Developing reflectiveness in mathematics classrooms – An aim to be reached in several ways

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Abstract: The importance of reflection has been emphasized in many conceptions of mathematics education and mathematical literacy. Beyond that, this article emphasizes the importance of reflectiveness, not as an activity or an ability, but as an attitude to be aimed at. The article shows examples and ways to approach this big aim of reflectiveness. It is not only a matter of initiating reflection on different levels, but also a matter of developing an individual disposition for questioning. For this, sense reflections and self reflections play a key role.

Kurzreferat: *Entwicklung von Nachdenklichkeit im Mathematikunterricht – Ein Ziel mit mehreren Wegen.* Dem Prozess des Nachdenkens wird in vielen Bildungskonzepten ein hoher Stellenwert eingeräumt, hier soll darüber hinaus auch die damit verbundene Haltung der Nachdenklichkeit als wichtiges Bildungsziel in den Blick genommen werden. In dem Beitrag sollen Beispiele und Wege aufgezeigt werden, dem Ziel Nachdenklichkeit näher zu kommen. Wichtig ist dabei nicht nur die gezielte Anregung von Reflexion auf unterschiedlichen Ebenen, sondern auch die Entwicklung einer Fragehaltung. Dazu werden Zugänge über Sinn- und Selbstreflexion vorgestellt.

ZDM-Classification: C20, D30

Introduction

It is an often stated empirical finding that German mathematics classrooms are too exclusively oriented at routine tasks whereas deeper intellectual activities and competencies are often neglected. As a consequence of the disappointing German results in international comparative student assessments like PISA (OECD, 1999), this problem is currently discussed in Germany for many school subjects in a similar way. This is emphasized by Annemarie van der Goeben, an influential German educator:

> "One of the most embarrassing results of TIMSS and PISA is the fact that German students are able to solve routine problems but fail when they are asked to reflect independently. [...] We need other, more reflective classrooms." (van der Groeben 2002, p. 45)

In this article, we follow van der Groeben in her understanding of reflective classrooms as those which succeed in "challenging and encouraging the learners to explore new issues in an active, experimental, researching, and evaluating way" (van der Groeben 2002, p. 45).

Although Germany is very far from offering such classrooms all over the country, there are nevertheless some well developed approaches in many subjects that put more emphasis on reflection. Some of the approaches for mathematics education will be shortly presented in this article and complemented by the important aspect of the development of an *attitude*: Most of the existing approaches focus on the *activity* of reflecting and the underlying intellectual *ability* to reflect. For activating this intellectual ability also in out-of-school situations, the *attitude of reflectiveness* is an important precondition.

By the attitude of reflectiveness, this article means a general attitude that can be characterized by the "alertness and interest for questions" (von Hentig, 1996), i.e. by the disposition and the self-confidence, that questions can be posed of one's own accord and answers can be found. The combination of ability and disposition of reflecting is what von Hentig has described as a *state of mind* that forms the core of his conception for *Bildung* (*Bildung* is the traditional German expression that can hardly be translated, the perhaps best translation might be the modern word *literacy* when this is meant in the wider sense):

> "Literacy [Bildung] is a state of mind, the result of a contemplative way of approaching principles and phenomena of one's own culture." (von Hentig 1980, p. 6)

This understanding of literacy is of great relevance also for mathematics. We can read this off the normative conception for mathematical literacy as it was defined by international consensus in the PISA-framework:

> "Mathematics Literacy is an individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded mathematical judgements and to engage in mathematics, in ways that meet the needs of that individual's current and future life as a constructive, concerned and reflected citizen." (OECD 1999, p. 41)

The PISA-framework has been under critique because in many test items, this deep and challenging understanding of mathematical literacy has simply been interpreted as the capacity to apply mathematics in non-mathematical situations. However, this formulation is the official consensus of all OECD-countries. If it is taken seriously, mathematics classrooms all over the world are asked to initiate not only high level problem solving processes, but also reflections about mathematics itself and its role in the world (cf. Jablonka, 2003 for a detailed discussion of this perspective).

This demand does not diminish the importance of mathematical challenges in the narrower sense. Of course, mathematical concepts, procedures and methods must be acquired and cannot be *substituted* by reflections. But the same is true for the inverse relation: reflections cannot be substituted by only working on the level of the mathematical objects themselves. The following sections will show that these different areas are not contradictory but complementary. For this, we will specify different levels of reflection in the following section and show that reflection can start with simple examples. The second section will summarize approaches of reflection-oriented tasks as means of initiating reflection

The development of reflectiveness can only succeed when the students adopt the questions as their own ones. In order to explain this thesis, Section 3 to 6 will develop the main idea of this article and show how the attitude of reflectiveness can be developed by a sense reflection and self reflection.

1. Levels of reflection in mathematics classrooms

In this article, reflectiveness in mathematics classrooms is only considered in the sense of *reflectiveness referring to mathematics*. Although Skovsmose (1998) and many others have convincingly shown how teaching mathematics can contribute to reflections about democratic issues or many other important aspects outside mathematics, this article is restricted to reflections on mathematics.

The demand of more reflection and reflectiveness with respect to mathematics is rarely explicitly denied. However, the author has met concerns in many discussions with teachers and mathematics educators. For example, many educators are afraid that the demand of reflection might intellectually ask too much of students ("Isn't that too complex for most of the students?") or they are concerned about missing mathematical basics ("Before we can reflect on these things, they have to master the simple skills. Otherwise, they do not know what they are talking about."). Beyond such comments, there is often the misunderstanding that reflections can only refer to deep epistemological or meta-mathematical discussions.

But reflection does not necessarily refer only to the big questions. Instead, it can start in modest forms and can be developed in parallel with the mathematical content. In his article *Reflecting as a Didaktik Construction*, Neubrand (2000) has structured the complex and widespread field of suggestions for areas of reflections by specifying four different levels of reflecting on and speaking about mathematics:

- The level of the mathematician
  "Speaking about mathematical subjects and problems themselves, for example, about the correctness of a proof, about the adequacy of the formulation of a definition, about logical dependencies, and so on."
- The level of the deliberately working mathematician
  "Speaking about specific mathematical ways of working, their value and meaning, for example, about heuristic techniques in problem solving; about various modes of concept formation in mathematics, about specific mathematics methods like systematisation, classification, or abstraction; about schemes and techniques of proof; and so forth […]."
- The level of the philosopher of mathematics
  "Speaking about mathematics as a whole with critical distance, for example about the roles of applications and their relation to mathematics concepts, about proofs as a characteristic issue in mathematics, and so on […]."
- The level of the epistemologist
  "Speaking about mathematics from an epistemological perspective, for example, about the characteristic distinctions between mathematics and other sciences, about the nature and the origin of mathematical knowledge, and so on […]"
  (Neubrand 2000, p. 255f)

The names of these levels might be misunderstood in the way that for example the level of the epistemologist is *not* adequate for the mathematician (and hence does not integratively belong to mathematics classrooms?). That is why this article uses the names of levels as they were originally given in Neubrand's German written articles (e.g. Neubrand, 1990). In order to make these levels more concrete and show how different questions can be reflected, some exemplary questions from various mathematical areas on each level are given here:

The level of epistemology:

- What is the nature of mathematical objects?
- What can we see through the "mathematical glasses", where are their limits?

The level of the philosophical base of mathematics:

- Why are we interested in logical dependencies between attributes of series? What do we gain by ordering them logically?
- Which kind of information about data can be given by the standard deviation? Which definitely not?
- Is this mathematization of the given problem adequate?

The level of deliberately working:

- How did we proceed? Which strategy is the most successful in this case? What idea is beyond that?
- What are the relevant concepts, theorems and procedures?
- Which rule does justify this transformation?

The level of mathematical contents:

- What characterizes exponential growth?
- Has the rule or the schema been applied correctly?

This list shows that there are many intermediate states between solving simple routine problems and philosophizing epistemologically, and each of these intermediate levels has its own important function in the process of developing mathematical literacy.

Reflective processes can be initiated first on the level of mathematical contents and the level of deliberatively working. A typical metacognitive activity on the level of deliberatively working is Polya's (1945) looking back on problem solving processes. It should be extended to longer units as well: "What were the most important issues, and how are the contents connected to each other?" (cf. Prediger, 2003).

But reflective mathematics classrooms cannot be restricted to the fist two levels since the treating of questions on the two higher levels are of major importance for developing mathematical literacy (we argued for this view in Lengnink & Prediger, 2000; see also Lengnink, 2005). Even more intensively than for reflections on the lower levels, this raises the methodical question how to initiate reflectiveness about questions on the higher level.

2. Reflection-oriented tasks as an approach to initiate reflection

How can the attitude of reflectiveness be *taught*? Freudenthal was very decisive about this point:

"A state of mind [...] cannot be taught by quickly telling how to behave." (Freudenthal 1982, p. 142).

In the light of constructivist learning theories, we are convinced today that this is true for *all* learning contents, and that individuals cannot even acquire knowledge by being simply taught. Instead, they have to construct their knowledge individually on the basis of their pre-experiences and existing cognitive schemas. However, the possible influence of teachers on this development is even more difficult for *attitudes* than it is for *knowledge*.

Freudenthal suggested to develop the state of mind by engaging students into activities. We follow this suggestion but enlarge the proposition by not only referring to the activity of problem solving but also on reflection about and beyond these processes. Although there are no simple techniques of teaching an attitude of reflectiveness, teachers can make well structured offers in order to initiate the activity of reflecting and this can contribute, in the long run, to the development of the attitude we aim at.

There already exist several didactical approaches, especially the Socratic method by Nelson (Loska, 1995) or Neubrand's approach of speaking about mathematics (1990, 2000). The actual German discussion on the development of classroom culture in mathematics has mainly emphasized the construction of open and reflection-oriented problems. The focus on problems and tasks is legitimated by the hope that changing the problems will change the classroom culture most directly (Henn, 1999). In this discussion, different formats of problems have been developed and are disseminated in the German reform movement (The contributions cited in parentheses give German examples that each represent a large variety of kindred propositions. For a careful analysis of what kind of reflection and metacognition each of these formats can initiate, see Kaune, 2001):

- open problems with several solutions or without narrow question (Winter, 1988; Becker & Shimada, 1997)
- open problems that demand mathematizations (Böer 2000), "Die etwas andere Aufgabe" ["The slightly different task"] (Herget & Scholz, 1998)
- cognition oriented tasks that initiate metacognitive reflections (Sjuts, 2002)
- searching for mistakes and misconceptions (Furdek, 2002)

Those kind of reflection-oriented tasks are very important since they let reflective questions on different levels be the object of learning in the classrooms. That is why construction of reflection-oriented problems (we talk about the "development of the problem culture" in Germany), is one important step on the way to more reflectiveness in classrooms. Questions that are explicitly formulated in written problems gain a greater relevance in the eyes of students. That is why it is an important step in the development of a reflective classroom culture that these kind of tasks have been integrated in the new generation of text books (e.g. Neue Wege, 2001). It is of equal importance that these kind of tasks also appear in the exams. "Is that part of the exam?" is the common question by which students evaluate the relevance of a subject in school.

But the development of reflection-oriented tasks and questions posed in classrooms alone cannot solve one important problem: Reflectiveness does not only include to answer the teacher's question, it also includes to formulate *proper questions* and to have the disposition and the self-confidence to go deeply into the matter. If only the official tasks are changed, the classrooms might still follow traditional patterns: students only answer questions they have never asked and on which the teacher knows the best answer.

Hence, although the construction of tasks is of major importance in order to establish certain issues as a content of mathematical classrooms, we have to go further and aim at students that really go into the matter and start asking questions on their own.

This affects the problem of motivation of students: We should not expect that all students in classrooms want to reflect carefully, e.g. because this might be much more exhausting than solving routine questions. This raises the question of how to enhance the *disposition* to reflectiveness. Only if students want to reflect, we can really speak about reflectiveness. Hence:

Being reflective means asking proper questions and following them.

3. The principle of situational reflection

The TIMSS video study has shown that in German classrooms practices, students rarely pose questions. In the observed and analysed German lessons, there was 7.5 questions of the teacher for 1 question of a student. Every third of these few questions of students was only concerned with organization issues (Begehr 2003, p. 94). These empirical insights in classroom practices show clearly that complementary to the "problem culture", the classroom cultures has to move towards more room for questions of students.

The attitude of reflectiveness can better be enhanced in an open, confidential classrooms culture, in which questions of students are explicitly welcome, in which mistakes are discussed constructively and in which curiosity and interest can evolve. Martin Winter has emphasized the importance of the atmosphere in classrooms, for that "students can adopt the problems as their own" (Winter, 1997). In the German reform movement, the development of an adequate classroom culture is of major importance (cf. e.g. Henn, 1999). With respect to the aims of this article this should mainly refer to ways of communication in which the logic of interaction does not substitute the logic of the mathematical content (see Voigt, 1985).

This is not only a matter of seriously engaging in the students' questions. Often, there appear chances for a communication that could not be foreseen nor by the teacher nor by the learner. Taking up such kind of chances for a reflective conversation is the core idea of the *principle of situational reflection* that has been developed and explained thoroughly in Prediger (2004a). It shall be explained here by an example (discussed under other perspectives in Prediger, 2004b):

Example 1: Katharina and the interpretation of division

"While doing her homework, Katharina orderly followed the rules for operating with fractions and divided the number 2 by ¼. Then, she came asking me since she was surprised of her result, 8. How could the quotient be greater than the dividend, although she had 'divided'? I tried to explain why it had to be like that (within the range of positive numbers) whenever we divide by numbers less than 1. As a counter-example, she confronted me with the fact that whenever she divides an apple 'in quarters', the pieces are smaller than the whole apple. I tried to point out the difference between 'divide in quarters' and 'divide by quarters'. Finally, she said: 'Okay, now I know how to calculate it. But you cannot tell me that thinking in mathematics is logical!'" (Heymann 1996, p. 206)

Katharina feels a discrepancy between her thinking and the expected mathematical thinking. This discrepancy is mainly based on diverging interpretations of the division. Katharina's interpretation of division as a sharing process (interpreted in the partitive model, cf. Fischbein et al., 1985) only covers a part of the mathematical meaning. The division by a fraction necessitates the quotative model of division, interpreting "2 : ¼" by "How many quarters do fit in two wholes?". In order to convince Katharina that this might also be "logical", it is necessary to talk about the quotative model of division.

Beyond that, her irritation is a very good occasion to reflect on the sense of generalizations of operations: Why do mathematicians use the same operation for different situations and models? Why do we lose the partitive model when we divide by fractions? In this way, Katharina's irritation about mathematicians being illogical could have become an interesting chance to reflect situationally on the level of deliberatively working and the level of the philosophical base of mathematics.

Picking up such kind of occasions to reflect can contribute immensely to the development of reflectiveness, not only because they give the opportunity for interesting considerations about mathematics, but also because students can make the experience that their questions are taken seriously. This can encourage to continue questioning. Especially for the reflection on higher levels, namely the epistemologist level and the level of the philosophical base of mathematics, these situational approaches are important in order to connect the reflections with the individual thinking of the learners. This is very important since many issues of philosophical reflections are by nature far away from students' interest and questions, at least at first sight. By the principle of situational reflection, it is not necessary to introduce all sequences of philosophical reflection by planning in advance. The more effective and pupil-oriented approach is to pick up situational chances for reflections that appear in the normal interaction in classrooms. How deep ontological questions can be connected with student's interests can be seen in the next example.

Example 2:
Lisa, Matt and the ontology of mathematical objects

Lesson about geometrically interpreted solutions of 2x2 linear equation systems in grade 9 (with about 15 year old students) of a colleague. Question: What happens if we have two equations of two parallel straight lines?

Lisa: Parallel lines do never meet, hence there cannot be a solution.

Matt: Oh yes, they do meet!

Lisa: No, they do not! Parallels cannot. Yours are perhaps not parallel. Look at these! [*draws parallel lines on the paper]*

Matt: They also meet, although very very far away *[draws a meeting point on the table outside the paper]*.

Having listened to this little controversy, the teacher mediated the conflict by adopting a meta-point of view and asked both students what ontological status they gave their lines:

Teacher: I have the impression that you are not talking about the same objects, aren't you? What exactly are those lines you are talking about? Where do they exist?

Initiated by this clarifying question, both students realized that for Lisa, parallel straight lines were theoretical constructs with idealized attributes (parallelism), whereas Matt focused on the drawn figures on the sheet of paper. Unlike the idealized constructs, drawn figures *have* an existence in the real world, and with this ontological status, they indeed nearly always intersect. By these considerations, the students were engaged in discussions about controversial ontological positions which are both well known in the philosophy of mathematics. Many questions can arise in such a discussion:

- About which kind of objects can we state anything in mathematics?
- In which one are we interested?
- Which ontology is suitable for the original interest on linear equation systems?

Starting from the inner mathematical problem (do parallel lines have a meeting point?), this situation led into interesting philosophical considerations. The situational picking up of chances for reflections got its dynamics by the fact that the ontological question about the nature of lines here was not "another stuff to learn" (as the students would say) but an important tool for clarifying an ongoing mathematical controversy.

The principle of situational reflection is supported by an interesting empirical result: In her analysis of learning situations with respect to the conditions for evolving interest, Bikner-Ahsbahs (2002) has found that situational interest often evolves in situations in which the teacher engages in the individual thinking of learners, when they work on their individual thoughts or questions and not only on the regular mathematical concepts.

4. Approaches to reflectiveness via involvement

The most important precondition for applying the described principle of situational reflections is the teacher's awareness of the reflective potential underlying a situation. If this awareness is not well developed, teachers will miss to exploit potentially reflection-rich situations.

This awareness can be increased by the knowledge about different forms of reflection that focus not only on mathematics itself but also on meanings and humans. Ludwig Bauer has specified four different forms of reflection which can serve as different approaches to reflection:

1. Content reflection, i.e. "reflecting on mathematical contents and issues, contemplatively going into mathematics, comprehensively doing mathematics."
2. object reflection, i.e. reflection "about lines of development, characteristics, fundamental questions of mathematics". This reflection is concentrated on the nature of the discipline mathematics.
3. reflection of meaning and sense, i.e. "reflection about possibilities and limits of mathematical thinking, about the meaning of mathematics and the sense of doing mathemat-

ics. Most central are issues of meaning, purposes and senses."
4. self reflection, i.e. the reflection about the meaning of mathematics for the person themselves. „Observe, analyze, evaluate on a meta level the proper relation to mathematics." (Bauer 1990, p. 6-7)

Bauer's and Neubrand's systematizations differ, not only in the groups they formulate, but also in their focus. Whereas Neubrand has systematized with respect to mathematics itself, Bauer has systematized a broader field and was especially concerned with the relation between mathematics, humans, and the world. In this perspective, sense and self reflections are of great importance. This article puts emphasis on these forms of reflections because they have the potential to increase the students' personal involvement. The main idea is condensed in the following thesis:

The development of an attitude of reflectiveness can be initiated and supported by sense and self reflection, because proper questions require authentic involvement.

This thesis should not be understood exclusively. Obviously, reflectiveness can exist without sense and self reflection. We know students who like to engage intensively in content and object reflection. These students might never pose any question concerning the self reflection since they experience their encounter with mathematics as unproblematic. But how to reach learners who avoid reflections, either because they do not want to or because they do not dare to? Sometimes, ways to reflectiveness for these individuals can be found via self or sense reflection. Most of them at least ask "Why shall I learn that? and "What has this got to do with me?".

These questions are often perceived as a signal of missing motivation or as a disturbance. And in some cases, they are really meant in this way. However, the majority of these questions can concern the core of the content. If these questions are followed instead of fended, they might offer good opportunities for reflection, and reflectiveness can evolve.

Roland Fischer has underlined the role of negotiations about individual sense constructions for developing mathematical literacy [*Allgemeinbildung*] (Fischer, 1982, 2001). According to him, discussions about sense are the most authentic opportunities for reflections we can construct.

"Subjective evaluations of relevance by teachers and students must play a role. Especially in the process of dealing with suggestions, by posing questions like 'What do these contents mean for me, what do they mean for the society, what do they mean for our learning group (as an intermediate community)?', this is the moment where *Bildung* takes place." (Fischer 2001, appendix)

Whereas sense reflection has found a certain place in some mathematics classrooms, self reflection is even less established in practice. The most important reason for this lack of self reflection in classrooms seems to be the widespread image of mathematics as a depersonalized culture without any place for negotiating the relation between individuals and mathematics (cf. Burton, 1995; Prediger, 2004a). That is why it is especially important to show how questions for sense and individual meaning can offer chances for reflection,. This is done by some examples in the next section.

5. Approaches to reflectiveness via sense and self reflection

Students usually pose the question for the individual meaning of contents in a very unspecified manner: "What has the exponential function got to do with me?" or "What do we need the exponential function for?" It is nearly impossible to find reasonable answers to those general questions. Hence, the way to reflectiveness should go via differentiating these broad questions. Students have to learn how to develop more concrete questions from these general ones. We can support this process by negotiating what exactly they mean and by offering adequate examples as prototypes for more concrete questions.

Similar to the reflection of the mathematical contents, these reflection can take place on all the different levels. In this point, the author of this article disagrees with Bauer who situates self reflection always on a high level of reflection and then warns that the philosophical areas should not be overemphasized (Bauer 1990, p. 8). In contrast, this article assumes that sense and self reflection can take place on each of Neubrand's levels.

Example 3:
"What has the exponential function got to do with me?"

Table 1 (next page) serves as an illustration for what sense reflection on the level of mathematical contents or self reflection on the epistemologist level might be. It summarizes and systematizes exemplary questions that can initiate reflections on the arbitrarily chosen field "exponential growth" on the different levels and by the different approaches. Bauer's content and subject reflection was gathered in one column because these two approaches are differentiated in the *lines* of the table. The table must not be understood as a canon that should be worked through cell by cell. It only offers a spectrum of exemplary issues on which reflectiveness concerning exponential growth can focus. The assignment of questions to single cells is never indubitable, but it is more important to see that every cell in such a table has its own important role in the learning process and can be filled with a fruitful question.

The second column shows the classical learning contents on exponential growth. These are the questions that are usually raised in units on exponential growth. But how can we succeed in really involving learners into these questions? For offering individual approaches, questions of the second and third column are very useful. In part, they offer just another formulation for the questions in the first column, others are more concrete variations of the broad question "What has this got to do with me?" or "What do we need it for?".

Once learners start to pose their own questions in the dimensions of self or sense reflection, they can lead to content reflection as well. This shall be shown with the fourth example.

Table 1. Ways to reflectiveness on four levels – exemplified for exponential growth

	Object and Content Reflection	Reflection of Sense and Meaning	Self Reflection
Level of the mathematical contents	What is exponential growth? What are differences to other patterns of growth? What is the suitable term for this function?	Where outside mathematics can we find processes of exponential growth? In what sense does exponential growth mathematize regular growth?	Where do I have difficulties with exponential growth? What is easy for me? Why?
Level of deliberatively working	How can exponential growth be represented and identified in different representations (table, graph, term, verbal explanation)?	What is it good for to identify a process as exponential? For what do we need a term description? What is the use of describing the same process in different representations?	Correction of an exam: "What I originally thought here, was… This is not congruent with mathematical approaches because……."
Level of the philosophical base of mathematics	What are chances and limits of characterizing real processes of growth by exponential functions? (e.g. prognoses)	Why do we want to describe processes of growth by mathematical functions? In what situations this has no relevance?	What connection is there between my individual thinking about processes of growth and their mathematical description? What is my conception of regular growth, and how is it described in mathematics?
Epistemologist level	Where is mathematics used to describe reality, where to construct reality?	What role does mathematics play in society, when prognoses are made by means of exponential functions?	What implications can it have for me when mathematicians use mathematical means for societal issues?

Example 4: Anne and the missing personal access to the algebraic calculus

While solving algebraic equations, the 16 year old Anne asked me, the teacher, in a frustrated mood: "What have all these transformations got to do *with me*?" Anne was achieving quite well in the technical doing but she was searching for a *personal access* to the rules and techniques. The teacher's first answer with the typical examples for applications (that can show the practical use of solving equations) could not convince her. So, we continued by discussing *why* she considered transforming equations not to be connected with her own person. In this way, we could approach the core of the problem: "When I solve equations, I feel like a machine. I do not even have to start thinking, really."

Following this idea of a machine, we reached an interesting field: We realized that it is an important characteristic of algebraic transformations that one can do them without "thinking", i.e., without any interpretation of the formal steps. They can indeed be drawn mechanically since they are rule-guided and independent of any meaning in a concrete context. That is why Krämer talks about "symbolic machines" and describes the historic development as a "long going and difficult history of the mechanical use of symbols, a history in which we learned to behave like machines when operating with symbols" (Krämer 1988, p.4, my translation). In a certain way, this mechanical calculus is de-humanized and hence there is hardly a personal access possible. But exactly this characteristic offers the very important opportunity to discharge our thinking.

Anne could experience this discharge directly when we reconsidered a geometrical question (concerning changes of areas) that we had solved by algebraic equations. We experienced that it is theoretically possible to interpret all the algebraic transformations in the geometrical context, but it is much more difficult to do so. Now, she began to value the possibility of discharge as a strategy of extending her own thinking. At the same time, Anne enormously took comfort in the knowledge that it was a long way in the history of algebra to develop this de-humanized calculus. (Personal experience in the author's class)

Unforeseen for me as the teacher in advance, the frustration about solving algebraic equations and the search for making sense of it took a fruitful development in this situation: Via a self reflection ("What exactly is it that I do not like in transforming equations?"), Anne came to the reflection of a fundamental idea of mathematics, algorithmizing, or more generally the rule-guided operating without interpretation. With reference to the concrete example of an algebraic equation, Anne has approached the philosophically important question on the relationship between human beings and mathematics and how it is threatened by its mechanization (see Prediger, 2004a for this philosophical question).

6. Summary

The main idea of this article can be summarized by three aspects: In mathematics classrooms, students should be allowed to be reflective, they should be willing to and should be able to.

Being allowed to be reflective

Developing reflectiveness needs a classroom culture in which students are allowed to be reflective. This is a matter of forms of interaction as well as a matter of the way teachers reply on reflective objections. Reflection-oriented tasks have proved to be important because they do not only allow reflectiveness but elevate reflective issues to the official learning content.

Being willing to be reflective

Being allowed to be reflective alone is not enough. Not all students can keep their inborn drive to ask questions as it is natural for all younger children. Hence, ways to reflectiveness should encourage to be (re-)willing to be reflective. This concerns the question of attitude that was centrally discussed in this article. Mathematics education cannot be restricted to the development of intellectual competencies. Its aims should additionally include attitudes, and especially the attitude of reflectiveness.

Like for every important educational aim, there is no pedagogical silver bullet that can guarantee success. However, the examples show some possible ways how sense and self reflection can offer approaches to initiate reflectiveness.

The threefold approach of content reflection, sense and self reflection does not only aim at enhancing motivation for reflection but also at focusing existing questions to fruitful ones. This belongs to the third aspect.

Being able to be reflective

The aspired attitude should be complemented by an intellectual competence to modify questions for sense and individual meaning in a way that allows fruitful answers. In the discussion about implicit knowledge of experts, this competence to specify *relevant* questions has been characterized as an important criterion by which experts of a field can be recognized (Breger 1992, p. 81). Obviously, this important competence can only be developed on the basis of well founded mathematical knowledge and abilities.

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