

Investigating and ordering Quadrilaterals and their analogies in space – problem fields with various aspects

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Abstract: Problem fields with one or two generating problems and possibilities of varying existing problems give a good chance for self-activities of students and can be used for reaching different general aims. In this paper some topics concerning quadrilaterals will be presented. I hope they will animate teachers for more problem orientation in mathematics education. First we will reflect about different types of convex and non-convex quadrilaterals and possibilities of ordering them. Then we focus on middle-quadrilaterals and types of quadrilaterals with special middle-quadrilaterals as well as their logical ordering. Finally we investigate the analogies in space to the parallelogram and its sub-types and order them in the “house of parallelepipeds”.

Kurzreferat: Problemfelder mit einem oder zwei Ausgangsproblemen und Möglichkeiten der Variation vorliegender Probleme ermöglichen Selbstaktivitäten von Schülerinnen und Schülern sowie den Erwerb bzw. die Festigung verschiedener allgemeiner Lernziele. Zur Anregung von Lehrerinnen und Lehrern für mehr Problemorientierung im Mathematikunterricht werden hier einige Themen rund ums Viereck dargestellt. Zuerst wird über verschiedene Typen konvexer und nicht-konvexer Vierecke und deren logische Ordnung reflektiert. Danach steht das Mittenviereck und Typen von Vierecken mit speziellen Mittenvierecken im Mittelpunkt der Überlegungen. Schließlich werden räumliche Analogie zum Parallelogramm und seinen Untertypen untersucht und im „Haus der Parallelepiped“ geordnet.

ZDM-Classification: A60, D50, G40

1. Introduction

With regard to learning theories and pedagogical knowledge I plead for mathematics education with a mixture of different methodical proceedings (instructions, exercises and self-determined learning). Roughly we could say that *one third of mathematics teaching should be spent on investigations and working on problems.*

The word “problem” we will use for a task which contains different aspects and cannot be solved at once. Moreover working on problems means that students mostly work by themselves, as individuals or in pairs or in groups. The role of the teacher mostly is to lead the selection of problems and to assistance by working on problems. But it is not my intention to discuss these general methodical aspects in detail. More important now is to notify that in classroom reality the problem orientated phases mostly do not occupy a third of mathematics teaching. Sometimes they do not appear at all. Therefore I think it is important to strengthen the

discussion about problem orientation in mathematics teaching and to present examples which can animate teachers to take them for problem orientated phases in the classroom.

My focus in this connection is working with *problem fields*; a problem field is a set of problems which are related to each other and have a generating problem (see e.g. Pehkonen 2001). This means that we do not work only with isolated problems (like in mathematical Olympiads) but with a mathematical field or a situation of everyday life where the students besides solving problems also can make investigations, pose problems and find connections, new insights or even mathematical theorems as well as discuss the ways and the limitations of modelling.

In the course of such problem orientation we can aspire to reach several *general aims*. The students first can learn a lot of mathematics on this way where the chance of remembrance is higher than in a course where the information is given only. Also the motivation for working with mathematics can be raised up and the beliefs about mathematics as well as mathematics education can become more positive. Secondly they see mathematics as a process and they learn that mathematics sometimes can be developed also by students. Last but not least they can train general goals like “abilities how to handle a problem and persevere on a problem as well as how to make investigations”, “abilities of problem solving and finding analogies“, „abilities of ordering and systematizing a given situation or finding new interlockings respecting theoretical aspects“, “positive self-confidence“, “ability of communicating mathematical facts and questions” and “team work abilities”.

2. Convex and non-convex quadrilaterals

Though children are confronted with quadrilaterals already in primary school (grade one to four) they normally have to deal with details and different types in grade six or seven. Mostly the instruction begins with special types like squares, rectangles, parallelograms and their different characteristics as well as their logical connections. Several points about this normally you find in text books.

But with respect to problem orientation you also could ask the pupils “*What is a quadrilateral? What characteristics of a quadrilateral in general you can name?*” According to experience different aspects like “four vertices”, “four sides” and “four angles” will be named but with this mostly the pupils only have convex quadrilaterals in their mind however with no concepts to describe convexity. Asking for a more precise description you often hear “First you fix four different points A, B, C, D and then you connect them with straight lines in that way that first A is connected with B, then B with C, C with D, and finally D with A.” In textbooks you also sometimes find a similar (but more sophisticated) definition “A quadrilateral is a closed sequence of straight lines with four vertices.”

If you use this usual definition then you have a problem. *Besides convex quadrilaterals you find also*

different types of non-convex quadrilaterals: such with one re-entrant angle, such with two sides cutting each other and also such with three co-linear vertices (see Fig. 1).

Also quadrilaterals in space (where the four points don't lie in a plane) are possible.

This is not a pity, rather an opportunity for investigations. First we can learn that also in mathematics different definitions are possible. Secondly we see that characterisations of a concept can lead to a generalization of this concept. Thirdly we are forced to find additional terms and characterizations to describe special types of a generalized concept.

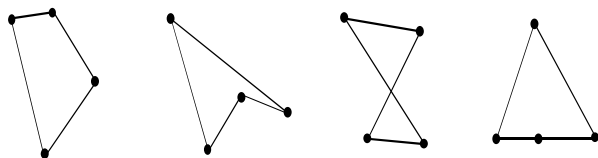


Fig. 1. Types of convex and non-convex quadrilaterals

Moreover we can discover *several mathematical characteristics of quadrilaterals*: E. g. convex quadrilaterals have diagonals which have an intersection while the others don't have, plane quadrilaterals which have no two sides cutting each other determine an area which is a sum of two triangle-areas. This easily can be generalized for quadrilaterals in space. For quadrilaterals with two sides cutting each other this is not so obvious but sometimes in literature it is done with using negative area belonging on the orientation of the triangles. Furthermore it is interesting to ask for the sum of the angle-measures. Also here we have to distinguish between positive and negative measures in respect to the orientation. Then we can get at least for all plane quadrilaterals the theorem that the sum of the four angle measures is 360° or 0° .

Another interesting investigation is the search for special non-convex quadrilaterals in analogy to the known types of convex quadrilaterals or by using symmetry. If we are e.g. looking out for quadrilaterals with one axis of symmetry we first have to distinguish between two cases.

First two vertices can be situated on the axis of symmetry. Then the connection of the other two vertices is bisected by the axis of symmetry and therefore all four vertices are situated in a plane. In respect to the distribution of the two vertices on the axis of symmetry we get a convex quadrilateral, the kite, or a symmetric quadrilateral with one re-entrant angle (see Fig. 2a). In the second case no vertex is situated on the axis of symmetry. Then there must be two sides which have the axis of symmetry as middle-orthogonal line. If all four vertices lie in a plane then these two sides are parallel. We therefore get as convex quadrilateral a trapezium and as non-convex quadrilateral a quadrilateral with two cutting sides and two parallel sides where the two cutting sides have the same length (see Fig. 2b).

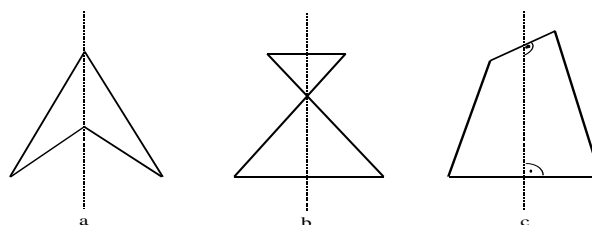


Fig. 2. Symmetric non-convex quadrilaterals

But also in space there exists a symmetric type (see Fig. 2c). We get it out of a trapezium by turning round the axis of symmetry one of the two parallel sides into space. It still has the characteristic that the other two sides have the same length.

3. Logical ordering of special types of quadrilaterals

By ordering the different types of convex quadrilaterals in respect to the relation "is special type of" we come to a problem field which is dealt with in classroom very often. Such a graph (tree) with different types of quadrilaterals is called a "house of quadrilaterals". In mathematics teaching mostly there is given only one way of ordering. However in literature you can find several ways (see e.g. Volkert 1999). I think it is a good way of learning mathematical thinking if we make a list of *different aspects quadrilaterals can be ordered with*. Such a list can focus the following points:

- *Sides* with equal length (two neighbouring or two opposite or three or four sides),
- sum of the length of two *sides* are equal (two neighbouring or two opposite sides),
- parallel *sides* (one pair of opposite or two pairs of opposite sides),
- *angles* with equal measure (one pair or two pairs of neighbouring or opposite angles, three angles or four angles),
- special *angle* measures (90° - perhaps 60° and 120° - with one, two, three or four angles),
- special sum of *angle* measures (two neighbouring or opposite angles lead to 180°),
- *diagonals* with equal length,
- orthogonal *diagonals*,
- one *diagonal* bisects the other one or each diagonal bisects the other one,
- *symmetry* (one, two or four axis' of symmetry where an axis connects two vertices or two side-midpoints, one or three rotation symmetry, one or two axis' of sloping-symmetry¹).

With the help of this list you can characterise all the well-known types of quadrilaterals but you also will find new types which are not used before. Whether these new types are of interest or have additional qualities can be discussed in the classroom.

¹ With a sloping-symmetry there exists a reflection - not absolutely necessary orthogonal to the axis - which maps the quadrilateral onto itself. For such a sloping reflection the connection of one point and its picture is bisected by the axis and all connections lines point-picture are parallel to each other.

If we look out for an *ordering system* (“house”) for different types first it is clear that the *square* is the quadrilateral with most specialisation. The square therefore has to stand at the bottom of an ordering graph. Then also it is clear that the *rectangle* is the next special quadrilateral. But also the *rhombus* is a generalisation of a square. From the two characteristics of the square (all sides have equal length *and* all angles are right angles) the rhombus holds only the first one while the rectangle holds only the second one. But both can be characterized as quadrilaterals with two axes of symmetry and one rotation symmetry (180° in respect to the midpoint) whereas with the rhombus the axes of symmetry meet the vertices and with the rectangle they meet the midpoints of the sides. The square however has four axes of symmetry (which are those of the rhombus and the rectangle) and three rotation symmetries (90° , 180° , 270° in respect to the midpoint).

Because there is no type which is situated between a square and a rectangle or a rhombus in respect of the relation “special case of” we find that there exists no quadrilateral with exactly three axes of symmetry or with exactly two symmetries of rotation. Also by looking at the types we just discussed we can see that with a quadrilateral of exactly two axes of symmetry these axes always must be perpendicular to each other. For the next step of generalisation there are several possibilities.

The most known types generalizing rectangle or rhombus are *symmetric trapezium*, *kite* and *parallelogram*. Kite and symmetric trapezium have only one axis of symmetry where the axis of symmetry of the symmetric trapezium meets two opposite midpoints of the sides (like rectangle) and the axis of symmetry of the kite meets two opposite vertices (like rhombus). The parallelogram has only one rotation symmetry like rectangle and rhombus). The other way round if we look out for convex quadrilaterals with one axis of symmetry or one rotation symmetry then we get only these three types.

But we also can see the generalisation in respect to sides with equal length and sides which are parallel. The kite and the parallelogram are characterized as quadrilaterals with two pairs of sides with equal length whereas the kite each two sides of equal length have a vertex in common and with the parallelogram they are in opposite position. The symmetric trapezium then is characterized as quadrilateral with one pair of sides with equal length and one pair of parallel sides while the parallelogram also can be characterized as quadrilateral with two pairs of parallel sides.

If we focus our generalisation on the angles of the quadrilaterals so we find as generalisation of the rectangle the parallelogram and the symmetric trapezium with two pairs of angles with equal measure. In the parallelogram the angles with equal measure are situated opposite to each other and in the symmetric trapezium they are situated next to each other. In a rhombus we have two pairs of opposite angles with equal measure. So a generalisation of a rhombus is a parallelogram or a kite. The kite has the characteristic of one pair of angles with equal measure which are situated opposite to each other.

A quadrilateral with three right angles is already a rectangle because of the theorem of the sum of the angle measures of a quadrilateral. But if we look out for a quadrilateral with two right angles we find the types *right angled trapezium* and *right angled kite* (respectively right angled tilted kite²).

For the next step of generalisation the situation is a little more complicated. First it should be mentioned that the characteristic of one pair of angles with equal measure and opposite does not only lead to the kite but also to a generalized kite called *tilted kite* because we can tilt one of the two angles (together with its sides) without changing the characteristic. Because the parallelogram could be characterized as quadrilateral with two pairs of equal angles opposite to each other the tilted kite is a generalization of the kite as well as of the parallelogram.

But we also can generalize the kite and the parallelogram to a quadrilateral where one diagonal bisects the other diagonal because this is true for the kite where in addition the diagonals are perpendicular and it is true for the parallelogram where both diagonals bisect the other one. A quadrilateral where at least one diagonal bisects the other one is called a *sloping-kite*; it has the additional characteristic that it has a sloping-symmetry³.

A generalization of the symmetric trapezium as well as of the parallelogram is well known. It is the (general) *trapezium* which can be characterized by one pair of parallel sides. Because the trapezium has two parallel sides it also has the characteristic of sloping-symmetry. As axis of sloping-symmetry we only have to take the connection of the two midpoints of the parallel sides. (Remember that at the sloping-kite the axis meets two vertices.)⁴

A generalisation only of the kite is e.g. a quadrilateral where at least we have one pair of neighbouring sides with same length. This type of quadrilateral is not very well known and therefore has no special name as far as I know.

A generalisation only of the parallelogram is e.g. a quadrilateral where at least we have one pair of opposite sides with same length. Also this quadrilateral has no special name. A generalisation only of a parallelogram there exist also two types which normally appear in school, the *string-quadrilateral* (all four vertices are situated on one circle so that all sides are strings of this circle) and the *tangent-quadrilateral* (all four sides are tangent to one circle). The string-quadrilaterals also can be characterized as quadrilateral where the sum of the measures of opposite angles is 180° while the tangent-quadrilateral can be characterized as quadrilateral where

² See coming section.

³ For the discussion about the two different generalisations of parallelogram and kite see also Neubrand 1981.

⁴ Besides our investigation for generalizations we lead to asking for sloping-symmetry at the types mentioned before. So we can discover that the parallelogram has four sloping-symmetries and that a quadrilateral with two axes of sloping-symmetry is already a parallelogram. From the trapezium to the symmetric trapezium the sloping-symmetry turns into a normal symmetry and also from the sloping-kite to the normal kite the sloping-symmetry turns into a normal symmetry.

sum of the length of two opposite sides is equal to the sum of the length of the other two opposite sides.

For an ordering tree concerning all named types see Fig 3.

An ordering tree which focuses on symmetry (including sloping-symmetry) is less difficult and also has a symmetric distribution (see Fig 4). It begins with the square at the bottom, the rhombus and rectangle in the

second row as well as the kite, the parallelogram and the symmetric trapezium in the third row and finally at the top the sloping-kite and the trapezium with connecting the kite and parallelogram on one hand and the parallelogram and symmetric trapezium on the other hand.

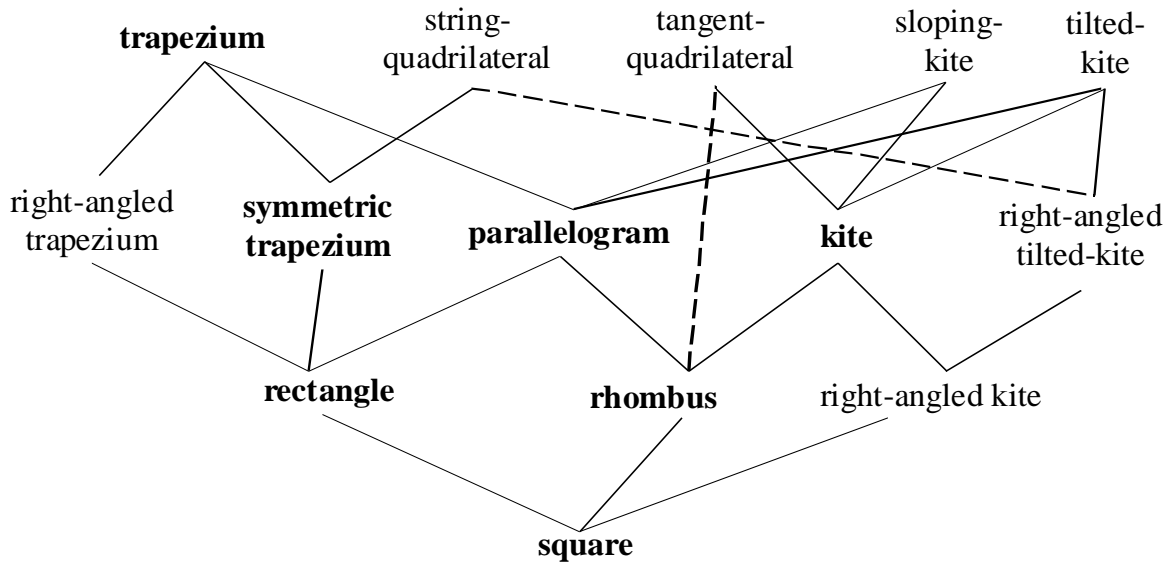


Fig 3. The “house of quadrilaterals” – An ordering tree concerning all important convex quadrilaterals

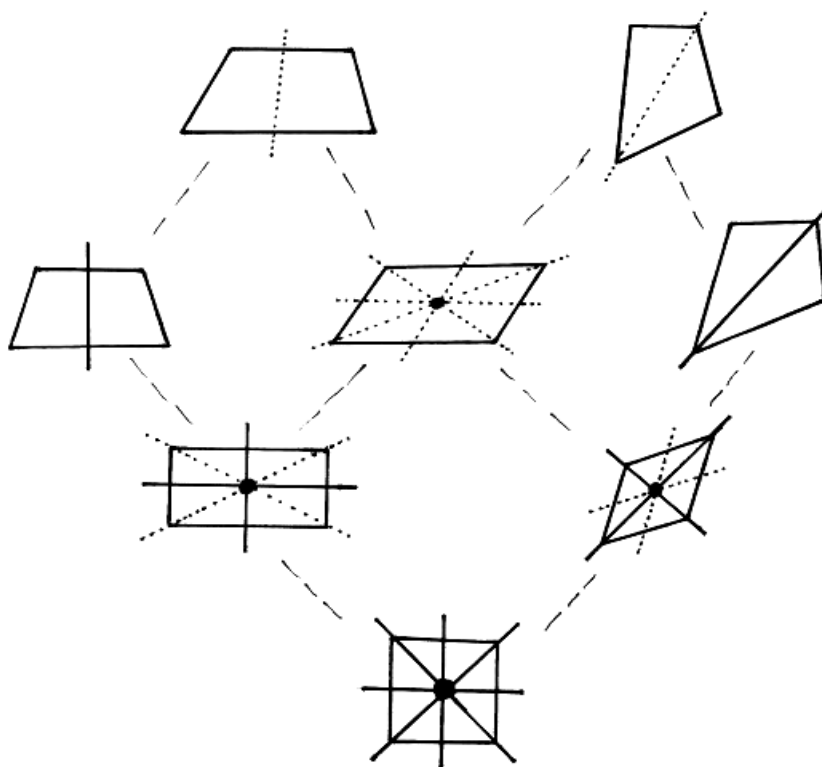


Fig. 4. The “house of symmetric and convex quadrilaterals” completed with quadrilaterals with sloping symmetry [The normal lines indicate normal (orthogonal) symmetry while the dotted lines indicate sloping symmetry.]

As already indicated there are also other ways of ordering the quadrilaterals. And we could look out for further types like those unusual types with orthogonal diagonal or diagonals of equal length⁵.

4. The middle-quadrilateral of a quadrilateral

If we take the midpoints of the sides of any polygon a vertices of a new polygon so we call this new polygon the middle-polygon of the given polygon. For a triangle an its middle-triangle we can find out that each side of the middle-triangle is parallel to one side of the original triangle. Moreover the ratio of the lengths of these two parallel straight lines is 1:2.

For any quadrilateral it comes out that its middle-quadrilateral is a parallelogram⁶. By looking at sharing triangles which are constructed out of two sides and one diagonal and applying our knowledge about middle-triangles we can find out this statement. Also with this we find out that the two parallel sides of the middle-quadrilateral are parallel to one diagonal of the given quadrilateral and have half the length of this diagonal. Moreover we can find out in this way that the area measure of the middle-quadrilateral is half of the area measure of the given quadrilateral.

For special types of quadrilaterals we also get special types of middle-quadrilaterals. It is easy to see that the middle-quadrilateral of a square is again a square. The middle-quadrilateral of a rectangle is a rhombus and that of a rhombus is a rectangle. The middle-quadrilateral of a kite is a rectangle and that of a symmetric trapezium is a rhombus.

Conversely if we ask for all convex quadrilaterals whose middle-quadrilateral is a rectangle we will be lead to a new type of quadrilaterals namely those with orthogonal diagonals. And also if we ask for all convex quadrilaterals whose middle-quadrilateral is a rhombus we will be lead to a new type of quadrilaterals namely those with diagonals of equal length. Combining both we get all convex quadrilaterals whose middle-quadrilateral is a square. Such quadrilaterals also build a type which is normally not well-known. They hold both characterizations: diagonals which are orthogonal and diagonals with equal length. For examples of these three types see Fig. 5.

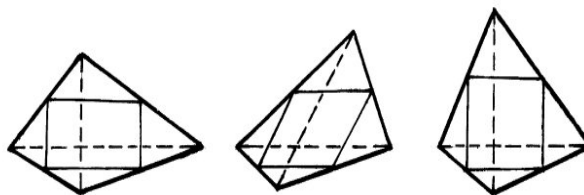


Fig 5. Quadrilaterals whose middle-quadrilateral is a rectangle or a rhombus or a square

With these reflections about middle-quadrilaterals we have got some new types of quadrilaterals which we could integrate into our house of quadrilaterals. But because our house now might become confusing we will present here only a sub-house focussing on the diagonals of quadrilaterals (see Fig. 6).

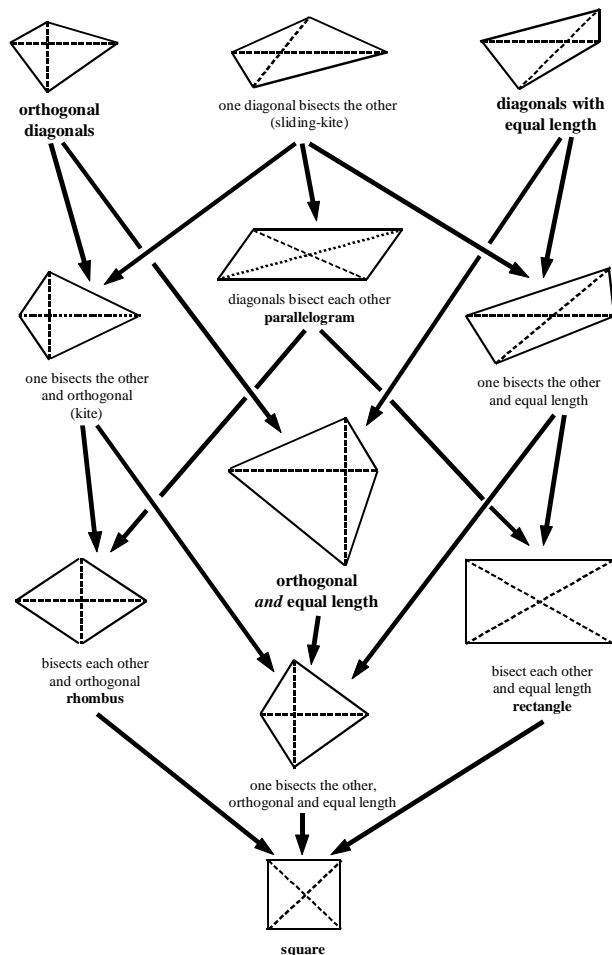


Fig. 6. A “house of quadrilaterals” concerning diagonals

If we look out for the middle-quadrilaterals of non-convex quadrilaterals we first can make the same considerations as above. Thus for all quadrilaterals, also the non-convex one, the middle-quadrilateral is a

⁵ Compare e.g. the graph with quadrilaterals next section.

⁶ In the case of a quadrilateral with crossing sides the parallelogram can degenerate to a straight line if two centre points are equal.

parallelogram, even for those in space. Also if we ask conversely for all non-convex quadrilaterals whose middle-quadrilateral is a rectangle, rhombus or square we are lead to the same conditions: diagonals with equal length, orthogonal diagonals or both conditions. Only by looking out for non-convex quadrilaterals with one of these conditions we have to make some new reflections. But this is a new nice little field for investigations.

Another extension of our investigations about middle-quadrilaterals comes out with *iterating* the process of building the middle-quadrilateral. Because the middle-quadrilateral of any quadrilateral is a parallelogram in any second step we have to build the middle-quadrilateral of a parallelogram. This of course is again a parallelogram. It has the property that its sides are bisected by the diagonals of the first parallelogram and its diagonals intersect in the same point the diagonals of the first parallelogram intersect. The same consideration of course holds for the middle-middle-parallelogram (middle-parallelogram of a middle-parallelogram), the middle-middle-middle-parallelogram and so on. Therefore the diagonals of all parallelograms in a sequence of middle-parallelograms coincident with the intersection-point of the diagonals of the first parallelogram. Also the middle-middle-parallelogram is similar to the first parallelogram and can be get by a stretching with the intersection-point of the diagonals as centre and factor $\frac{1}{2}$.

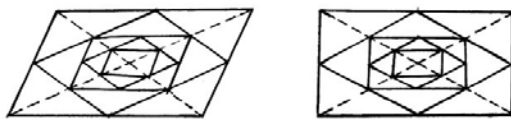


Fig. 7. Iterating the middle-quadrilateral of a parallelogram or a rectangle

So beginning with a parallelogram we have a sequence of parallelograms which build two classes of similar parallelograms where all parallelograms have the same point of intersection of the diagonals. Thus the limit of this sequence of parallelograms is this common point of the diagonals.

If we start with a rectangle or rhombus we will get a sequence changing between rectangles and rhombuses where all rectangles are similar to each other and all rhombuses are similar to each other.

Variations of our problem field concerning middle-quadrilaterals we will get if we look out for middle-polygons and the iteration of such middle-polygons with simultaneously stretching them with an appropriate factor so that the sequence does not collapse into a point. From regular polygons we will find such a factor for any fixed number of vertices. It then comes out there exists a limit of such a sequence of polygons which is a polygon with same number of vertices and some additional characteristics⁷.

⁷ Such a characteristic e.g. is that each diagonal is parallel to one side. See Tischel 1999.

Another variation we will get if we don't use middle-points but third-points or quarter-points and so on. This means that we cut the sides into three, four or more equal parts and then use such a subdivision-point of each side (mostly with same ordering).

Also looking out for analogies in space will lead us to interesting investigations. Concerning middle-points with polyhedrons we have to distinguish between middle-points of the edges and middle-points of the surface-sides. For the five platonic solids (convex regular polyhedrons) we will get semi-regular polyhedrons by using the middle-points of the edges. By using the middle-points of the surface-sides we get the well-known dualism between the platonic solids: dodecahedron-icosahedron, cube-octahedron, tetrahedron-tetrahedron. For polyhedrons with non-regular surfaces we first have to decide which point we want to call the middle-point.

But we don't want to go in details for these variations. Here we rather want to focus on analogies in space for some types of quadrilaterals we have already used above.

5. Analogies in space to plane figures out of the "sub-house of parallelograms"

The looking out for analogies between figures in plane and space leads us to an interesting problem field where students can deepen their knowledge and comprehension about figures in plane and space.

First of all it is clear for everybody that *the square and the cube* are analogous forms. Both have the characteristic that all sides respectively all edges have the same length and all angles in the surface-sides are right ones. On the other hand the cube is the only polyhedron which has only squares as surface-sides.

If we look out for an analogous solid to the *rectangle* we will come to the *cuboid*. The main characteristics of both, rectangle and cuboid, are the right angles between the sides respectively between edges and surface-sides. And the cuboid is the only polyhedron which has only rectangles as surface-sides. Moreover the rectangle has axis' of symmetry through each two midpoints of opposite sides whereas the cuboid has axis' of symmetry (in the space) through each two midpoints of opposite surface-sides (not paying attention to planes of symmetry in analogy to the axis' of symmetry concerning the rectangle).

A next consideration brings us to the *analogy of parallelogram and parallelepiped*. Both have the characteristic that opposite sides respectively opposite edges and opposite surface-sides are parallel. And the parallelepiped is a polyhedron which has only parallelograms as surface-sides. Moreover both figures are polygons respectively polyhedrons with point-symmetry. Furthermore we can construct a parallelogram out of two linear independent vectors \vec{a} , \vec{b} (where the vertices of the parallelogram are determined by \vec{o} , \vec{a} , \vec{b} , $\vec{a} + \vec{b}$) and we can construct a parallelepiped out of three linear independent vectors \vec{a} , \vec{b} , \vec{c} (where the vertices are determined by \vec{o} , \vec{a} , \vec{b} , \vec{c} , $\vec{a} + \vec{b}$, $\vec{a} + \vec{c}$, $\vec{b} + \vec{c}$, $\vec{a} + \vec{b} + \vec{c}$).

In particular for a cuboid we have the additional conditions $\vec{a} \perp \vec{b} \perp \vec{c} \perp \vec{a}$ and for a cube we have the additional conditions $\vec{a} \perp \vec{b} \perp \vec{c} \perp \vec{a}$ and $|\vec{a}| = |\vec{b}| = |\vec{c}|$.

By looking at the house of quadrilaterals there will now come the question about the *analogous polyhedron to a rhombus*. It is not so well known as cube and cuboid. But knowing a parallelepiped it is easy to describe. It is a parallelepiped with all edges of same length. We call such a solid a *rhombus-parallelepiped*. It can be characterized as parallelepiped which surface-sides are only rhombuses too. Moreover the rhombus has axis' of symmetry through each two opposite vertices whereas the rhombus-parallelepiped has axis' of symmetry (in space) through each two opposite vertices (not paying attention to planes of symmetry in analogy to the axis' of symmetry concerning the rhombus). The determining three vectors of a rhombus-parallelepiped have the additional condition that they have the same length, i.e. $|\vec{a}| = |\vec{b}| = |\vec{c}|$.

With these results we did find all analogies to figures of the "house of parallelograms". But in the "house of parallelepipeds" we can find more than the above named four types of polyhedrons because in space we have a little bit more freedom (more variables) so that there do exist also some *mixture types* of two "neighboured" types.

For example between the cube and the cuboid there exists also a type with squares as well as rectangles on its surface. This type of polyhedron has congruent squares on the bottom and on the top whereas the four other sides are congruent rectangles. We call such a figure a *square-prism*. We also can see this figure built out of a cube by stretching it in one direction perpendicular to a surface-side. The determining vectors of a square-prism are three vectors which are perpendicular to each other and of which two (only) have the same length, e.g. $\vec{a} \perp \vec{b} \perp \vec{c} \perp \vec{a}$ and $|\vec{a}| = |\vec{b}|$.

Between the cube and the rhombus-parallelepiped we find a type which has in its surface two pairs of congruent squares and one pair of congruent rhombuses. We can build it out of the cube by fixing the bottom-square and moving the parallel surface-side first parallel to the bottom and after that pushing it down so that one pair of opposite surface-sides remain as squares while the last two surface-side turn to congruent rhombuses. For this type of polyhedron there is no special name as far as I now but I will call it a *tilted-cube*. The determining three vectors of this tilted-cube all have the same length and two pairs of them are perpendicular, e.g. $\vec{a} \perp \vec{b} \perp \vec{c}$ and $|\vec{a}| = |\vec{b}| = |\vec{c}|$.

But also between the tilted-cube and the rhombus-parallelepiped we have another mixture type which can be built out of the tilted-cube by moving the square on top again, first parallel to the bottom in a direction orthogonal to the first moving direction and then pushing it down so that all edges keep their equal length. It then has two squares in parallel surface-sides and four surface-sides with the shape of a rhombus. For this type there is also no known name but I will call it a *double-tilted-cube*.

The determining three vectors of this parallelepiped all have the same length and one pair of them is perpendicular, e.g. $\vec{a} \perp \vec{b}$ and $|\vec{a}| = |\vec{b}| = |\vec{c}|$.

Between the rhombus-parallelepiped and the (general) parallelepiped we then can find one mixture type which has one pair of parallel rhombuses as surface-sides whereas the other surface-sides are (general) parallelograms. It can be built out of a rhombus-parallelepiped by stretching it in the direction of the connection of the midpoints of two opposite surface-sides. Therefore I will call it a *stretched-rhombus-parallelepiped*. Its three determining vectors have the characteristic that (only) two of them have the same length, e.g. $|\vec{a}| = |\vec{b}|$.

Between the cuboid and the parallelepiped we first find a type which I will call *tilted-cuboid*. It can arise out of a cuboid by tilting it in the direction of the connection of the midpoints of two opposite surface-sides. On its surface we have four rectangles and two parallelograms. Its three determining vectors have the property of two pairs of perpendicular vectors, e.g. $\vec{a} \perp \vec{b}$ and $\vec{b} \perp \vec{c}$.

According to the double-tilted-cube we now find a type between the tilted-cuboid and the (general) parallelepiped which I will call *double-tilted-cuboid*. It can arise from the tilted-cuboid by tilting it in the direction of the connection of the midpoints of the two other parallel surface-sides of the tilted-cuboid. The surface of the double-tilted-cuboid contains two rectangles and four parallelograms and among its three determining vectors we have one pair of perpendicular vectors, e.g. $\vec{a} \perp \vec{b}$.

But the above named mixture types are not all of them. We also find mixture types directly between a square-prism or a tilted-cube and a parallelepiped.

As a *generalisation of a square-prism and a tilted-cube* we find two different new types which I will call *tilted-square-prism* and *rhombus-prism*. The first one arises by tilting a square-prism parallel to the squares whereas the second one arises by shifting one rectangular surface-side or by stretching the squares of the tilted-cube. The surface of a tilted-square-prism consists of two squares, two rectangles and two rhombuses while the surface of a rhombus-prism consists of four rectangles and two rhombuses. Among the three determining vectors for both types we find one pair of vectors with equal length and two pairs of perpendicular vectors. With the tilted-square-prism the one pair of equal length is also one of the two pairs of perpendicular vectors (e.g. $\vec{a} \perp \vec{b} \perp \vec{c}$ and $|\vec{a}| = |\vec{b}|$) while with the rhombus-prism the one pair of vectors with equal length and the two pairs with perpendicular vectors are three different pairs (e.g. $\vec{a} \perp \vec{b} \perp \vec{c}$ and $|\vec{a}| = |\vec{c}|$).

As a generalisation of a double-tilted-cube and a tilted-square-prism or a rhombus-prism we finally find two types which I will call *double-tilted-square-prism* and *tilted-rhombus-prism*. The double-tilted-square-prism has in its surface two squares and four parallelograms while the tilted-rhombus-prism has in its surface two rectangles two rhombuses and two parallelograms. Among the three determining vectors we have one pair of vectors with

equal length and one pair of perpendicular vectors. With the double-tilted-square-prism the two pairs are the same (e.g. $\vec{a} \perp \vec{b}$ and $|\vec{a}| = |\vec{b}|$) while with the tilted-rhombus-prism they are different (e.g. $\vec{a} \perp \vec{b}$ and $|\vec{a}| \neq |\vec{b}|$).

As you can find out with systematic combinations of pairs of vectors with equal length and pairs of perpendicular vectors among the three determining vectors of a parallelepiped the above named special types of a parallelepiped are all sub-types which are possible. Also by looking at these pairs and its combination we find the following “house of parallelepipeds” (see Fig. 8).

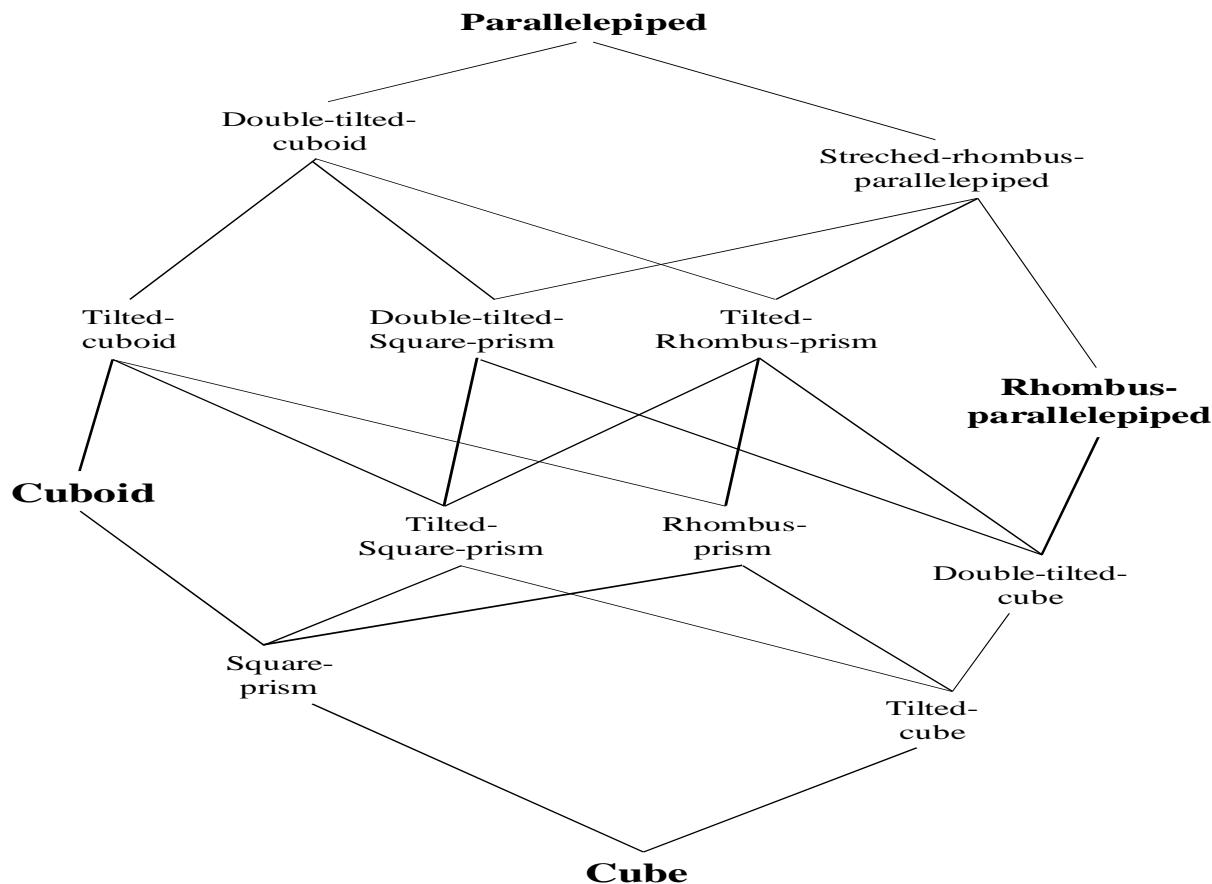


Fig. 8. The “house of parallelepipeds”

Besides this finding and ordering of special types of parallelepipeds we could make more investigations about all these types of parallelepipeds with help of vector geometry. Especially with scalar- and vector-product we could look at the measure of the surface and the volume as well as the height of the solid. Also the symmetries of all these types built a new interesting little problem field.

Another extension of this problem field we will meet if we look out for analogies in space for all symmetric quadrilaterals or a house of symmetric polyhedrons. As analogy of a kite or a symmetric trapezium e.g. we could look out for polyhedrons with eight vertices and one plane of symmetry. Already for this question we will find different solutions. But we also could look out for a polyhedron with one plane of symmetry and four, five or six vertices.

Other questions are those of the analogy of a triangle, a quadrangle or a (general) polygon. First it is clear that the analogy of a triangle must be a triangular pyramid (a polyhedron with four vertices and only triangles in its surface). Can we find and order analogies in space to the

special types of the house of triangles? And what should be the analogy of a quadrilateral? What about the analogies of convex (or non-convex) regular polygons in space?

I think beyond the problem fields I have described you can find a lot more geometrical problem fields⁸ which can animate students for working problems in geometry. But it is enough to begin with here.

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⁸ About problem fields with triangles, a systematic of regular polygons, polygons in space or trigonometric functions see e.g. Graumann 2001, 2002a and 2002b.

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