leitbilder mathematischer und informatischer Aktiviäten. Hildesheim: Franzbecker. pp. 57-67
Wittmann, E.C. (2001): Developing Mathematics Education in a Systemic Process. – In: Educational Studies in Mathematics, 48 (H. 1), pp.1-20
Wittmann, E.C. (2003): Was ist Mathematik und welche pädagogische Bedeutung hat das wohlverstandene Fach auch für den Mathematikunterricht der Grundschule? – In: M. Baum & H. Wielpütz (Hrsg.), Mathematik in der Grundschule. Ein Arbeitsbuch. Seelze: Kallmeyer. pp. 18-46
Wittmann, E.C. (2005): Eine Leitlinie für die Unterrichtsentwicklung vom Fach aus: (Elementar-)Mathematik als Wissenschaft von Mustern. In: Der Mathematikunterricht, 51 (H. 2/3), pp. 5-22

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