On the teaching and learning of Dienes' principles

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Abstract: Zoltan Dienes' principles of mathematical learning have been an integral part of mathematics education literature and applied both to the teaching and learning of mathematics as well as research on processes such as abstraction and generalization of mathematical structures. Most extant textbooks of cognitive learning theories in mathematics education include a treatment of Dienes' seminal contributions. Yet, there are no available studies at the tertiary level on how students internalise the meaning of Dienes' principles. This paper explores post-graduate mathematics education student's understanding of Dienes' principles and their ability to reflexively apply the principles to their own thinking on structurally similar problems. Some implications are offered for university educators engaged in the training of future researchers in the field.

Kurzreferat: Zoltan Dienes' Grundzüge des Mathematiklernens sind mittlerweile ein nicht mehr wegzudenkender Beitrag in der fachdidaktischen Literatur; sie sind bedeutsam sowohl für Lehr- und Lernprozessen als auch für die Forschung, die sich mit der Abstraktion und der Verallgemeinerung von mathematischen Strukturen beschäftigt. Auch die meisten Lehrbücher zur Kognition von Lernprozessen erwähnen Dienes' Beiträge. Es ist allerdings erstaunlich, dass bislang Studien fehlen, die sich unter Zugrundelegung von Dienes' Ansätzen mit dem tertiären Bereich beschäftigen. In dem vorliegenden Aufsatz wird der Versuch unternommen, diese Prinzipien von postgraduierten Mathematikern reflektieren und dabei auf eigene, strukturell ähnliche Probleme wie bei Dienes anwenden zu lassen. Diese Analyse hat erste interessante Folgerungen für spätere Untersuchungen deutlich gemacht, was die universitäre Ausbildung von Mathematikdidaktikern anbetrifft.

ZDM-Classifikation: B50, C30

1. A Brief Review of Dienes' Principles

The work of Zoltan Dienes is by no means new in mathematics education literature. In Mura's (1998) survey of mathematics educators in Canadian universities, among the most influential mathematics education authors or theorists cited most frequently were the names of Piaget, Dienes, Freudenthal and Bruner. Dienes (1960) originally postulated four principles of mathematical learning through which educators could foster mathematics experiences resulting in students discovering mathematical structures. The first principle, namely *the construction principle* suggests that reflective abstraction on physical and mental actions on concrete (manipulative) materials result in the formation of mathematical relations. *The multiple embodiment principle* posits that by varying the contexts, situations and frames in which isomorphic structures occur, the learner is presented opportunities via which structural (conceptual) mathematical similarities can be abstracted. *The Dynamic principle* states that transformations within one model correspond to transformations in an isomorphic model although the embodiments of these models are different. Finally the *Perceptual variability principle* recommends that when presenting problem situations one should include perceptual distractors, i.e., one should vary the perceptual details of the problem but include some common structural characteristics so that students have an opportunity to link structurally similar problems.

A considerable body of mathematics education literature can be traced back to the seminal ideas of Dienes. For example Dienes' principles influenced the design of some items in the Rational Number Project (Lesh, Post & Behr, 1987) Another example is Mitchelmore's (1993) theory of *abstract-apart* and *abstract-general* concepts in the formulation of mathematical generalizations. Abstract-apart concepts are formed when a learner does not link structural similarities in perceptually varying situations, whereas abstract-general concepts emerge when successful linkage occurs across embodiments, and perceptually varying situations. Most recently Dienes' principles have been adapted by Lesh & Doerr (2003) in the design of authentic modelling activities.

2. Conceptualizing Dienes Principles: Study Design

In the remainder of the paper, the difficulties of internalising and applying these principles will be discussed in the context of a graduate mathematics education course. This is an especially important but under-investigated area of mathematics education with major implications for the field. Batanero et al (1992) suggested that in order to develop any kind of a theoretical foundation for the field of mathematics education, one of the most crucial tasks would be to focus on the preparation of researchers in the field. In the United States, mathematics education doctoral programs are still in their infancy compared to established programs in mathematics. Typically among the various requirements for graduating with an advanced degree in mathematics education is university coursework. The present discussion is situated in one such course on cognitive learning theories. The five students in this course, taught by the first author were in preliminary stages of designing research to obtain either their M.S or Ph.D degrees. Students were exposed to Dienes' principles of instruction via numerous assigned readings in the course and the ensuing classroom discussion about the readings. The two main texts used in the course were English & Halford (1995) and Lesh & Doerr (2003). In addition, students participated in a problem-solving study as an integral part of the course.

The problem-solving study was designed as follows. Students were individually presented with numerous problem-solving situations and asked to "talk-aloud" about their impressions of a given problem and possible strategies to solve them. Although students were not asked explicitly to solve the problems, all five students invariably attempted to solve them. After the first problem-solving session, students were given all the problems and asked to solve and reflect on problems they had experienced difficulties with during the problem-solving session. After 6 weeks the students participated in a second problem solving session, which involved them 'talking-aloud" about the problems they had attempted. In this second session students were presented with some new problems, many of which were isomorphic to the original problems but perceptually different. These problem-solving sessions took place in the first author's office and were approximately one hour each. The sessions were audio taped, transcribed and used in conjunction with students' written artefacts to assess their understanding of Dienes' principles as well as their ability to reflexively apply it to their own thinking. At the end of the course, the students were presented with all the problems on the final assessment (exam) and explicitly asked to apply Dienes' principles to link problems they thought were structurally connected. Triangulation of data sources was achieved by collecting data from different sources such as (1) the interview transcripts, (2) student problem-solving artefacts and (3) written final in-class assessment. For reliability purposes, classroom discourse throughout out the semester was audio taped, transcribed and used to check for consistencies in student's responses during the interviews and classroom assessments. Since the author was an integral part of the course and the interviews, the ethnomethodolgical approach (Holstein & Gubrium, 1994) was most appropriate to interpret events in the classroom and the interviews. The data from the discourse and interview transcripts were analyzed in five iterative cycles for emergent student understanding of Dienes' principles.

3. The Problems

The problems chosen for the study were mostly combinatorial in nature, given the mathematical sophistication of the post-graduate students and their prior exposure to counting techniques in previous mathematics courses. In the first problem solving session, problems A and B were among the four problems used. In the second problem solving session problems C and D were the two new problems presented to the students. In the final in-class assessment all of these problems were presented to the students.

Session 1

A. One of Santa's helpers makes at least one toy everyday but not more than 730 toys in a year (including leap years). Prove that for any given positive integer n, the elf makes exactly n toys over some string of consecutive days. (Lozansky & Rousseau, 1996)

B. What is the last digit (i.e., the digit in the units place) when you expand 7^{365} ?

Session 2

C. Prove (or disprove) that there exist two powers of 7 whose difference is divisible by 2004.

D. Suppose you are given a set of seven distinct integers, prove that there must exist two integers in this set whose sum or difference is a multiple of 10. (Fomin, Genkin & Itenberg, 1991).

The reader will note that problem C looks perceptually similar to B but is structurally very different and infact similar to A. Problem D on the other hand looks perceptally different from B but is structurally similar to A (and C). It was hoped that the structural similarity of problems A, C and D would become evident to the students when they realized that these three problems required the delineation of two classes of objects in the solution process and the application of the Dirichlet principle to prove the given statements. That is, if 'n+1'items are to be distributed among 'n' boxes, then at least one box will contain more than one item.

4. Student Understanding of Dienes' Principles

For the sake of brevity, the emergent understanding of one student will be re-constructed and analysed within the framework of the problems and the four principles. First consider the student's attempt on problem A.

Student X [Work from Interview Artefacts: Session 1]

Let x_i *= toys on day i ;* $x_i \in Z, x_i \geq 1$

$$\sum_{i=k}^{l} x_i = n$$

365 days in a year, 366 (leap year).
x toys per day, $x \geq 1$*;* $366x \leq 730$
$366 \leq 366x \leq 730$*, so* $1 \leq x \leq$ *less than 2*

Student X [Interview: Session 1]
A= Author; X = Student

A: Have you seen a problem of this type?

X: Not really…Well, I am starting to look at it from a linear programming standpoint… You know thinking about putting restrictions on the variable and what can happen. Then going from there. I think I want to combine these two to get a cap on x. So lets see. This one if I multiply it by 366, I get that. So 366 has to be less than or equal to 366x which is less than or equal to 730. Maybe I like it in terms of x but I didn't want to divide 730 by 366 without a calculator. It is somewhere around 2. So x is between 1 and 2. That doesn't seem very productive. But close to 2.

A: So are you saying that he can't make three toys one day based on the current conditions?

X: If he has to make the same amount of toys every day.

A: Does it say anywhere that he has to make the same amount of toys?

X: No it doesn't. So if he had to make the same amount of toys per day it would have to be, he couldn't make three. He could make at most two, I think. I just have to multiply 366 here. So he can't quite make two even. Ok, prove that for any given positive integer n the elf makes exactly n toys every some string of consecutive days. So I would have to try to make whatever this fraction is, so.. it is a little less than two. I can do that division. But if he has to make exactly n toys over some string of days. I don't… well, I don't have an exact number of what x has to be. So I don't think I can prove that in some amount of days he has to make exactly n toys. I think I could put a bound on how many toys he has to make. But I don't think, for instance if n was 10, I don't think I could tell you how many days it would take him to make exactly 10 toys.

Commentary
The student interpreted the question to be asking for an explicit formula (or function) which would give the string of consecutive days for any chosen n, as opposed to just proving the existence of such strings for any given n. After the student had worked on the problem for several weeks, the intent of the problem became clear. In the second interview, the student talked about the attempt to prove the given statement.

Student X [Interview: Session 2]

A: Ok, do you want to look at the Santa problem? So the last time you were here you tried to set up some kind of a linear programming approach.

X: Well, I think at that point I was concluding that I didn't know how to prove it. I still don't think that ...it is provable. I think that it is probably true.

A: So could you elicit the reasons why it is true?

X: Well, I started to try a proof by induction and I can get n=1 because there is some day in a year where he has to make one toy. If everyday was greater than or equal to 2 he would make too many toys in a leap year.

A: So the leap year kind of came into play all of a sudden.

X: Because I had to use that to get n = 1. But then where do you go? I mean if you have consecutive days that you make n toys, how does that relate to consecutive days where they make n + 1 toys? Well, I don't know that they make one toy on a day right before or right after those consecutive, so I don't know that you can use that. I am trying to induct on that. Specifically I tried to think of an example that was coming to mind of a worse case scenario. So I thought, ok, this Santa's helper, this little elf guy, he makes one toy per day up until the last day of the year. On December 31st he makes 365 toys in a leap year. That makes his 730 at the most.

A: So he meets his quota.

X: Right. I mean it could be anything for that matter. I could be 200 because it said not more than 730. And he repeats this every year making just the right amount on the last day. So on non leap years he makes 366 on the last day, if he does that, it says for any positive integer n, so that must mean I am considering more than one year obviously. So if I string all these years together I can come up with every number. Because I can get the numbers 1 through 364, then I get 365 the next day and I can go up by 1 everyday after that. That was almost like a worst case scenario to me. Like it would jump too severely too suddenly. But even then the induction can't happen on that because to do 364, well all the way up until there I just bumped on the next day. But when I hit 365 I threw all of those days out and I just looked at one new day and I kept going from that. So it was like the pattern wasn't the same.

A: So what makes you believe that an induction argument would work here?

X: I tried it because I was looking at this formula and I was thinking, boy you know that would be really handy if I could just assume it was true for n and come up with the next one. Because I could find 1. So I wanted to say, so lets find 2 then or something like that.

A: So then what did you do?

X: So then I stopped because I got frustrated. But I want to look at more, the thing I would look at is more patterns. I want to try to find, I am still searching for a counter example. I don't think it exists, but I am still searching for a pattern that he can use somehow.

Commentary
As is evident from the preceding interview artefacts and vignettes, Student X initially took a formal approach to the problem based on the representation. In the second interview (after 6 weeks), student X had moved away from interpreting the problem as asking for an explicit function and attempted a proof of the statement. The student resorted to mathematical induction as a viable approach based on the perception created by the representation. The student was later presented with problems C and D and asked to compare the solution attempt on these new problems to the previous problems.

A: Do you think that there might be any kind of connection between these problems (C and D) and the Santa problem (problem A)?

X: I don't see one, right off.

A: So you are saying that these two problems are obviously very different.

X: I think so. Just because in this one (problem A) I am thinking of the strict sum and I want them to be equal to a certain number and in this case (problem D) I only have to worry about whether it is a multiple of 10 or not. Just because they are asking you to prove something doesn't mean that they are the same. I think that there are some major differences. Although this one (B) could possibly be a subset of this (C).

A: So did you think that any of the principles of Dienes come into play at all? The reason that I am asking you that is that I noticed that when I asked you to do this (problem C) you immediately went to this (problem B).

X: Well the only thing I can think of is that he (Dienes) talked about having more than one situation to deal with the problem. Thinking about more than one. I mean they were still dealing with last digits because that is what the problem asked, but they were very different problems. This one was simply looking for a pattern and the other one is a lot more algebraic in nature. So I was using a couple of different approaches to make a concrete determination about the problem.

Commentary
The student initially used the perceptual similarities in B and C as they both dealt with powers of 7 and classified one as a sub-problem of the other since the notion of powers appeared in two different situations. However when attempting to solve problem C, the algebraic nature of the solution mechanics employed by the student led to the conclusion that the problem was a bit different from B. At this stage, the student understood that one of Dienes' principles was applicable to the similar content in the two problems.

Final In-class assessment
As mentioned earlier, all the problems were given on the final assessment and students were asked to apply the four principles of Dienes to classify the problems. In this assessment student X wrote:

I think problems B and C are similar because they both involve looking for patterns – the Dynamic principle.

The student did not link problem A to either C or D.

5. Discussion

The problem-solving experience of Student X poses the following pedagogical conundrum. First, it indicates the difficulty of reflexively applying (or being aware of) Dienes principles when confronted with problem-solving situations whose solutions hide an underlying mathematical structure. In other words a university educator can teach the four principles of Dienes and provide several mathematical examples of structurally similar problems in varying contexts/embodiments that lend themselves to an application of the principles. Yet, the learning that a student takes from such a treatment and exposure to Dienes' principles is typically superficial. In the present context, arguably, a student that was able to "solve" the problems by applying the Dirichlet principle would be able to discern the underlying structural similarity in the perceptually varying contexts. However the point that is being made here is that underlying mathematical structures in varying problem-solving contexts are not easily discernable! (English, 1999). On the other hand the student was able to apply Dienes' principles after having attempted problems B and C to conclude that they were structurally different although particular features of B were embodied in C. The formal notational use in problem A led student X to believe that an induction argument would work. This is also a case of linking the given situation to a situation from prior mathematical experiences where a summation proof is easily resolved by induction.

These findings with post graduate students are similar to findings in related studies with young children (English, 1999) as well studies with high school students (approx.14 year olds) (Sriraman, 2003, 2004). While English (1999) found that young children need a lot of explicit instruction in what they were supposed to be looking for, Sriraman (2003, 2004) found that some high school students, particularly the able students, were able to discern structural similarities independently with minimal amount of probing.

There are several research and pedagogical implications that can be drawn from this particular case study. In order to do so, we need to re-examine the starting conditions of this experiment. The subject was a post-graduate student with a fairly sophisticated background in mathematics. The objective was to mediate problem- solving conditions whereby the subject could reflexively apply Dienes' principles and discover/link underlying structures. The purpose of the author having a graduate student meet such an objective was to internalise the applicability of Dienes' principles. In order to mediate problem-solving conditions a class of problems were designed which had perceptual variability but structural similarities (namely the Dirichlet structure). Problems with superficial distractors (e.g., problem B) were also included in this class of problems. It should be noted that the class of structurally similar problems A, C and D represent an extreme case of perceptual variability as they can also be classified according to the types of arguments involved to solve them. While problem A requires knowldege of sophisticated counting facts such as partial sums, subsets of a set, partial column decompositions, problems C and D require the use of a number theoretic divisibility arguments. Although the Dirichlet structure eventually manisfests, the mathematical content is quite different. Therefore from a problem content standpoint, the students mutually exclusive classification of A, C and D is valid.

As seen from the case reported in this paper, an extreme scenario of perceptual variability in a problem solving context was created. It is hoped that university educators can easily perturb the starting conditions which result in accomplishing the objectives of graduate students internalising and reflexively applying Dienes' principles to their own problem solving. It is one thing for students to read about Dienes' principles in the literature and be exposed to examples formulated *a priori* that validate the principles. However, it is an entirely different thing for students to experience the applicabity of these principles to their own mathematical experiences during problem solving. The former leads to a superficial pedantic awareness whereas the latter leads to a deeper, more beautiful experience of the presence of underlying structures and conveys the difficulty of experiencing what Dienes principles are suggesting.

6. Concluding Points

One pedagogical implication is that one should control the variability in the class of problems depending on the mathematical sophistication of the students. For instance in courses taken by practicing middle (grades 5-8) and high school teachers (grades 9-12), the author has controlled problem sequencing and variability as follows. Problem A could be reformulated from a declarative statement into an interrogative statement, and presented as:

A*. Given that one of Santa's helpers makes at least one toy everyday but not more than 730 toys in a year (including leap years). Is it possible for the elf to make n toys over some string of consecutive days?

This interrogative statement is simply asking for one existence solution. Assuming (safely) that students with even a basic background in mathematics are able to do this, the university educator has the discretion to not use distractors such as B, and instead formulate a perceptually different but more structurally similar problem such as :

B*. A person takes at least one aspirin a day for 30 days. Suppose he takes 45 aspirin altogether. Is it possible that in some sequence of consecutive days he takes exactly 14 aspirin? (Gardner, 1997).

Formally speaking these two new (reformulated) problems are the weak version of a general problem, which states: Given starting conditions such as in A* and B*, there exists a sequence $a_1, a_2, a_3, ...a_k$ such that each $a_i \geq 1$ and $a_1 + a_2 + ... + a_k = M$, and for some subsequence $\Sigma a_n = N$ for $n = i+1$ to j. The strong version on the other hand states that for every sequence $a_1, a_2, a_3, ...a_k$, where each $a_i \geq 1$ and $a_1 + a_2 + ... + a_k = M$, prove there exists a subsequence such that $\Sigma a_n = N$ for $n = i+1$ to j. Alternatively, the strong version of both problems can be used depending on the sophistication of the students. In the strong version the items come from two sets. The first batch of items are the partial sums, that is the tally counts starting from day 1 and going through day n. The counts vary between 1 and 45 or 730 or in general M. In essence these problems really ask if two partial sums can be found which differ by an amount (14 in the case of B* and n in general). To answer this a second batch of items are obtained by adding 14 (or N) to each partial sum in the first list. We eventually get n+1 sums (the items) that lie between 1 and n (the boxes), thus illuminating the true underlying structure.

From the standpoint of training future researchers in the field, it is important that university educators create mathematical experiences that result in students actually **experiencing the theories they are being exposed to**. It is hoped that such experiences will create an appreciation for the seminal theories in the field and their wide ranging applicability to mathematical content.

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