Algebra in Elementary School: Developing Relational Thinking[1]

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Abstract: We have characterized what we call *relational thinking* to include looking at expressions and equations in their entirety rather than as procedures to be carried out step by step. For the last 8 years, we have been studying how to provide opportunities for students to engage in relational thinking in elementary classrooms and how to use relational thinking to learn arithmetic. In this article, we present interviews with two third-grade students from classrooms that foster the use of relational thinking. In both cases, we focus on the distributive property. The first example illustrates how a teacher scaffolds a sequence of number sentences to help a student begin to relate multiplication number facts using the distributive property. The second example shows another student who is already using the distributive property and the extent of his knowledge.

Kurzreferat: Wir gehen in dem Beitrag von einer Charakterisierung von relationalen Denken aus und beschreiben, was es bedeutet, die Analyse von Termen und Gleichungen mehr ganzheitlich zu betrachten als als schrittweise auszuführende Prozeduren. Während der letzten 8 Jahre haben wir untersucht, wie Lernenden in Grundschulklassen Möglichkeiten zum Betreiben und Anwenden von relationalen Denken beim Lernen von Arithmetik angeboten werden können. In diesem Beitrag stellen wir Interviews mit zwei Schülern aus dritten Jahrgangsklassen vor, in denen die Anwendung von relationalen Denken gezielt gefördert wird. In beiden Fällen kommt es uns besonders auf die distributive Eigenschaft der Multiplikation an. Das erste Beispiel zeigt deutlich, wie eine Lehrperson eine Folge von Zahlensätzen aufbaut, um einem Schüler dabei zu helfen, zu beginnen, multiplikative Zahleneigenschaften in Beziehung zu setzen unter Verwendung der Distributionseigenschaft. Das zweite Beispiel zeigt einen weiteren Schüler, der bereits die Distributionseigenschaften anwendet sowie seinen Wissensstand.

ZDM-Classification: D40, H20

1. Introduction

Current reform recommendations propose that algebra should be taught throughout the grades beginning early in elementary school (National Council of Teachers of Mathematics 1998, 2000). The goal is not, however, to push the traditional high school mathematics curriculum down into the elementary grades. We would accomplish little if we simply added more topics to the elementary mathematics curriculum without fundamentally reforming how we teach arithmetic. In the current American curriculum, there is a serious discontinuity between the arithmetic that students learn in elementary school and the algebra that they are expected to learn in the upper grades.

In this article we consider how arithmetic concepts and skills that students learn in elementary school can be better aligned with the concepts and skills that they need to learn algebra. By attending to relations and fundamental properties of arithmetic operations (what we call *relational thinking*) rather than focusing exclusively on procedures for calculating answers, learning and instruction can be made more consistent with the kinds of knowledge that support the learning of algebra, while at the same time supporting and enhancing the learning of arithmetic.

For the last 8 years, we have been studying how to provide opportunities for students to engage in relational thinking in elementary classrooms and how to use relational thinking to learn arithmetic. Drawing on the work of Robert Davis (1964), we have used true/false and open number sentences to engage students in relational thinking by focusing on specific properties and ways of thinking about number operations. For example, a number sentence like 38 + 47 = 47 + 38 explicitly focuses on the commutative property of addition. Students might figure out that the sentence is true by carrying out the addition on each side of the equal sign, but more commonly they immediately conclude that the sentence is true because only the order of the numbers has been changed. This can lead to a discussion of whether this relation generalizes to all numbers and whether it is true for other operations.

Our research has focused on understanding children's conceptions and misconceptions related to relational thinking, how conceptions develop, how teachers might foster the development of relational thinking and the use of relational thinking to learn arithmetic, and how professional development can support the teaching of relational thinking. The studies have included design experiments with classes and small groups of children (e.g. Carpenter & Levi 1999, Koehler 2004, Valentine & Carpenter 2004) and large-scale studies (Carpenter, et al. 2003, Jacobs, Franke, Carpenter, & Levi 2005).

In this paper, we illustrate elementary school children's use of relations and properties of operations as a basis for learning arithmetic. We begin with a general discussion of relational thinking, and then illustrate children's use of relational thinking with interviews with two third-grade students that show: (a) how children can use the distributive property to learn multiplication number facts and (b) the kinds of scaffolding and sequencing of problems that support this kind of learning.

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2. Focusing on relations rather than calculating answers

Traditional arithmetic has been almost exclusively concerned with calculating answers. Addition, subtraction, multiplication, and division have generally been portrayed as processes that involve doing something. In arithmetic, calculations result in closure; a collection of numbers is operated on in a progression of steps to generate a single number, which is the answer to the calculation.

In algebra, on the other hand, the focus is on relations. Certainly the solution of equations has been a fundamental part of high school algebra, but even solving equations has a different character than applying an algorithm to a collection of numbers. An algebraic equation is solved by successive transformations of the equation, and the final transformation results in an equation expressing a relation (x = some number or numbers), rather than a single, isolated number.

Our point is not that arithmetic entails a mindless sequence of procedures whereas algebra is necessarily learned with meaning. Both arithmetic algorithms and procedures for solving algebraic equations can be taught as rote procedures or with meaning. In fact, we propose that the more that meaning comes into play, the closer the connection between arithmetic and algebra. For example, consider the addition problem 40 + 50. If students think of the problem as adding 4 tens and 5 tens, the calculation is based on the same principles as adding 4y + 5y. In both cases, the distributive property is the basis for adding 4 and 5. The operations involved in these calculations can be represented as:

$40 + 50 = 4 \square 10 + 5 \square 10 = (4 + 5) \square 10 = 9 \square 10$ and

$4y + 5y = (4 + 5)y = 9y$

2.1 Equality as a relation

The way that the equal sign is used in these examples illustrates a fundamental distinction between a computational perspective of arithmetic and a perspective of arithmetic and algebra in which relations are prominent. The way that the equal sign is thought of by most children throughout the elementary grades is to signify that the answer comes next.

A more appropriate conception of the equal sign and one that is critical for learning algebra is that the equal sign expresses a relation. This distinction is clearly illustrated by children's responses to the open number sentence 8 + 4 = _ + 5. Most children throughout elementary school treat the equal sign in this number sentence as an indication to carry out the preceding calculation (Behr, Erlwanger, & Nichols 1980, Carpenter, Franke, & Levi 2003, Falkner, Levi, & Carpenter 1999, Saenz-Ludlow & Walgamuth 1998). They respond that 12 should go in the box. Other children add all the numbers and put 17 in the box. Only a few children in traditional elementary school classes recognize that the equal sign represents a relation so that the expressions on each side of the equal sign should represent the same number. Recognizing that the equal sign expresses a relation is critical for learning algebra (Kieran 1992, Matz 1982), and the flexibility that using the equal sign to express a relation affords can provide students a notation for representing important ideas in learning arithmetic (Carpenter et al. 2003). Thus, understanding that the equal sign represents a relation is a critical benchmark in learning to think about mathematical relations, but there is more to thinking about relations than using the equal sign appropriately.

2.2 Thinking relationally

We have characterized what we call *relational thinking* to include looking at expressions and equations in their entirety rather than as a process to be carried out step by step. Relational thinking involves using fundamental properties of number and operations to transform mathematical expressions rather than simply calculating an answer following a prescribed sequence of procedures. This implies some level of awareness of the properties, but not necessarily a complete understanding of them or knowledge of formal definitions. As we will illustrate later in this paper, students may have a good operational sense of when and how it is appropriate to use the distributive property, but not be able to express the property in words or symbols.

To illustrate this notion of relational thinking, consider the open number discussed above: 8 + 4 = _ + 5. Students can find the correct number to put in the box by adding 8 and 4 and then figuring out what to add to 5 to get 12. This is a perfectly valid solution to the problem that deals appropriately with the equal sign as expressing a relation. Although the solution is correct, it still relies on computations specified in the problem to calculate the answer. A student who considered the equation as a whole might have recognized that 5 is one more than 4 so the number in the box needs to be one less than 8. This student used these relationships to solve this problem:

$8 + 4 = (7 + 1) + 4 = 7 + (1 + 4)$

In other words the student at least implicitly used the associative property of addition to transform the equation.

These sorts of transformations can frequently make computation easier. For example, the number sentence 87 + 56 – 56 = _ is a great deal easier to solve if one recognizes that it is possible to first subtract 56 from 56 rather than carrying out the calculation from left to right. Unfortunately many children do not recognize that even a simple reordering of procedures such as this can often be used to simplify computation (Jacobs et al. 2005).

Various mathematics curricula have included instruction in procedures to help students perform certain calculations in their heads. These mental math activities often included a number of tricks for transforming problems to simplify the calculation. Although these "tricks" were based on fundamental properties of arithmetic, they were not always learned that way. We want to be very clear that focusing on relational thinking is very different from learning a collection of computational tricks. Although knowing that one can subtract 56 from 56 in the problem 87 + 56 – 56 = _ can simplify the calculation, this specific knowledge is not our major goal. Calculations of this form do not come up very often. Understanding why this is possible, however, entails understanding important ideas about the relation between addition and subtraction and applying some fundamental properties of addition and subtraction.

In point of fact, all computational algorithms are based on a small collection of fundamental properties of arithmetic. Essentially what algorithms do is to use basic properties of number operations to reduce complex calculations to simpler calculations that can be carried out using well-known procedures or knowledge of number facts (Carpenter et al. 2003). Traditional computational algorithms, however, are designed for efficiency, and the transformations involved in the calculations are generally hidden in the notational elegance of the algorithms. As a consequence, many students complete elementary and middle school having had very little opportunity to do anything but follow routine, well-specified sequences of procedures. They do not understand how basic properties of number operations are applied in their computations, and, as a consequence, they do not recognize that arithmetic and algebra are based on the same fundamental ideas. Furthermore, by failing to take advantage of the structure of the number system, the learning of arithmetic has been made much harder (See Kilpatrick, Swafford, & Findell 2001).

3. Developing relational thinking

In the next two sections of this paper, we present interviews with two third-grade students from classrooms in which students were encouraged to engage in relational thinking. The interviews were conducted by one of the researchers to illustrate the kind of thinking that children in these classes were engaged in and the kinds of questioning, scaffolding, and sequencing of problems that support the development of relational thinking. The classes from which these two students were drawn were taught by classroom teachers who had participated in sustained professional development on teaching for relational thinking. Additional examples of the types interactions that occurred in these classes can be found in Carpenter et al. (2003) and Koehler (2004).

In both of the cases reported below, we focus on the distributive property. The distributive property is often considered one of the more difficult properties for students to learn, even at the high school level, but it is fundamental in understanding multiplication. Furthermore, an implicit understanding of the distributive property can provide students a framework for learning multiplication number facts by relating unknown facts to known facts. The first example illustrates how a teacher could scaffold a sequence of number sentences to help a student begin to relate multiplication number facts using the distributive property. The second example shows another student who is already using the distributive property and the extent of his knowledge.

3.1 Learning to use the distributive property

In the first interview, Ms L worked with Kelly, a third grader in the middle of the school year. Ever since the first grade, Kelly's teachers had provided her with opportunities to engage in relational thinking and had expected her to explain her thinking. Kelly was a fairly typical student in the class and was actually struggling more than most of the students, but she was learning with understanding.

In the following exchange, Ms L posed problems to Kelly designed to engage her in relating multiplication and addition. The first example simply represented multiplication as repeated addition. Subsequent problems explored different ways of breaking multiplication down, implicitly drawing on the distributive property. The interview shows a gradual progression of problems that allowed Kelly to build on previous problems to figure out a multiplication fact that she did not know (4×7) by relating it to a fact she did know (2×7). The first few problems called attention to specific relations. In the final problem, Kelly used what she had learned from the earlier problems to generate a number fact that she did not know by recall. Kelly actually noticed relations at several levels. She attended to the relations between multiplication and addition within each problem, and she built on the relation between the problem she was solving and previous problems. The goal of the careful sequencing of problems illustrated in this interview was to draw out Kelly's implicit knowledge of the relation between multiplication and addition with problems that specifically called her attention to those relations so that subsequently she could use relational thinking to generate a number fact that she did not know. The first problem Ms L posed was the true number sentence, $3 \times 7 = 7 + 7 + 7$.

Ms L: "Could you read that number sentence for me and tell me if it is true or false"?

Kelly: "Three times 7 is the same as 7 plus 7 plus 7. That's true, because times means groups of and there are 3 groups of 7, 3 times 7 just says it in a shorter way".

Ms L: "OK, nice explanation".

Kelly's explanation for why this number sentence was true indicated that she understood the relationship between addition and multiplication. At this point, it was not clear whether she had an understanding of multiplication that extended beyond combining individual groups (repeated addition).

Ms L: "How about this, $3 \times 7 = 14 + 7$, is that true or false"?

Kelly: "It's true".

Ms L: "Wow, that was quick, how do you know that is true"?

Kelly: "Can we go back up here [pointing to $3 \times 7 = 7 + 7 + 7$]"?

Ms L: "Sure".

Kelly: "Seven and 7 is 14, that is right here [drawing a line connecting two 7s in the first number sentence and writing 14 under them]. Fourteen went right into here [pointing to the 14 in the second number sentence]. Then there is one 7 left pointing to the third 7 in the first number sentence], and that went right here [pointing to the last 7 in the second number sentence]".

Again, Kelly used relational thinking to reason about this problem. She did not simply do the calculation on each side of the equal sign and compare the sums. She knew that since 14 was the same as two 7s, 14 plus 7 would be the same as three 7s. This type of reasoning suggests that Kelly might have been beginning to understand multiplication as something beyond repeated addition. She might have been beginning to see that when she multiplied $n \times m$, she could think of chunks of m's until she ended up with n m's altogether. Kelly also clearly saw the relationship between these two problems. In an attempt to see if Kelly could further build on these relationships, Ms L posed the following problem: 4 × 6 = 12 + 12.

Ms L: "OK, I have another one for you 4×6 = 12 + 12, true or false"?

Kelly: "That is true".

Ms L: "OK, how did you get that one so quickly"?

Kelly: "Six plus 6 is 12, in this case, there are 4 groups of 6, so it is like this [writing 6 + 6 + 6 + 6]. Six and 6 is 12, that leaves another 6 and 6, and that equals 12. So one 12 is here and one 12 went here [indicating the two 12s in the problem]. What I'm trying to say is there are four 6s and you broke them in half and made them into two 12s".

Ms L: "Nice! Kelly, do you know right away what 4 times 6 is"?

Kelly: "Yes".

Ms L: "What is it?"

Kelly: "It's [pause] thirty- [long pause] two."

Ms L: "OK, do you know what 12 plus 12 is"?

Kelly: "Yeah. That is the same thing, 32".

Ms L:" Do you have a way of doing 12 plus 12, to check it"?

Kelly: "Well, there are two 10s, 20- oh wait, I was thinking of a different one"!

Ms L: "You were thinking of a different multiplication problem"?

Kelly: "Yes. 4 times 6 is 24, because 10 and 10 is 20, and 2 and 2 is 4, put those together and its 24".

It is interesting to see what happened when Ms L asks Kelly if she knew "right away" what 4 times 6 was. Although Kelly knew that Ms L valued solving problems with understanding, Kelly either thought or wanted to think that she knew this product right away. It is interesting to see that Kelly quickly abandoned her answer of 32 when she added 12 and 12. This suggests that she really did believe that 12 and 12 is the same as 4×6. Thinking back on how Kelly reasoned about the initial number sentence, we see that Kelly did not compute both expressions to reason about this equation. She quickly saw that 12 was two 6s, and therefore knew that she was comparing four 6s to four 6s, which of course are equal. This is a very basic example of the distributive property [4×6 = (2×6) + (2×6)]. Although Kelly knew this number sentence was true, it was not clear if she would use the distributive property to solve problems such as this. Therefore, Ms L asked Kelly to solve a purely computational problem (4×7 = _) to see if she would use the distributive property. Ms L purposely stuck with 4 groups to see if Kelly would use a strategy similar to that of the previous number sentence.

Ms L: "OK, here is another one. Four times 7 equals box. I want you tell me what you would put in the box to make this a true number sentence".

Kelly: "That would be [short pause] 28".

Ms L: "OK, how did you get 28"?

Kelly: "Well, I kinda had other problems… that went into this problem. If you go up here [pointing to 3×7 = 7 + 7 + 7] 3 times 7 is the same as 7 plus 7 plus 7. That problem helped me and I used it with this problem, [pointing to 3×7 = 14 + 7] 3 sevens is the same as 14 and 7… You add one more seven and that goes right to here. [Then she points to 4×6 = 12 + 12.] This problem also helped me because 4×7 is like… My mind went back up to here [pointing to 3×7 = 14 + 7], and I said, there is another 7 so I could put those two 7s together, that's 14, and there are two 14s, 10 and 10 is 20, 4 and 4 is 8, 28".

When Kelly was asked to make 4 × 7 = _ true, she used the distributive property. She knew that two 7s was 14 and she could add 14 and 14 to get four 7s. She got the answer 28, very quickly. It took her less time than it did for her to remember (incorrectly) that 4 × 6 was 32.

It is not clear how Kelly would have found 4 × 7 if she had not worked through the first three number sentences. The goal of the sequence of problems was to get Kelly to reflect specifically on the distributive property in contexts that would support the generation of number facts by breaking down the multiplication into products that she knew. Kelly did a good job of explaining how these problems were related and how she used the problems she had already solved to solve 4 × 7. The goal for subsequent lessons was to get Kelly to use the distributive property on her own to support her learning of number facts and multi-digit multiplication.

3.2 Probing a student's understanding of the distributive property

Alex, the student in the next interview, already used the distributive property to generate multiplication number facts that he did not know. In this interview, Ms L explored Alex's understanding of the distributive property with larger numbers and in different contexts. The interview occurred near the end of third grade. Alex had spent first- through third-grades in classes that provided him opportunity to use relational thinking. In these classes he was expected to solve problems using his own strategies and to share his thinking with his teacher and fellow students. This year Alex had some opportunity to use the distributive property, but several of the problems in the following interview were new to him. Although Alex used more complex relational thinking than many of his classmates, five or six other students in the class demonstrated a similar level of understanding of fundamental properties.

The exchange started with Ms L asking Alex to compute 8 × 4 and 8 × 6, two very traditional arithmetic problems. Ms L noticed that Alex solved these problems using the distributive property, and decided to pose some true, false and open number sentences to draw out Alex's understanding of the distributive property.

Ms L: "Alex, here is a multiplication problem for you, 8×4".

Alex: "8×4, well, that is just like 4×4 twice, so 4×4 plus 4×4, so 16 plus 16, 32".

Ms L: "OK, here is another one, 8×6".

Alex: "I know 6×6 would equal 36, then I could do 36 plus 12, cause that would be having two more sixes, I get 48".

Ms L: "OK, good. Now I have a question for you. I asked you to find 8×6. What you did is figure 6×6 and 2×6. I am going to write this for you here [writes 8×6 = (6×6) + (2×6)]".

Ms L: "Is that what you did"?

Alex: "Yeah".

Ms L: "Is that a true number sentence"?

Alex: "Yep".

Ms L: "Why does that work"?

Alex: "Here there are 8 groups with 6 in each. Here is 6 groups and each has 6. There is always going to be the same amount in each group. I know that 8 is 2 more than 6, so I did 2×6 and added it on to the 6×6 gives me 8×6".

When Ms L presented a number sentence to represent Alex's use of relational thinking, Alex agreed that the number sentence represented his thinking and that the number sentence was true. Alex's justification for the truth of 8 × 6 = (6 × 6) + (2 × 6) was closely tied to the specific numbers used. At this point it was not clear if his understanding of the distributive property was tied to small numbers that he could compute with or would generalize to different situations with larger numbers. Ms L decided to pose a problem with larger numbers to see if Alex's understanding would generalize.

Ms L: "I am going to ask you an open number sentence now [Writes (7×156) + (9×156) = (j×156)]. My question for you is what would j have to equal for that to be a true number sentence"?

Alex: "The amount in each group stays the same, so I don't have to worry about that. All I have to think about is the number of groups. Seven and 9, I add those together so there will be 16 groups of 156, j is 16".

Alex was able to use the distributive property to reason about this number sentence. With these larger numbers Alex's reasoning for what would make the number sentence true was more general. He did not mention the 156 until the very end, but rather noted that the amount in each group was the same. It seems that Alex had a general understanding of the distributive property for number sentences such as this where the amount in each group stays constant. Next Ms L wanted to see if Alex's understanding of the distributive property could extend to situations where the number of groups is the same but the amount in each group differs. These are quite different problems, and the fact that a student applies the distributive property correctly in one situation does not mean that the student understands how the property applies in the other.

Ms L: "OK, here is another one. [Writes 52×11 = (52×10) + m]. What would m have to equal to make that a true number sentence"?

Alex: "The amount of groups are the same. But there is a different amount in each group. So what I am thinking is I change this m to a 1 and have a 1 go in each group. Like if these were candies and I add one more to each group of candies I would have 11 candies in each group".

Ms L: "So what would m have to equal"?

Alex: "One".

Ms L: "OK, so let's see, are you adding one to each ten"?

Alex: "Yes, one to each group".

Ms L: "Alex let me show you how we would write what you are doing [writes 52×11 = 52 × (10 + 1)]".

Alex: "Yes, I add the 1 to the 10 first and then multiply by 52".

Ms L: "Excellent, I see just what you are saying. But, when I write it this way [Writes (52×11) = (52×10) + m], it means I want to have 52 groups of 10 and then add something to it".

Up to this point Alex had interpreted the number sentence in a non-standard way. His solution made sense and appropriately used the distributive property. However, his interpretation of the number sentence made the problem simpler than the one Ms L had posed. Ms L returned to the original problem with parentheses to indicate that 52 was multiplied by 10 before m was added:

(52×11) = (52×10) + m.

Ms L: "Can you figure out what m would have to be to make this true"?

Alex: "So there are 52 tens and 52 elevens. I have to add something to the 52 tens to make it equal. Well, I am not so sure [pause]. I am about to think that I should add 52 ones".

Ms L: "So what could m equal to make that true"?

Alex: "I could add 52 and then take 1 off 52 and put it in each group".

Ms L: "So what would m be"?

Alex: "Fifty-two".

Ms L: “Alex, excellent work”.

Alex clearly was able to use relational thinking based on the distributive property to reason about these number sentence. In our experiences teaching high school, we ran across many students who did not hold such an understanding. Understanding the distributive property helps this 9-year-old in many ways. First let’s consider his learning of arithmetic. Within the next year after this interview, he would be expected to become efficient with his multiplication facts. In this interview, we already see evidence that he was learning his facts in relationship to other facts [e.g. 8×4 is 4×4 twice, $8 \times 6 = (6 \times 6) + (2 \times 6)$] Relational thinking appeared to be helping him to learn his number facts at the same time that learning his number facts was increasing his use of relational thinking. Furthermore, this knowledge might also be expected to support his learning of strategies to multiply large numbers, as almost all efficient multi-digit multiplication procedures are based on the distributive property.

Now consider how what Alex knew could support his learning of algebra. Alex showed evidence of already beginning to think about the distributive property in general terms. His success in reasoning about the problems $(7 \times 156) + (9 \times 156) = (j \times 156)$ and $(52 \times 11) = (52 \times 10) + m$ was not tied to the specific numbers used. He was beginning to generalize his understanding of the distributive property. We hypothesize that if Alex is given continued opportunity to develop and use relational thinking, it will be less likely that he will have to memorize a series of steps to reason about number sentences such as $a \times (b + 1) = a \times b + x$ or will make errors such as $(x + y)^2 = x^2 + y^2$ when he begins to study formal algebra.

4. Conclusion

The kinds of activity and thinking illustrated by these two cases are not isolated examples, and they do not represent mathematics that should be reserved for only a limited number of students or as supplementary enrichment (Carpenter et al. 2003). The results of a recent study by Koehler (2004) documents that students of a wide range of abilities are able to learn to think about relations involving the distributive property and that instruction that focuses on relational thinking as illustrated in these examples can support the learning of basic arithmetic concepts and skills.

We started this article by distinguishing between the practices of traditional arithmetic instruction and the kinds of knowledge that students need to learn algebra with understanding. We are not proposing to abandon the teaching of arithmetic procedures. Procedural fluency continues to be a fundamental goal of the mathematics curriculum (Kilpatrick et al. 2001, National Council of Teachers of Mathematics 2000), but procedural fluency entails more than routine calculation. Procedural fluency includes being flexible in choosing how and when to use procedures. That is precisely what is involved in relational thinking.

Furthermore, procedural fluency is only one strand of mathematical proficiency; another critical strand is developing understanding (Kilpatrick et al. 2001). One of the defining characteristics of learning with understanding is that knowledge is connected (Bransford, Brown, & Cocking 1999, Carpenter & Lehrer 1999, Greeno, Collins, & Resnick 1996, Hiebert & Carpenter 1992, Kilpatrick et al. 2001). Not all connections, however, are of equal value. Students who engage in relational thinking are using a relatively small set of fundamental principles of mathematics to establish relations. Thus, relational thinking as we have described it can be considered to be one way of specifying the kinds of connections that are productive in learning with understanding.

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