Structural stability and dynamic geometry: Some ideas on situated proofs

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Abstract: In this paper we survey the historical and contemporary connections in mathematics between classical "conceptual" tools versus modern computing tools. In this process we highlight the interplay between the inductive and deductive, experimental and theoretical, and propose the notion of *situated proofs* as a didactic tool for the teaching of geometry in the 21st century.

Kurzreferat:. Dieser Artikel konfrontiert vor einem historischem und aktuellen Hintergrund überblicksartig die Spannung zwischen ‚klassischen' und modernen Computerwerkzeugen. In diesem Zusammenhang wird das Zusammenspiel zwischen Induktion und Deduktion, zwischen Experimentellem und Theorie herausgearbeitet, und für ein Konzept des kontextuellen Beweisens als didaktisches Werkzeug der Geometrielehre im 21. Jahrhundert plädiert.

ZDM-Classification: E50, G10, R20

1. Methods in Mathematics: Classical and Modern Considerations

Pre-Euclidean Greek geometry evolved as a model of physical space. An important example is Erathostenes' method to measure the radius of the Earth. It appears that these examples, in which it is impossible to establish empirically their validity —due, for instance, to the dimensions involved—, were instrumental to develop the deductive method as a validation strategy. This way, it was possible to reach more than intuitive knowledge on physical space *without touching it*. The history of mathematics shows a continuous evolution of conceptual tool building. It is interesting to notice that even if the new technologies are not yet fully integrated within the mathematical thinking, their presence will eventually erode the mathematical way of thinking. The blending of mathematical symbol and computers has given way to an *internal mathematical universe* that works as the reference fields to the mathematical signifiers living in the screens of computers. This takes abstraction a large step further.

According to Balacheff & Kaput (1996), the main impact of information technology on educational systems is epistemological and cognitive, because it has contributed to the production of a new form of realism in mathematical objects. This new form of realism depends on the interpretative resources provided by the socio-cultural environment. Thus, technology has the power to become an educational agent for change but this process of change is complex.

The virtual versions of mathematical objects produce the sensation of material existence, given the possibility of changing them where they manifest themselves, that is, on the screen. Students' growing familiarization with computational tools allows these tools to be transformed into mathematical *instruments* (Guin & Trouche, 1999; Rabardel, 1995) in the sense that computational resources are gradually incorporated into the student's activity. For example, when secondary school students are asked to explore the relationships between the inscribed angle in an arc and the corresponding central angle, we see two behaviors in the classroom: students remain immobilized by the question (we think this is because they are not able to mobilize their expressive resources) or, when they have computational resources at their disposal (for example, dynamic geometry), they are led to draw up comparative tables between angles and to eventually realize that that the central angle is "nearly double" the inscribed angle in the same arc. The students' strategy, taking the inscribed angle from the central angle is possible thanks to the expressive power the students acquire through the computational tools (Moreno & Block, 2002).

In the absence of these, as we have already mentioned, it is not feasible for students to carry out the numerical comparison between the angles and to establish a conjecture. Nor are the students capable of producing a formulation associated with their explorations and express it in the language of the computational medium in which they are working. The computing environment is an *abstraction domain* (Noss & Hoyles, 1996), which can be understood as a scenario in which students can make it possible for their informal ideas to begin coordinating with their more formalized ideas on a subject. In the example of dynamic geometry, we can put it this way: The exploration of drawings and of their properties gives rise, through the semiotic mediation of dynamic geometry, to the recognition of a system of geometric relationships, which in the final analysis constitute the geometric object.

One of the aims of this paper is to conduct a historical survey that shows our natural tendency for mathematical tool building, conceptual and technological, and use this to understand how the implementation of the new technology should be conducted. We know that the first stage could entail working within the framework of a pre-established mathematical thinking. At that point, it becomes fundamental to understand the nature of knowledge that emerges from their interactions with those mediating tools. In other words, it is important to understand the epistemological role of the new tools when they are lodged into a previous mathematical-thinking infrastructure.

Working with computational tools leads us to face their incorporation from two different angles (Berger, 1998): as *amplifying* tools and as *cognitive re-conceptualizing* tools. These amplification and re-conceptualization processes can be illustrated in the following way: The amplification process is similar to the function of a

magnifying glass. Through this lens, we can enlarge objects visible at first sight. Magnification does not change the structure of the objects that are being observed, however, on the other hand, the reorganization process can be compared to the act of seeing through a microscope. The microscope allows us to observe what is not visible at first sight and, therefore, to enter a new plane of reality. In this way, the possibility of studying something new and of accessing new knowledge arises.
Computing environments provide a window for studying the evolving conceptions –caused by the presence of these new tools. This can be done analyzing the use of these tools to generate knowledge within a computational environment.

Our work with computational tools has led us to consider the phenomenology one can observe on the screens of calculators and computers. The screen is a space controlled from the keyboard, but that control is one of action at a distance. The desire to interact with virtual objects living on the screen provides a motivation for struggling with the complexities of a computational environment (Pimm, 1995).

During the time that passes while the graph is being drawn on the screen, the student observes the characteristics of the function that are reflected in its construction. We propose, therefore, that the student has the opportunity to transform the graph into an object of knowledge. This is similar to what the Greeks did with writing. They used the writing system not only as an external memory but also as a device to produce texts on which to reflect.

At first, students might make some observations *situated* within the computational environment they are exploring, and they could be able to express their observations by means of the tools and activities devised in that environment. That is the case, for instance, when the students try to invalidate (e.g., by dragging) a property of a geometric figure and they are unable to do so. That property becomes a theorem expressed via the tools and facilitated by the environment. It is an example of *situated proof*. Another example is that of Pick's theorem[1], which can easily be illustrated/discovered via the use of a simple java applet. By using technology one pick co-ordinates with a mouse click (that virtually simulates the physical action of marking pegs on a geoboard with a rubber band) and very quickly examine the pathological cases that one must cast out in order to discover the theorem (formula) inductively. The discovery of this result within a technological medium should call into question the necessity of a mathematical proof. The important didactic question to consider is the nature of this proof-should it be deductive or could it be situated within the technological medium.

[1] Let P be a lattice polygon. Assume there are I(P) lattice points in the interior of P, and B(P) lattice points on its boundary. Let A(P) denote the area of P. Then
$A(P) = I(P) + B(P)/2 - 1$ (Varberg, 1985).

2. Situated Proofs versus Deduction

A situated proof is the result of a systematic exploration within an (computational) environment. It could be used to build a bridge between situated knowledge and some kind of formalization. Students purposely exploited the tools provided by the computing environment to explore mathematical relationships and to "prove" theorems (in the sense of situated proofs). As a new epistemology emerges from the lodging of these computational tools into the heart of today's mathematics, we will be able to take off the quotations marks from "prove" in the foregoing paragraph.
Ruler and compass provided a mathematical technology that found its epistemological limits in the three classical Greek problems (trisection of an angle, duplication of the cube, and the nature of π). Ruler and compass embody certain normative criteria for validating mathematical knowledge. And more general, they are an example of how an expressive medium determines the ways to validate the propositions that can be stated there.
Now we can ask: What kind of propositions and objects are embodied within dynamical mathematical environments? The way of looking at the problem of formal reasoning within a dynamical environment is of instrumental importance. What we propose as a *situated proof* is a way to deal with a transitional stage. We cannot close the eyes to the epistemological impact coming from the computational technologies, unless we are not willing to arrive at new knowledge but only at *new education*. An impossible goal, indeed.

Representing the space on a plane and obtaining, *deductively*, empirically valid results on its structure, showed that plane pictures and the linguistic descriptions associated with those pictures, were able to capture, at least partially, the essence of space. This methodology upturned the relations between mathematics and the physical space: From then on, *mathematics would explain physical space*. Perhaps this is the kind of viewpoint that centuries later led Galileo to write that the book of nature was written in the language of mathematics and that, to understand that book, one had to speak that language. Today, no doubt, the persistence of a version of this conception has been instrumental for the development of modern science.

The deductive evolution of Greek mathematics took a long time but when finally it informed Euclid's ***Elements***, its fate was sealed. Geometers discovered that axiomatic method provided an efficient strategy "to save" mathematical knowledge. Deduction from first principles and within the system, is the key to make the mathematical seeds to germinate.

During the 17th century mathematicians adhered to the inductive approach to knowledge and considered that Greek attachment to axiomatics was exagerated. As a consequence, deductive method led room to more intense inductive times. The flow of discovery was taken as evidence that induction was the correct way to develop mathematics. But later, as has always been the case, the need to organize the bulk of discoveries led again to the consideration of deductive method. During the 19th century, mainly in the hands of Cauchy, Bolzano, and Weierstrass this return to the logical organization of mathematics, gave birth to what has been called the *Arithmetization of Calculus* (Bottazzini, 1986).

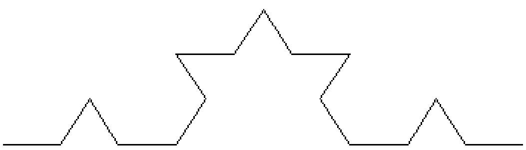
Weiertrass viewpoint opened the door, among other results, to the analytic proof of the existence of continuous non-differentiable functions. This result was instrumental to abandon visual intuition as a secure guide in the development of mathematics. But not all were happy with the new state of affairs. In 1904, the Swedish mathematician Helge Von Koch (1870-1924), published a paper in which he disapproved the exceedingly analytic approach followed by Weierstrass.

Von Koch wrote:

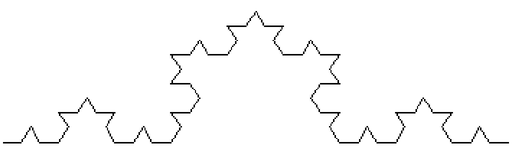
> *Until Weierstrass constructed a continuous function not differentiable at any value of its argument it was widely believed in the scientific community that every continuous curve had a well determined tangent...Even though the example of Weierstrass has corrected this misconception once and for all, it seems to me that his example is not satisfactory from the geometrical point of view since the function is defined by an analytic expression that hides the geometrical nature of the corresponding curve...This is why we asked ourselves —and we believe that this question is of importance also as a didactic point in analysis and geometry— whether one could find a curve without tangents for which the geometrical aspect is in agreement with the facts.*

Von Koch approach to this problem (i.e. the existence of a continuous non-differentiable function) was genuinely geometrical. Today it is an icon in the world of fractals. It is simple to understand the construction process employed by Von Koch from the following figures:

Stage 1



Stage 2



Stage 3

The Italian mathematician Ernesto Cesaro (1859-1906) recognized the fractal nature of this curve and wrote:

It is this similarity between the whole and its parts, even infinitesimal ones, that makes us consider this curve of von Koch as a line truly marvelous among all. If it were gifted with life, it would not be possible to destroy it without anihilating it whole, for it would be continually reborn from the depths of its triangles, just as life in the universe is. (in the *Atti d. R. Accademia d. Scienze d. Napoli,* 2, XII, number 15):

The need to take into account the dual role of induction and deduction, of discovery and proof in mathematics, has been fully recognized. For instance, written in collaboration with Cohn-Vossen, Hilbert (1999) expressed:

> *In mathematics as in any other scientific research, we find two tendencies present. On the one hand, the tendency towards abstraction seeks to crystallize the logical relations inherent in the maze of material that is being studied, and to correlate the material in a systematic and orderly manner. On the other hand, the tendency towards intuitive understanding fosters a more inmediate grasp of the objects one studies, a live rapport with them, so to speak, which stresses the concrete meaning of their realtions...it is still as true today as it ever was that intuitive understanding plays a major role in geometry.*

Courant and Robbins (2001) called attention on the risks run by mathematics if, inadvertently, the balance between inductive and deductive thinking is broken:

> *There seems to be a great danger in the prevailing overemphasis on the deductive-postulational character of mathematics. True, the element of constructive invention, of directing and motivating intuition... remains the core of any mathematical achievement, even in the most abstract fields. If the crystallized deductive form is the goal, intuition and construction are at least the driving forces.*

More recently, Lakatos (1976) has insisted on this perspective.In his classic treatise he unfolds the power of analogy, of systematic experimentation during the process of discovery that leads to a mathematical theorem. Lakatos (1976) considers that (op. cit. P. 142):

> *The deductivist style hides the struggle, hides the adventure. The whole story vanishes, the succesive tentative formulations of the theorem in the course of the proof-procedure are doomed to oblivion while the end result is exalted into sacred infallibility.*

As it has already been mentioned, along the historical development of the discipline, the mathematical pendulum oscilates from inductive approaches to deductive ones. Like if this were the result of a natural law. Yet, this is quite natural as mathematics cannot be other thing but a human activity and, as such, reflects the quandaries of human thinking.

Historical examples that convey the interplay between the inductive and deductive can be seen in the marriage between non-Euclidean geometries and modern Physics. If one considers Weyl's (1918) mathematical formulation of the general theory of relativity by using the parallel displacement of vectors to derive the Riemann tensor, one observes the interplay between the experimental (inductive) and the deductive (the constructed object). The continued evolution of the notion of tensors in physics/Riemmanian geometry can be viewed as a culmination or a result of the flaws discovered in Euclidean geometry. Although, the sheer beauty of the general theory was tarnished by the numerous refutations that people came up with when the general theory was proposed, one cannot deny the present day value of the mathematics resulting from the interplay of the inductive and the deductive.

According to Bailey & Borwein (2001), Gauss , used to say *I have the result but I do not yet know how to get it* Besides, he considered that to obtain the result a period of *systematic experimentation* was necessary. There is no doubt then, that Gauss made a clear distinction between *mathematical experiment* and *proof*.
In fact, as Gauss expressed, we can reach a level of high certitude concerning a mathematical fact, before the proof and at that moment we can decide to look for a proof. Many of Euler's results on infinite series have been proven correct according to modern standards of rigor. They were already established as valid results in Euler's work. Then, what has remained and what has changed in these theorems? If instead of looking at foundations we choose to look at mathematical results, as resulting from a human activity that is increasingly refined, then we could find a way to answer that difficult question. This perspective coheres with the view that mathematical ideas can be thought through successive levels of formalizations. *The theorem is the embodied idea*: the proof reflects the level of understanding of successive generations of mathematicians. Different proofs of a theorem cast light on different faces of the embodied idea.
Nowadays, the computer (the tool that "speaks mathematics" in Lynn Steen able expression) is responsible for a new face from this old tension. In 1976, when Appel and Haken proved the Four Color Theorem using computers in a crucial way, they were far from imagining the harsh reaction of a substantial part of the mathematical community. Theirs was not a proof acording to the classical definition. It was not the case of using a computer to help mathematicians in their quest for truth. Yet, with the new proof, cognition up to a considerable amount, had been transferred to a machine. The computer appeared as a cognitive partner, on equal terms, with the humans. The challenge cast by this new partner could not be ignored: The Gauss' *mathematical experiments* turned into a new kind, thanks to the computer. Since then, the role of the computer in mathematical investigation has increased, but this does not mean that its role is accepted by all. More recently, Thomas Hales, announced his proof of Kepler's conjecture on the packing of spheres, which states that the densest arrangement of spheres is one in which they are stacked in a pyramid —as grocers arrange oranges. Hales' proof is supposed to appear this year in the journal *Annals of Mathematics* with a cautionary note by the editors, explaining that they are 99% percent sure that the proof is correct but that "proofs of this type, which involve the use of computers to check a large number of mathematical statements, may be impossible to review in full" (*Nature*, 424 12-13, 2003). This is a very delicate matter that has to be thought with the utmost care as it involves deep epistemological questions. Let us remind some excerpts from the letter written by Archimedes —addressed to Erathostenes—, to introduce his newly found *Mechanical Method* to obtain, among other results, his formula for the volume of the sphere:

> *Certain things became clear to me by a* ***mechanical method****, although they had to be demonstrated by geometry afterwards because their investigation by the said* ***mechanical method*** *did not furnish an actual demonstration. But it is of course easier, when we have previously acquired, by* ***the method****, some knowledge of the questions, to supply the proof than it is without any previous knowledge.* (Peitgen, Jürgen y Saupe, 1992)

If instead of the ***bold*** expressions we write "computer", we obtain the common viewpoint of a substantial part of the mathematical community with respect to the computer. That is, the computer is at most a tool to discover, never to proof. But the Four Color Theorem, Hales; proof of Kepler's conjecture are examples that something is beginning to change in the heart of that human activity, that is, in mathematics. Emphasizing mathematical activity as the heart of mathematics, V.I. Arnold (2000) one of the most distinguished mathematicians of the last decades, has said:

> *Proofs are to mathematics what spelling (or even calligraphy) is to poetry. Mathematical works consists of proofs as poems consist of characters.*

Arnold is essentially trying to establish a very sharp distinction between the activity and the dried out results. In the same paper, Arnold (2000) quotes Sylvester saying that:

> *A mathematical idea should not be petrified in a formalised axiomatic setting, but should be considered instead as flowing as a river*. (p. 404)

One should always be ready preserving the informal idea. In these days, numerical algorithms have been designed that allow the computation of a numerical answer with a precision beyond one hundred thousand decimal figures (Bailey &Borwein, 2000, p. 53, op.cit.). One can ask again, if we are not entering a new era wherein the previous relationships between exploration and justification are being qualitatively changed by the mediation of computational tools. In particular, it is crucial to understand the impact of these tools in the learning of mathematics which means that we must elaborate on the relations between the computer and mathematical thinking within educational settings.

3. Tools and their mediation in cognition: the need of a long term perspective

Viewed from an evolutionary perspective, tools shape the learner... by webbing internal and external resources.
Webbing and the co-evolution of knowledge, tool and activity means that we need new methodologies and … ways to think about the meanings students develop.

The computational versions of mathematical objects produce the feeling of material existence, as we have the potential to change them wherein they make manifest their existence, that is, on the screen. Students' growing familiarization with computational tools allows these tools to be transformed into mathematical *instruments* (Guin & Trouche, 1999) in the sense that computational resources are gradually incorporated into the student's cognitive activity. An instrument can be considered as a functional organ made up of a tool component and a psychological component. This organ construction, named *instrumental genesis*, is a complex process, *needing time,* and linked to the tool characteristics and to the subject's activity, his/her knowledge, and former method of working. We suggest, then, that exploring with computational tools eventually allows students to realize how the mediational role of these tools helps them reorganize their problem-solving strategies.

As stated previously explorations within an *abstraction domain* facilitate the understanding of the character *situated* in the propositions and the situatedness of its proofs. *Situated theorems* refer to the understanding and articulation of processes within the context in which they have been explored. Let us explain: At first, students might make some observations situated within the computational environment they are exploring, and they could be able to express their observations by means of the tools and activities devised in that environment. That is the case, for instance, when the students try to invalidate (e.g., by dragging) a property of a geometric figure and they are unable to do so. A situated theorem is the result of a systematic exploration within an (computational) environment. It could be used to build a bridge between situated knowledge and some kind of formalization. It is a kind of "border object".

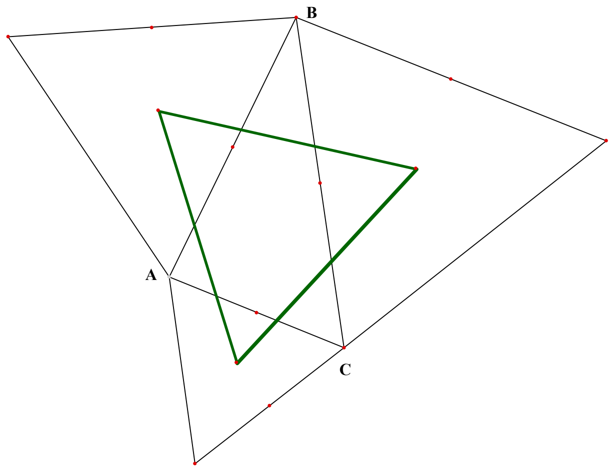
As Noss and Hoyles (1996) explained, while discussing related ideas, students can generate and articulate relationships that are general to the computational environment in which they are working. This means students can develop an ability to state general propositions in the language of the environment (i.e.: the can develop a sense of *situated abstraction*). We can say that these computational environments derive their educational power from the posibility to manipulate and externalize abstract ideas while working within their borders.

Situated abstractions (S.A's) are meanings which emerge through webbing with tools that 'embed' mathematical knowledge.S.A's are an expression of invariant relationships perceived within a setting. That is, they are expressed within the tools and language of the setting. It is a case of abstraction within, rather than away from, context

4. Formal reasoning within a computational environment

What is the nature of the co-evolution of knowledge, tool and activity?
The use of technology offers great potential for students to search for invariants and to propose corresponding conjectures. We have illustrated how students build dynamic environments to represent problems that eventually lead them to propose conjectures and *prove* them using the tools of the environment. Thus, the software becomes a tool for students to look for and document the behavior of objects and relationships and explore their structural nature. Our next example has to see with *features of mathematical proof* privileged via the use of technology. The mathematical discussion involving this example was much more subtle and students (teachers included) had difficulties when trying to understand it. But it was to introduce mathematical proofs within a computational environment. In a sense we can interpret this part of our work as a *teaching* experiment. At first we discussed the notion of macro construction within the Cabri dynamic environment. After a while, it became clear for the students that a geometric object built using a macro was a genuine geometric object living in the Cabri universe. This way we could answer the question: To what extent mathematical arguments or ways to approach problems within a Cabriworld vary from the traditional approaches with paper and pencil?
We all know how controversial can be to discuss the place of computers and calculators in the field of mathematical proofs. And how important it is for students to understand what a proof is.
Can we *prove* a geometry theorem using Dynamic Geometry? We want to illustrate how this can be made feasible. Let us study Napoleon theorem:

Given an arbitrary triangle, construct on each side the corresponding outer equilateral triangle. Then the triangle that results by joining the centroids of these three triangles is always an equilateral triangle.



The thick triangle is the Napoleon triangle corresponding to triangle ABC. Let us recall how we proceed with this construction. We built two Cabri-macros:

Given two vertices, the first macro determines the third one so that we have an equilateral triangle.

The other macro produces the centroid of a given triangle.

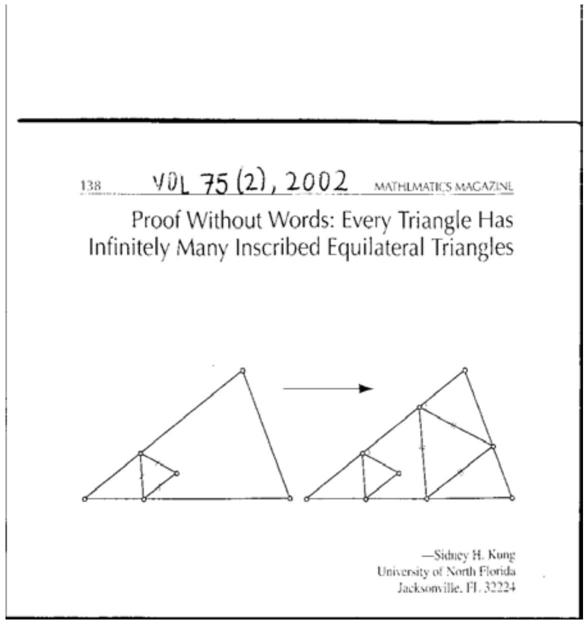
Then, after playing with the construction trying to "destroy" the equilateral triangle (Napoleon's triangle) one "has to accept" the validity of the proposition. This is a natural approach if we are exploring a (possible) theorem within a dynamic environment. That is, we try to make sure that the claim made is an invariant with respect to dragging. But we can go farther than that: we can give a proof within the Cabri world. We mean, a *situated* proof.

In fact, we design a macro that enables us to construct equilateral triangles and each time we use it, the result is a genuine equilateral triangle. Likewise, as we have a macro that determines the centroid of a triangle, each time we use it, the result is the genuine centroid of the given triangle. Taking this into account, we realize that when we point out a vertex of the Napoleon triangle we can read the question: "what object?" We can answer "equilateral" or "centroid" and that means that the vertices of the Napoleon triangle always coincide with the centroids. We know then, that the Napoleon triangle is always equilateral. It is important to remark that this kind of reasoning takes us beyond the perceptual level: this is precisely the case when we intend, for instance, with paper and pencil, to prove a geometrical assertion. Working within a computational environment forces us to adopt a different strategy: we have to resort to the nature of the mediating tools we have at our disposal. Of course, we cannot loose sight of the internal mathematical universe residing in the innards of the computer.

Exploring and justifying: Other examples

Let us consider the following problem:

Is it possible to inscribe an equilateral triangle in an arbitrary triangle ABC? It is interesting to recall the "proof without words" solution: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Proof Without Words: Every Triangle Has Infinitely Many Inscribed Equilateral Triangles

—Sidney H. Kung
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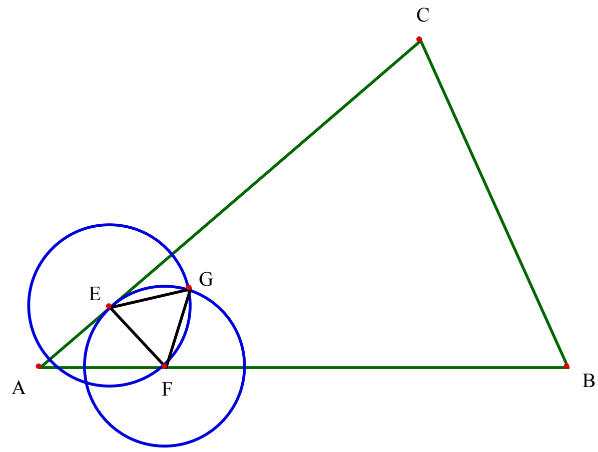
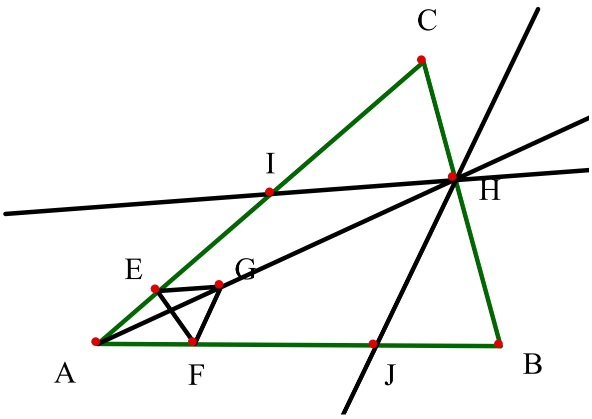
Jacksonville, FL 32224

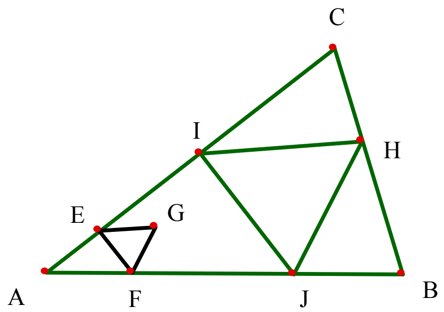
Let us see how we can work this example with the mediation of Cabri. First we have the original triangle ABC and we construct an equilateral triangle as shown in the following figure:



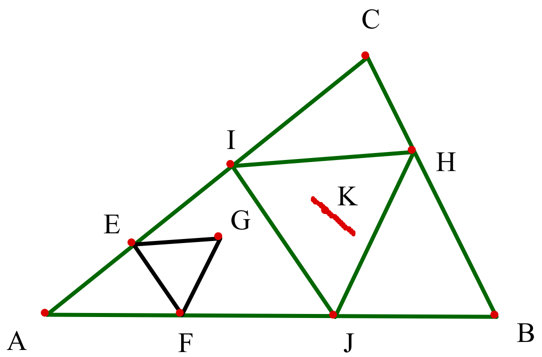
That is, we construct the segment EF and then the third vertex G as the intersection point of circles, as shown. This is the classical construction of an equilateral triangle. Now, we construct the ray connecting A and G and name H the intersection point with side BC. Through H we construct paralell lines to EG and FG and name I, J the intersection points of the paralells with sides AC and AB respectively.



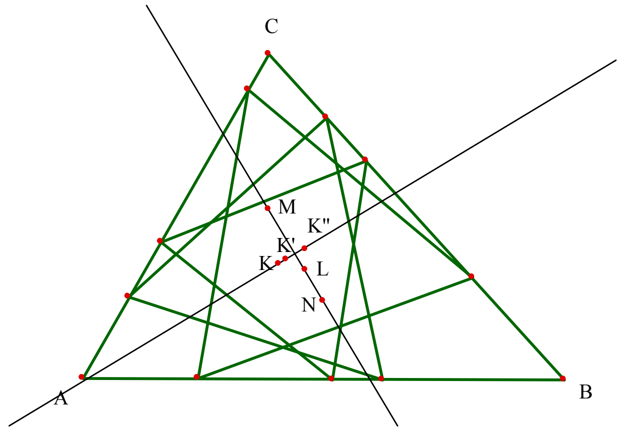
The triangle JHI is equilateral and is inscribed in the original triangle ABC. But now comes the gift form Cabri: we can drag points F or E and obtain with each new position of any of these points(E or F) a new equilateral triangle inscribed in ABC. We get all these inscribed triangles in a dynamic fashion from the initial one, JHI as point F (or E) moves on side AB (or AC).



having solved this problem we can explore the following proposition: Given three arbitrary inscribed triangles in ABC, the centroids of them are always collinear.
We will first explore dynamically the problem and then extract a formal solution, all within cabriworld.



If K is the centroid of triangle JHI, then <u>**we**</u> ask Cabri to give the locus of K as F (or E) moves, the answer is always the red (**thick**) segment we see in this last figure: it does not matter where we place F or E, (re-placing F or E produces a new equilateral triangle JHI) we always obtain the same red **(thick)** segment. This suggest that all centroids are collinear. But the exploration gives us something else: it gives a Cabri method to formally solve the problem at hand.
How are the lines determined by the centroids and the Euler line of triangle ABC? (We wish to thanks David Benitez for this example.)
The answer is: they are perpendicular. The following figure illustrates this property:



K, K' and K'' are the centroids of the three inscribed equilateral triangles and L,M,and N are respectively, the centroid, the orthocenter and the circumcenter of triangle ABC. Of course, the line these last three points determine, is the Euler line[2] of ABC. As in the case of Napoleon theorem, using macros, constructed with specific purposes, will enable us to **prove** the propositions within the Cabriworld.

Now consider the example of the Morley Triangle, a delightful geometric object associated with the Morley Trisector theorem, which is easily discovered and maniputable in a dynamic environment like Cabri. The theorem states that:
The three points of intersections of adjacents trisectors of the angles of any triangle form an equilateral triangle.

In general, angle trisection is impossible with the classical Greek methods of compass and straightedge, which is perhaps one of the reasons this theorem was only discovered about a 100 or so years ago. However trisectors are constructable in a dynamic medium such as Cabri via the use of transformational geometry features. This opens up the possibility for students to dynamically explore the existence of this intriguing object, which hitherto would have been difficult to realize in the classical static medium. As is indicated in the next two constructions (Swicegood, 2004) in a dynamic medium, the proof of the validity of Morley's Theorem does not involve any mathematical knowledge beyond elementary high school geometry.

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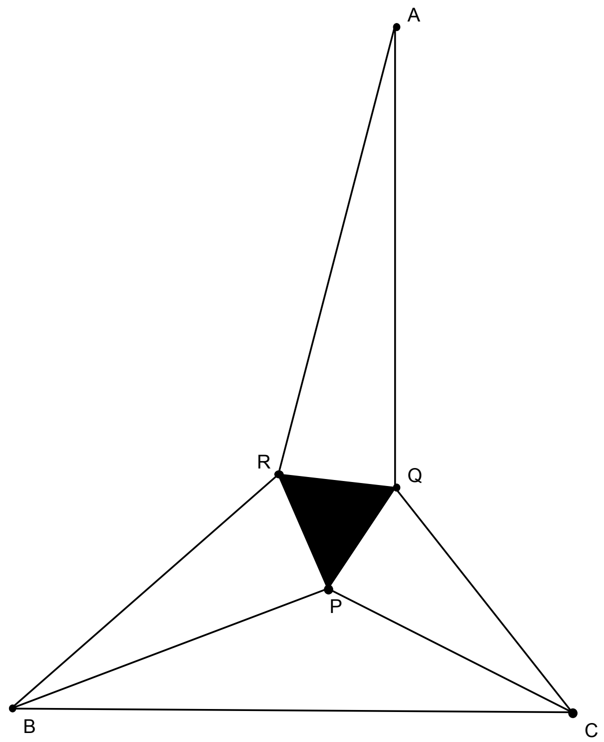
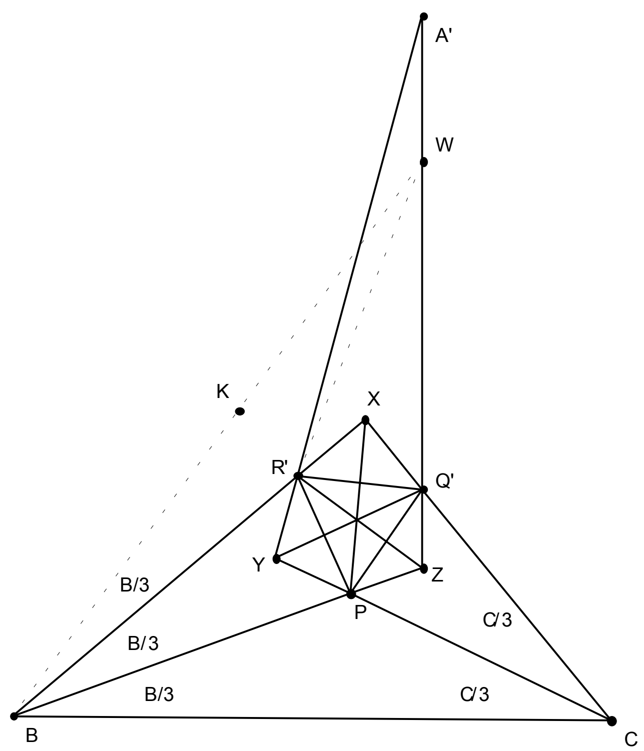
[2] A note on Euler line
The points L,M,and N are collinear. We can dynamically explore this property using the geometric transformation *dilation.*



Construction #1 of Morley's Trisector Theorem



Construction #2

5. Final remarks

Explorations within a computational environment eventually allow students to generate and articulate relationships that are general in the environment in which they are working. Those relationships which encapsulate general statements have been called *situated abstractions*, precisely because they are bound into the medium in which they are expressed (Noss&Hoyles, 1996). What we have introduced in the last section is a kind of proof we could call *situated proof.* In a sense, *every proof is situated* but emphasizing the situatedness while working within a computational environment pays an extra bonus. In our study, whose goal was to explore how students "proved" a mathematical proposition within a computationsl environment, we worked with 17-18 years olds, trained in dynamic geometry. For the development of the activities, teams of two or three students were formed. In this, as in other related cases, students became aware of invariants and they could express the relevant ideas *but only within the expressive medium* made feasible by the computer.

We remember our encounters as students of mathematics with Non-Euclidean Geometry and the semiotic mediation of exterior differential forms that made possible to extend the geometry of surfaces to Riemannian manifolds. It gradually, slowly, and painfully became clear, through the study of this kind of geometry, that visualization was not a matter of looking at but looking through the objects that were born into our perceptual field.

Objectifying and the quality of the *perceptual* were different. Symbolizing made the difference. Perhaps that is what Poincaré was trying to say when he introduced his orthogonal circles model of Hyperbolic Geometry: that one could *perceive* the orthogonal circles as orthogonal circles or one could objectify them as lines from a Non-Euclidean world. That happens with every model: one looks *through* it. And at that moment, the semiotic mediation of the model operates its magic: it generates meaning.

But, what happens when one tries to teach? Teaching is communication with all the complexity that communication involves. For all of us, it is very sad to see frustration in the eyes of our students and even in ourselves, when one discovers that certain beautiful idea is eluding our students' understanding: as if there were no path between the minds of the people in the classroom. But things have been changing in the last years. As teachers of mathematics, as researchers in the field of mathematics education, we are very fortunate to have Cabri at our disposal.

In the request for a proof, a mathematician analyzes the present problem, conjecture, and some previous proofs and examines whether the conjecture is true and how it could be deduced from known theorems. This process ends by finding a proof or refuting the conjecture. In the former case, the analysis is followed by the organization of the proof: the process of arranging the results of analysis into a deductively articulated argument. The deductively organized proof is explained to other people through lectures or publications. However, a proof becomes a proof only when it is accepted in a mathematical community (Manin 1977).

Therefore, explaining a proof to other people is not just presenting a proof. The mathematician has to try to convince other people. Here attention is required as to the social context of explaining. The original proposition to be proved may be modified through the interaction with other members of the community. Thus, the discovery of a proposition does not necessarily precede its proof.
The preceding three processes overlap and affect each other. Sometimes, any one of these processes is called a proof. Depending on the emphasis, the labels of proving are different. The nature of proof is described as a test or thought experiment when the analysis is meant a verification or justification when the synthesis is meant, and convincing people or a message when the explanation to other people is meant.

According to Lakatos's case study, Cauchy's proof of Euler's conjecture on polyhedra gave birth to the rubber sheet theory, which formed a view of polyhedra completely different from the theory of solids. Unlike the latter, the former does not concern angles, lengths, ratios, shapes of lines and faces, and so forth but connections among vertices. Moebius's seminal ideas eventually contributed towards the generalization of the Euler formula to the general topological formula N(min)-N(saddle) + N (max) = 2 -2g, where g is the genus of the surface. 2-2g, named the Euler characteristic of the surface and is very similar (almost analogous) to Euler's formula mentioned in the paper. Finally, Morse theory connects these two formulas and is the generalization of the ideas that Moebius developed nearly 130 years before.

Wittgenstein (1978) points out that the proof changes the grammar of our language, changes our concepts. It makes new connections, and it creates the concept of these connections. As a result, a conjecture is understood differently than before. In terms of Lakatos (1976), the conjecture has been proved in several different ways, it receives several different interpretations. Thus, different proofs yield different theorems.
Once the conjecture having been proved faces refutations or criticism, those assumptions constituting the interpretations are disclosed and articulated (Von Glasersfeld 1983). Because those differences in understanding of a conjecture are not formulated explicitly, they are usually unnoticed. In Lakatos's case study on Euler's conjecture mentioned previously, for example, such assumptions as simple and simply-connected, inherent in the rubber sheet theory of Cauchy's proof, where isolated and developed into basic constructions of theory after counterexamples were proposed.
Psychological, structural, and social perspective of proof has different positions. Criticism of a proof was not conceived in the psychological perspective of proof as well as structural perspective of proof. Because the model of mathematical practice in traditional mathematical philosophy is one ideal mathematician, absolute criteria and infallible intuition are supposed for judging the validity of proof. However, from the sociological perspective, the criteria and intuition for judge of proof are community dependent!

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