With a focus on 'Grundvorstellungen' Part 2: 'Grundvorstellungen' as a theoretical and empirical criterion

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Abstract: The first part of this serial pointed out the integration of the German concept 'Grundvorstellungen' into current concepts, especially its central position as a mediator between reality and mathematics. The next stage is therefore to explain the use of the proportion and percentage calculations within this concept and how it can be used as a criterion to detect the demands of mathematical problems. Firstly, we will take a look at a classification of mathematical items. This classification shows the complexity of the mathematical item in respect of 'Grundvorstellungen'. The consequences of this consideration have hierarchical levels of demand on these items. Furthermore to show how to describe and interpret these results on the basis of these levels, we refer to selected results of the PISA 2000 comparative study.

Kurzreferat: Im ersten Teil dieser Reihe wurde die zentrale Bedeutung mathematischer Grundvorstellungen als Brückenglied zwischen realen Sachkontexten und mathematischen Inhalten betont. In diesem Teil soll nun am Beispiel der Proportionalität und Prozentrechnung dargestellt werden, wie das theoretische Konstrukt als Kriterium bei der Anforderungsanalyse von Mathematikaufgaben genutzt werden kann. Das heißt, für den Bereich der Proportionalität und Prozentrechnung wird aufgezeigt, wie man Aufgaben danach klassifizieren kann, wie komplex der zu deren erfolgreichen Bearbeitung notwendige Einsatz mathematischer Grundvorstellungen ist. Es ergeben sich hierarchisch aufeinander aufbauende Anforderungsstufen. Vor diesem Hintergrund soll abschließend auf ausgewählte Befunde aus der Vergleichsstudie PISA verwiesen werden, zu deren Interpretation und Beschreibung die Anforderungsstufen herangezogen werden konnten.

ZDM-Classification: B10, D70, F80, M10

1. Introduction

In part one of this series we take a look at mathematical competences as an understandable use of mathematical modelling in real-life situations. Thereby, we understand mathematical competences and modelling within the meaning of mathematical literacy according to OECD (2003) and vom Hofe, Pekrun, Kleine and Goetz (2002). In this process of modelling there is a special focus whilst working with problems on the transition between real situations and the mathematical level. Mental objects are necessary for these transitions for they mediate between reality and mathematics. These mental objects are called 'Grundvorstellungen'. When one considers this concept in contention with current concepts within educational sciences, 'Grundvorstellungen' can be described as mental models for mathematical objects which are constructed on the basis of schematic structures. Referring to this theoretical framework it is the aim of this article to deal with the question how 'Grundvorstellungen' can be used (1) as a normative criterion for the construction of items and (2) as an empirical criterion in comparative studies.

2. 'Grundvorstellungen' as a theoretical criterion in selected fields

Firstly, we must take a closer look at *proportions* which are the basics of the following considerations. The importance of this mathematical content can be seen in the preparation of a simple mathematical model to mathematise many application-related situations. According to Kirsch (1969, 2002), proportions are only understood, if they are apprehended as a transformation which maintain the structure between two quantities. Thereby a transformation f between two quantities G_1 and G_2 is called *proportional*, if there is for any $n \in \mathbb{N}$ and any $a \in G_1$:

$$f(n \cdot a) = n \cdot f(a).$$

In ideal cases, these kinds of transformations between reality and mathematics are based on two 'Grundvorstellungen', abbreviated to GV (cf. Malle, 2000; Vollrath, 1993):

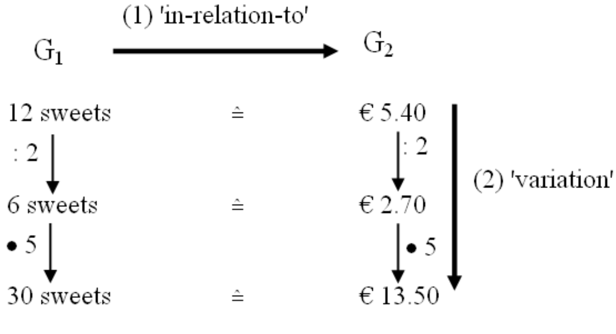
1. A connection between quantities can be described, founded or researched. Thereby, the elements of one quantity are *in relation to* elements of another quantity.
2. The effect of the *variation* of elements of one quantity in relation to the elements of the other quantity is described or observed.

In figure 1 we can see a simple example for a situation with a proportional context. The modelling structure of this item is the basis for the cognitive demand in these kinds of situations. It is due to this, that we can define the first cognitive level.

Figure 1. Identification of 'Grundvorstellungen' in an item at level 1 by the "rule of three".

Example 1:
If 12 sweets cost € 5.40, find the cost of 30 sweets.

Possible solution:
"rule of three"



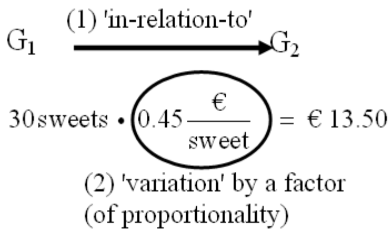
Level 1: (1) *Relations* between different quantities must be identified (on a "horizontal perspective"), i.e. the element of quantity G_1 is in relation to the element of quantity G_2. (2) The quantities are varied due to the characteristics of proportional transformations (on a "vertical perspective"). Koerber (2000) has called this kind of variation "scalar" to emphasise that this variation has no units. We have to pay attention however, that the transformation is based on the characteristics of proportionality.

The 'Grundvorstellungen' can not only be integrated such as in figure 1's solution; it is also possible to activate the 'Grundvorstellungen' for example by a factor of proportionality (figure 2a), by an equality of ratio (figure 2b) or by an equality of quotient (figure 2c).

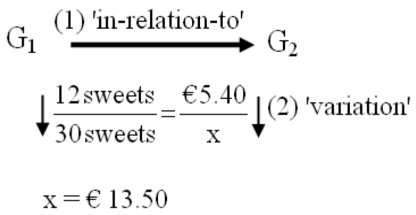
Figure 2. Identification of 'Grundvorstellungen' in an item at level 1 a) by a operator, b) by a ratio, c) by a quotient.

Example 1:
If 12 sweets cost € 5.40, find the cost of 30 sweets.

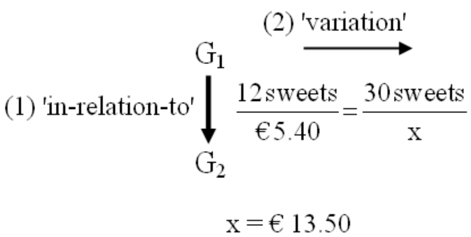
Possible solutions:
a) operator



b) ratio



c) quotient



The shown solutions in figures 1 and 2 are ideal types of solution processes; in practical work we can find different mixtures of these solutions (cf. Kurth, 1992; Stöckel, 1992). In all solutions however, it is possible to identify the 'Grundvorstellungen' "in relation to" and "variation". Thus we can understand the cognitive demand on this level as modules, whereby processes of variations and relations are combined into solution schemes. A reason for this point of view is the fact that the relation is mostly elementary and moves along with the other modules. These considerations are analogue in the field of inverse proportionality.

On the basis of this thought 'Grundvorstellungen' can be rediscovered in the following (logical equivalent) characteristics of proportionality (for more details see Jordan, Kleine, Wynands & Flade, 2004):

P1. If you *multiply* the elements of one quantity with a *number n*, than the related elements of the other quantity are also multiplied with the same number n (see figure 1). This aspect is a direct conclusion from the definition of proportionality.

P2. You can get the element of a quantity by multipliying the related element of the other quantity with a constant operator c (*factor of proportionality*) (see figure 2a): $f(a) = c \cdot a$.

P3. The *quotient* of one element of a quantity and the related element of the other quantity are always the same (see figure 2b):
$$\frac{f(a)}{a} = \frac{f(b)}{b}.$$

P4. The *ratio* of elements of one quantity is equal to the ratio of the related elements of the other quantity (see figure 2c):
$$\frac{f(a)}{f(b)} = \frac{a}{b};$$
i.e. quantities increase in the same ratio.

Furthermore, we have the *addition* of a fifth characteristic:

P5. The sum of two elements of one quantity is the same as the sum of the related elements of the other quantity: $f(a + b) = f(a) + f(b)$.

According to Griesel (1997), if we take a look at the demand in *percentage calculations* on level 1 we can understand percentage quotations as special kinds of quantities. Percentages can be understood as a *unit*. Our previous thoughts about solution processes are also valid for percentage and interest calculations.

We must note admittedly that there are additional characteristic 'Grundvorstellungen' in the field of percentage calculations. In the following paragraph we aim to explain the 'Grundvorstellungen' of *fractions* as a basis for percentage calculations.

According to Freudenthal (1986), fractions are an important part of mathematics in school because, on the one hand, they are constantly used in everyday speech, and on the other hand, their implementations are depicted. Padberg (1995) and Postel (1981) pointed out different aspects of the fraction-concept whilst Blum, Wiegand (1998) and Hefendehl-Hebeker (1996) worked out necessary 'Grundvorstellungen' for this concept:

GV-Fraction 1: Fractions can be understood as commensurable *parts of one whole* or multi wholes. For example, if one divides up three pizzas for four children equally then every child will get $\frac{3}{4}$ pizza. That means: every child will get three quarters of one pizza or the fourth part of three pizzas.

GV-Fraction 2: Fractions can be a (multiplicative) *calculation statement such as an operator*. For example, one rotation of the large hand on a watch is 60 minutes. A $\frac{3}{4}$ rotation is $\frac{3}{4} \cdot 60$ minutes = 45 minutes.

GV-Fraction 3: Fractions describe relations between two elements of the same quantity as a *ratio*. For example, in a pearl necklace $\frac{3}{4}$ of the pearls are black and $\frac{1}{4}$ of the pearls are white. We can phrase the same facts by saying that "the ratio of black to white pearls is 3:1", "for each white pearl we have three black pearls", "three out of four pearls are black".

These three 'Grundvorstellungen' of fractions are also necessary for percentage calculations. Particularly with regard to percentage as a special case of fractions, we have obtained the following 'Grundvorstellungen' for percentage calculations (cf. Blum, vom Hofe, Jordan & Kleine, 2004):

GV-Percentage 1. This first one is a special kind of 'GV-Fraction 1': *The whole is divided into one hundred equal-sized sections*. For example, $43\% = \frac{43}{100}$ of the students live in the town near to the school. By connecting the 'Grundvorstellungen' with an operation we talk about the *hundredth-operator-GV* (see GV-Fraction 2).

GV-Percentage 2. According to Hefendehl-Hebeker (1996) this case is a special kind of 'GV-Fraction 3': It is a statistical point of view whereby a *basic set is divided into subsets with the cardinal number of 100*. For example, 43% of the students live in the town near to the school means: for every 100 students of the school, 43 students live in the same town as the school.

GV-Percentage 3. As mentioned above, percentages can be understood as a *unit*. Therefore, the unit and the value as a whole are a special kind of quantity.

Connected with these specific 'Grundvorstellungen' the basic items for percentage calculations can be solved similarly to the mentioned solution processes above.

<u>Level 2:</u> We can increase the demand of items by adding further 'Grundvorstellungen' to the former demand in level 1. This addition is not a trivial combination of 'Grundvorstellungen'. Thus resulting in level 2 as proportion and percentage calculations with a higher cognitive demand than in level 1. Typical items on this level are the combination of the 'Grundvorstellungen' of percentage calculations with arithmetical ones, such as 'addition' or 'subtraction' (cf. Padberg, 1996; part 1 of this series). Figure 3 shows possible solutions for items using the process of level 1, whereby a structure of subtraction is (a) prefixed and (b) post fixed.

Figure 3. Identification of 'Grundvorstellungen' in an item at level 2 using different solution processes.

Example 2:
In order to attract customers, a shopkeeper advertises a discount of 20% an all goods in the shop.
How much would an item marked at € 129 cost now?

Possible solutions:
a) prefixed subtraction

100% - 20% = 80%
(using 'Grundvorstellungen' of subtraction)

'Grundvorstellungen' percentage calculations (see level 1)

operator
$€\ 129.00\ □\ \frac{80}{100} = €\ 103.20$

"rule of three"
100% □ € 129.00
20% □ € 25.80
80% □ € 103.20

…

b) post fixed subtraction

'Grundvorstellungen' percentage calculations (see level 1)

operator
$€\ 129.00\ □\ \frac{20}{100} = €\ 25.80$

"rule of three"
100% □ € 129.00
20% □ € 25.80

…

€ 129.00 - € 25.80 = € 103.80
(using 'Grundvorstellungen' of subtraction)

A non-trivial combination can also exist due to the repeated use of the demands from level 1.

<u>Level 3:</u> An additional increase in the demand can be achieved by the repeated combination of the aspects of level 2. For example in compound interest calculations or the use of factors of growth in percentage calculations 'Grundvorstellungen' from level 2 have to be arranged in a non-trivial way. Figure 4 shows an example of such a combination.

In practical work we can expect different kinds of concrete solution processes analogue to the former levels. From a theoretical point of view however, we can describe the demands as the interlinking of 'Grundvorstellungen'. The repetition in level 3 can be combined several times (e.g. within the compound interest calculation). This demand can have an influence on the used solution method: Even with items which have multiple linking processes the use of operators is superior to other methods because of clarity and cognitive economy.

Figure 4. Identification of 'Grundvorstellungen' in an item at level 3 using different solution processes.

Example 3:
A CD-player costs € 150 at wholesale price. A customer must add 16% government tax. Payment in cash reduces the proce (inclusive of government tax) by 3%.
How much does the player cost a customer who pays cash?.

Possible solutions:
operator

100% + 16% = 116%

€ 150.00 · 1.16 = € 174.00

100% - 3% = 97%

€ 174.00 · 0.97 = € 168.78

repeated combinations of the demand of level 2

or just: € 150.00 · 1.16 · 0.97 = € 168.78

"rule of three"

100% + 16% = 116%

100% € 150.00
4% € 6.00
116% € 174.00

100% - 3% = 97%

100% € 174.00
1% € 1.74
97% € 168.78

repeated combinations of the demand of level 2

Now it is possible to classify items on the basis of the implemented 'Grundvorstellungen'. The levels can be the basis for the individual support of students (cf. vom Hofe, 1995). In the following section we want to study how one recognises the concept of 'Grundvorstellungen' within comparative studies especially those which are based on the concept of mathematical literacy (see part 1 of this series).

3. 'Grundvorstellungen' as an empirical criterion

In the following section we want to prove how far the theoretical considerations which we have already discussed can be used in an empirical analysis. For this, we need to take a look at selected results from the PISA 2000 report; for closer inspection we refer to detailed analyses of corresponding articles. For our focus we are only interested in results of item-groups working with mathematics in concrete terms in the area of application. On the basis of the mentioned hierarchical levels we use the results under two different headings: (a) with regression analysis the difficulties of items have to be reconnoitred, (b) the achievement of students should be classified particularly in regard to the hierarchical levels.

At first, we will show the relationship between the demands of items regarding to 'Grundvorstellungen' and the empirical difficulty of items within the PISA 2000 report. On the basis of the hierarchical levels Blum, vom Hofe, Jordan and Kleine (2004) define a variable *'intensity of Grundvorstellungen'* with the value 0 (no 'Grundvorstellungen' necessary), 1 (elemental 'Grundvorstellungen'), 2 (extended 'Grundvorstellungen') und 3 (complex 'Grundvorstellungen'). With regard to the hierarchical levels of proportions and percentage calculations we do not have any complex 'Grundvorstellungen' with the value 3 (an example of Blum et al. for this value is the concept of derivation as a combination of the concepts of limit and gradient). Our implemented level 1 corresponds with value 1 of the variable as elemental 'Grundvorstellungen'. Level 2 and 3 correspond with value 2 (extended 'Grundvorstellungen') whereby the higher demand of level 3 gets its results from the cognitive complexity due to the repeated combination of activated 'Grundvorstellungen'. With technical items we do not need any 'Grundvorstellungen' per definitionem (value 0).

In the analysis Blum et al. (2004) refers to 31 items from the international section of PISA 2000 and 86 items of the German national section of PISA 2000. For items which can be described by the term 'procedural modelling' the authors can clarify 45.7% of these difficulties by the variable 'intensity of Grundvorstellungen'. It is still possible to clarify 35.3% of the difficulties by our used variable for items which can be termed as 'conceptual modelling' (see table 1). For a detailed account of the concepts and terms refer to Neubrand (2004).

Table 1. Clarification of difficulties within selected item-groups (working with mathematics in concrete terms in the area of application) in PISA 2000 by the variable 'intensity of Grundvorstellungen' (Blum et al. 2004, p. 155).

	standardised coefficient beta	sig.	clarified variance (R^2)
procedural modelling	.686	.000	.457
conceptual modelling	.616	.001	.353

The authors use multiple regression analysis (cf. Tabachnick & Fidell, 1996) in order to clarify the difficulty. One can observe that the levels of demand regarding 'Grundvorstellungen' are not only theoretical considerations; they are also empirical criteria which show the difficulties of items.

In a different analysis Jordan et al. (2004) adopt the hierarchical levels of proportions and percentage calculations to classify the achievement of students. They use the data N = 31747 students who are taking part in the German national PISA 2000 test. According to Blum et al. (2004) they show that the empirical order of the difficulties of items is approximately analogue to the hierarchical structure of the different levels. On the basis of these results the authors subdivide the students into three corresponding groups of competence. In each group i (i = 1, 2, 3) the students have problems dealing with the items of the corresponding level I with certainty. However they have no problems dealing with items from lower levels (i-1 or i-2).

They now have the opportunity to interpret the competences of students in the groups, on the basis of the demands within each level. For example, on the one side, over 40% of all students fall into group 3, which is the highest category. Most of these students go to grammar schools. On the other side, they have discovered that 25% of students in year 9 (approximately 15 year olds) are members of group 1. This means that, without doubt, one quarter of all students in this year have problems solving the basic items of proportion and percentage calculations; thus they have problems activating the corresponding 'Grundvorstellungen' from level 1. These results prompt the authors to carry out more research in educational sciences to force methods and knowledge of learning and teaching structures, especially in regards to a cognitive and temporal stability of mathematical skills and abilities.

4. Summery

This article has aimed to further develop the theoretical fundaments of the concept 'Grundvorstellungen'. We have shown that it is possible to use this concept as a theoretical criterion in order to describe items according to the extent of the 'Grundvorstellungen'. We have used the field of proportions and percentage calculations to illustrate of our thoughts and ideas. We have developed in this field a hierarchical structure of three levels, whereby the demands on each level are determined by the extent of the used 'Grundvorstellungen'. The activation of 'Grundvorstellungen' can proceed with different types of solution processes. These different processes can be evaluated by comparing them to different aspects of "quality", i.e. they are developed in different mathematical fields and they have a different range of application. However, with regard to the theoretical criterion of 'Grundvorstellungen', the items on the different levels refer to similar aspects.

Furthermore, the theoretical aspect can be used to evaluate the empirical results from comparative studies such as PISA 2000. We have shown a detailed analysis into the different ways to implement the concept 'Grundvorstellungen' as an empirical criterion. On the one hand, the empirical difficulties of items dealing with mathematics in the area of application can be described significantly by the criterion 'Grundvorstellungen'. On the other hand, it is possible to compare the empirical achievement of different groups of students in proportions and percentage calculations with the theoretical criterion. This achievement can thus be described in terms of activated 'Grundvorstellungen'.

In conclusion, we have shown in this series that the concept 'Grundvorstellungen' holds a central position for the modelling of items with real-life situations. This concept can be used both theoretically for the construction of items and the analysis of their demands, and even for the consideration of empirical results in comparative studies.

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