

Introduction: The development of students' algebraic thinking in earlier grades from curricular, instructional and learning perspectives¹

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1. Introduction

Algebra readiness has been characterized as the most important “gatekeeper” in school mathematics. Given its gatekeeper role as well as growing concern about students' inadequate understandings and preparation in algebra, algebra curricula and instruction have become focal points of mathematics education research (e.g., Bednarz, Kieran, & Lee 1996, Lacampagne, Blair & Kaput 1995). It is widely accepted that to achieve the goal of “algebra for all”, students in elementary and middle school must have experiences that better prepare them for more formal study of algebra in the later grades (e.g., National Council of Teachers of Mathematics [NCTM] 2000). Yet, curriculum developers, educational researchers, teachers, and policy makers are just beginning to think about and explore the kinds of mathematical experiences and knowledge elementary and middle school students need to be successfully prepared for the formal study of algebra in the later grades (Cai 2004). The goal of this special issue is to present recent work that addresses issues related to early algebra. In particular, the eight articles that comprise this special issue reflect the breadth of focus and perspectives that characterize research on the development of students' algebraic thinking in the elementary and middle school grades.

2. Algebraic thinking in earlier grades

Traditionally, most school mathematics curricula separate the study of arithmetic and algebra—arithmetic being the primary focus of elementary school mathematics and algebra the primary focus of middle and high school mathematics. There is a growing consensus, however, that this separation makes it more difficult for students to learn algebra in the later grades (Kieran 1992, 2004). Moreover, based on recent research on learning, there are many obvious and widely accepted reasons for developing algebraic ideas in the earlier grades (Cai et al. in this issue). The means for developing algebraic ideas in earlier grades, however, is not to simply push the traditional secondary school algebra curriculum down

into the elementary school mathematics curriculum. Rather, developing algebraic ideas in the earlier grades requires fundamentally reforming how arithmetic should be taught (Carpenter et al. in this issue) as well as a better understanding of the various factors that make the transition from arithmetic to algebra difficult for students.

The transition from arithmetic to algebra is difficult for many students as it requires them to make many adjustments, even those students who are quite proficient in arithmetic (Kieran 2004, Kilpatrick et al. 2001). Kieran, for example, suggested the following adjustments: (1) A focus on relations and not merely on the calculation of a numerical answer; (2) A focus on operations as well as their inverses, and on the related idea of doing/undoing; (3) A focus on both representing and solving a problem rather than on merely solving it; (4) A focus on both numbers and letters, rather than on numbers alone; and (5) A refocusing of the meaning of the equal sign. These five adjustments certainly fall within the domain of arithmetic, yet, they also represent a shift toward developing ideas fundamental to the study of algebra.

What is algebraic thinking in earlier grades then? While the authors in this volume do not address this question per se, all seem to agree that developing algebraic thinking in earlier grades should go beyond arithmetic and computational fluency to attend to the deeper underlying structure of mathematics. The development of algebraic thinking in the earlier grades requires the development of particular ways of thinking, including analyzing relationships between quantities, noticing structure, studying change, generalizing, problem solving, modeling, justifying, proving, and predicting (Kieran 2004). The authors in this volume focused their attention on many of these ways of thinking algebraically and how such thinking can be developed in earlier grades.

3. Development of algebraic thinking in earlier grades

The authors of the special issue articles address the development of students' algebraic thinking in the elementary and middle school grades from multiple perspectives. Broadly speaking, the articles consider issues related to curricula, teaching, and learning. In what follows, we briefly provide the reader with a snapshot of each article, highlighting the focus and major points of each article.

3.1 The curricular perspective

Curricula have a significant influence on what students learn (NCTM 2000) and have been found to contribute to mathematical performance differences between U.S. and Asian students (McKnight et al. 1987, Schmidt, McKnight, & Raizen 1996). Accordingly, the examination of curricula from other nations can provide a broader point of view regarding curricular approaches to integrating algebraic ideas into earlier grades as well as provide insights regarding the development of students' algebraic thinking. The first three articles by Cai and colleagues, Schmittau, and Wong concern issues related to the curricular treatment and development of algebraic

¹ As the guest editors, we are pleased to present this Special Issue of the journal to the mathematics education research community. We are very grateful for the support of Gabriele Kaiser in the 18-months process, from the initial idea of the issue to the final editing and from formatting to the substance of each article. We also would like to thank all of the authors for their contributions.

ideas in elementary school mathematics curricula.

In the first article, Cai et al. report results from a cross-national comparative study of the treatment and development of algebraic ideas in the curricular programs of several countries. In particular, the authors conducted case studies of the national mathematics curriculum in China, Singapore and South Korea, a reform-based curriculum from the United States, and a Russian curriculum based on Davydov's theory. Cai et al. examined four critical areas related to the development of algebraic thinking in earlier grades: the extent to which curricula expect students to think algebraically, the level of formalism expected of students, the nature of support provided to help students make a smooth transition from arithmetic to algebraic thinking, and the role of authentic applications in developing students' algebraic thinking. In all five curricula, the main goal for learning algebraic concepts is to deepen students' understanding of quantitative relationships, however, the emphases and approaches to helping students deepen their understanding of quantitative relationships are very different across the five curricula. The results offer valuable insights regarding the design of curricula intended to support the development of students' algebraic thinking.

In the second article, Schmittau presents a description of an interesting curricular approach to elementary school mathematics based on the work of Russian psychologists Lev Vygotsky and V.V. Davydov. In contrast to curricular approaches that view number as foundational in children's early mathematical development, the curricular approach Schmittau describes views algebraic structure as foundational. In this approach, "traditional" arithmetic understanding is developed as students apply their algebraic understanding to concrete numerical instances. As Schmittau points out, this approach is very different from recent reform approaches that seek to introduce elements of algebra into the study of arithmetic. She presents compelling examples from the curriculum as well as of students' thinking to illustrate important characteristics of the approach. In thinking about the implications of this work, we could not help but wonder what such an approach to early elementary school mathematics curriculum might require of elementary school teachers in terms of their knowledge of mathematics. In addition, given the curriculum is a three-year program (grades 1-3), we also wonder what happens as students transition to 4th grade and beyond.

In the third article, Wong takes a historical look at the positioning of algebraic ideas in the elementary school mathematics curriculum in Hong Kong. He describes not only the specific algebraic ideas that were introduced in various versions of the Hong Kong elementary school mathematics curriculum, but also the underlying reasons for the inclusion of these ideas as well as for the evolution of the curriculum. He illustrates his historical analysis with examples from the various versions of the curricular materials as well as with excerpts from interviews with key personnel involved in the various curricular revisions. In light of the success of Hong Kong students on mathematics assessments such as

TIMSS, his analysis promises to provide valuable insights into the design of curriculum that seemingly supports the development of students' algebraic thinking. Further, it is also interesting to compare the various versions of the Hong Kong curriculum Wong describes with the various curricula examined in the Cai et al. article.

3.2 The instructional perspective

Although curricula can provide elementary and middle school students with opportunities to develop their algebraic thinking, teachers are arguably the most important influence on what students actually learn. Thus, the success of efforts to develop students' algebraic thinking rests largely with the ability of teachers to foster such thinking. The second set of articles—Blanton and Kaput, Carpenter and colleagues, and Borko and colleagues—focus on issues related to instruction; in particular, the first two articles focus on instructional practices that foster the development of algebraic thinking, while the third article focuses on professional development designed to better prepare teachers to enact such practices.

In the fourth article, Blanton and Kaput examine how children think about functions and how instructional materials and school activities can be extended to support students' functional thinking. The authors argue that elementary grades mathematics should, from the start of formal schooling, extend beyond the fairly common focus on recursive patterning to include curriculum and instruction that deliberately attends to how two or more quantities vary in relation to each other. Blanton and Kaput provide specific suggestions about how teachers can transform and extend their current resources so that arithmetic content can provide opportunities for pattern building, conjecturing, generalizing, and justifying mathematical relationships between quantities and how teachers might embed this mathematics to help children build mathematical generality.

In the fifth article, Carpenter et al. describe instructional practices that provide opportunities for students to engage in relational thinking in elementary school classrooms. They characterize relational thinking to include examining expressions and equations in their entirety rather than as procedures to be carried out step by step. In their paper, Carpenter et al. note that both arithmetic algorithms and procedures for solving algebraic equations can be taught as rote procedures or with meaning. According to the authors, the more that meaning comes into play, the closer the connection between arithmetic and algebra. Interview excerpts from two elementary school students provide vivid illustrations of children who are developing relational thinking, of different points on a developmental trajectory of relational thinking, and of instructional moves that support the development of both arithmetic and algebraic ideas. Carpenter et al. clearly send the message that we have to help students develop procedural fluency, but developing procedural fluency should go beyond routine calculation. They contend that relational thinking is a powerful means of developing procedural fluency as well as developing students' algebraic thinking.

In the sixth article, Borko and her colleagues focus on professional development designed to foster teachers' abilities to develop middle school students' algebraic thinking. Their program design is based on the premise that the quality of mathematics instruction depends fundamentally on what teachers do with students and what teachers can do depends fundamentally on their knowledge of mathematics (RAND 2003). In their article, Borko et al. present a conceptual framework that informed the structure of their professional development program—a program that focuses on cultivating teachers' understanding of algebraic thinking, learning, and teaching. They report preliminary findings from the first component of their multi-year program (an intensive two-week summer workshop) that highlight the impact of the professional development on the participating teachers' learning. As efforts to support teachers' implementation of practices that foster the development of students' algebraic thinking (practices of the nature described in the previous two articles) continues, there is much to learn from research such as that being undertaken by Borko and colleagues.

3.3 The learning perspective

The design of curricula and professional development programs as well as the enactment of instructional practices intended to support the development of students' algebraic thinking are all dependent, to a great extent, on what we know about students' algebraic thinking and its development. The final set of articles—Izsák and Findell, and Knuth and colleagues—focus on issues related to students' algebra learning and highlight the need to capitalize on curricular and instructional opportunities to develop middle school students' algebraic thinking.

In the seventh article, Izsák and Findell report on work that examines opportunities to build continuity between middle school students' experiences in arithmetic and in algebra. Their particular focus is on the notion of adaptive interpretation, that is, the coordination of shifts in perspective about problem situations involving covarying quantities and corresponding shifts in perspective regarding external representations used in solving such problems. In contrast to previous research that often focuses on connections across multiple representations (e.g., tables, graphs), the authors focus on students' uses of a single representation to analyze relationships and patterns of change between two covarying quantities. They present examples that illustrate students engaging in adaptive interpretation as well as curricular opportunities to explicitly develop adaptive interpretation as students represent and solve problems in arithmetic.

In the final article, Knuth and his colleagues focus on middle school students' understanding of two core algebraic ideas, equality and variable, and the relationship of their understanding to performance on problems that require use of these two ideas. They report results that suggest students' understanding of these core ideas influences their success in solving problems, the strategies they use in their solution processes, and the justifications they provide for their solutions. Although

the number of problems presented to students was relatively small, the results highlight the need for continued attention to these ideas during students' middle school mathematics education. Knuth et al. acknowledge that the nature of students' pre-algebraic experiences lays the foundation for more formal study of algebra and they argue that much of this foundation can be laid as well as strengthened in the middle school grades.

4. Final remarks

As we look across this set of articles with their variety of foci and perspectives, two cross-cutting themes surfaced. First is the importance of better integrating into current school mathematics practices opportunities for students to develop their algebraic thinking. These opportunities include both the design of curricula, at the elementary school level in particular, that pays explicit attention to making connections between arithmetic and algebra, and the recognition of opportunities to strengthen these connections as students progress through middle school. The second theme to emerge is the importance of supporting teachers' efforts to implement practices that foster the development of students' algebraic thinking. If future generations of students are to become better prepared for more formal study of algebra in the later grades, then likewise teachers must also be better prepared. As a set, the articles in this special issue provide guidance and suggestions for continued work in the area of early algebra research and professional development.

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