From the Principle of Bijection to the Isomorphism of Structures: An Analysis of Some Teaching Paradigms in Discrete Mathematics

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Abstract: This paper is concerned with the teaching of Discrete Mathematics to university undergraduate students. Two to three decades ago this course became a requirement for math and computer science students in most universities world wide. Today this course is taken by students in many other disciplines as well. The paper begins with a discussion of a few topics that we feel should be included in the syllabus for any course in Discrete Mathematics, independent of the audience. We then discuss several potential models for teaching the course, depending upon the interests and mathematical background of the audience. We also investigate various educational links with other components of the curriculum, consider pedagogical issues associated with the teaching of discrete mathematics, and discuss some logistical and psychological difficulties that must be overcome. A special emphasis is placed on the role of textbooks.

Kurzreferat: Dieser Artikel beschäftigt sich mit dem Lehren diskreter Mathematik an einer Universität. Vor zwanzig bis dreissig Jahren wurde solch ein Kurs Pflicht für Mathematik- und Informatikstudenten. Heutzutage wird er auch von Studenten anderer Fachrichtungen belegt. Dieser Artikel beginnt mit der Diskussion einiger Punkte, die unserer Meinung nach im Lehrstoff eines jeden Kurses in diskreter Mathematik enthalten sein sollten, unabhängig von der Zuhörerschaft. Anschliessend diskutieren wir verschiedene Modelle diesen Kurs zu unterrichten. Diese hängen vom Interesse und dem mathematischen Ausbildungsstand der Zuhörer ab. Wir untersuchen ebenfalls verschiedene Verbindungen mit anderen Komponenten des Lehrplans. Dazu gehören sowohl pädagogische Sachverhalte in Verbindung mit diskreter Mathematik, als auch logistische und psychische Schwierigkeiten, die überwunden werden müssen. Spezielle Betonung wurde auf die Rolle des Lehrbuchs gelegt.

ZDM-Classification: N75, K25

1 Introduction

Today mathematics education as a research domain is a well-established interdisciplinary field of science. It borders mathematics itself on one side, and philosophy and social science on the other side (see [SieK98]). The main target of this science is pre-college mathematics education (kindergarten through grade 12 in the USA). At this level scientists are involved in various kinds of activities, ranging from concrete teaching strategies to global politics. A recent paper [Cuo03] provides a nice survey on the constant struggle for the improvement in mathematics education in U.S. public schools.

However, as soon as the target shifts to the college or university level, the investigations become much more narrowly defined and oriented primarily on methodological issues in certain areas of mathematics, such as algebra, geometry, and calculus. This is paradoxical at first sight. Indeed, the current mathematical community consists of thousands of professionals, most of them continually adding to their collective experience in the teaching of university level mathematics. While most university professors are happy to share with one another interesting problems, new proofs of known results, or even novel approaches to certain topics, almost none wants their teaching methodology questioned or reviewed. This is somehow viewed as an infringement on "academic freedom". Unfortunately, relatively few seem to have much interest in pedagogical issues in general.

This paper is concerned with the teaching of Discrete Mathematics (DM, for short) to undergraduate students. During the last few decades, DM has become a recognized area within mathematics. Simultaneously, DM has become an important part of undergraduate (and graduate) mathematics education. In the early stages, this process was accompanied by visible attention in the literature. The Sloan Foundation in the U.S. funded several colleges and universities to develop a two year curriculum that balanced continuous and discrete mathematics for entering students. In particular, there were a lot of papers devoted to the teaching of DM as part of the development of a general curriculum for computer science students.

Our attempt here is to resume this discussion of the educational aspects of DM. In this article we try to attract the reader's attention to the following general questions:

- What is the essence and spirit of DM?
- Which models for courses in DM appear most frequently?
- How does one deal with teaching DM to audiences with widely differing interests and mathematical backgrounds?
- What are the "meta-goals" in teaching DM?
- What specific skills should students acquire when taking DM?
- How does one handle the associated logistical problems in teaching DM (grading strategy, exam structure, textbook selection, teaching assistant roles, etc.)?

Although we do not pretend to provide complete, convincing answers to all the posed questions, we do hope that the somewhat unusual combination of our teaching interests and experiences allows us to provoke some much needed dialogue on these issues.

We address this paper to two kinds of readers. On the one hand, we hope that experts in the teaching of DM will find the entire article of great interest, including Section 2 where we describe a few "gourmet meals" from our "teaching kitchen". On the other hand, for a general mathematical educator, probably Sections 4 and 5 will be of most interest. These sections describe the place and role of DM in the global picture of mathematical education.

The paper consists of six sections. In Section 2 a few topics from a standard syllabus in DM are reviewed. Sections 3 and 4 form a global outline of various educational aspects of DM. In particular, we discuss common and distinct features in various models for DM courses, and we look at important links with other parts of the mathematical curriculum. Section 5 is devoted to what may be called "teaching logistics''. Concluding remarks are found in Section 6.

2 A Few Topics from the Syllabus

In this section we briefly outline a few important topics from a course in DM which belong to its "kernel"; that is, which typically appear in almost every syllabus independent of the institution, the specific audience, or the level of experience of the students. This kernel includes set theory and logic, binary relations and functions, integers, mathematical induction, elementary combinatorics, recursions and recurrence relations, and elements of graph theory. These topics can be taken as our definition of DM.

2.1 Set Theory and Logic

Set theory and logic comprise the backbone of any course in DM. In many universities worldwide, these topics, together with relations and functions, form the basis for an introductory course in mathematics that is obligatory for mathematics and computer science majors as well as for students with majors in engineering and economics. This course may be taught over two semesters, where the above-mentioned topics form an integral part of the first semester. However, even if taught in one semester (with either 3 or 4 credit hours), any serious syllabus for such a required course will contain a non-trivial introduction to sets and logic.

An obvious question is what should be considered first: sets or logic? It seems to us that there is no simple answer to this question. One can make an argument for either order. On the one hand, beginning with logic is less formal because the initial notion of a simple statement (or proposition) may be introduced with the aid of several convincing and interesting examples from both real life and mathematics. Taking this approach, the central logical connectives appear as a natural formalization of the "traditional" rules of logic.

On the other hand, at least for future mathematicians, it might be more suitable to begin with the elements of naïve set theory as a background for all the mathematical constructions that they might later study. Students learn to accept the concept of a set as a fundamental idea which cannot be formulated in terms of simpler ones, but one which may be explained through analogies and detailed examples. The instructor then quickly switches to a more formal presentation, thereby providing an opportunity to meet one of the meta-goals of DM: introducing students to the language of mathematics. In any case, logic and set theory should go together, in a sense "shoulder to shoulder" (see examples below).

The main laws of set theory (more rigorously, the axioms of Boolean algebra) must be dealt with early on. Ideally, a teacher should introduce all the axioms (including the absorption laws), although consideration of most of the axioms at the very beginning is probably sufficient. It is important to clarify as soon as possible the "rules of the game": namely, all other equivalences in Boolean algebra should be proven based on these given axioms.

At this stage the instructor has an opportunity to discuss the isomorphism between set-theoretical and logical interpretations of the axioms of Boolean algebra. For a more advanced audience (where a deep course in linear algebra is a prerequisite) one may actually use this term, appealing to the students' experience with isomorphisms between various models of linear spaces. However, as a rule, one should initially avoid a strict explanation of isomorphisms, using instead a nice analogy between disjunction, conjunction and negation in logic on the one hand, and intersection, union and complementation in set theory on the other hand.

Proofs in set theory (and in logic) provide a natural training ground for introducing students to the activity of proving theorems in general. For example, proving the equality of two given sets is a particularly good place to begin. Here we distinguish four kinds of proofs.

I. ***Proof by Definition*** According to the definition, two sets A and B are defined to be *equal* if and only if each is a subset of the other. In the early stages, this may create some difficulty for beginning students, due to the formal nature of such proofs. However, this is a good introduction to formal proofs in other areas of mathematics.
II. ***Naïve or Intuitive Proofs*** The discussion now turns to proofs via Euler-Venn Diagrams. Advantages of this approach are evident; namely, students have a visual method for proving results concerning set theory and logic. The two main disadvantages are these: first, this approach is practically restricted to at most three variables, although some artificial extensions may be designed for, say, 4 variables (see [Gru84a], [Gru84b]); second, this method depends on the comparison of two diagrams and deciding when they are identical (which requires some careful formalization which is beyond the scope of DM).
III. ***"Algebraic" Proofs*** By this terminology, we mean the use of step-by-step transformations, each one using one of the axioms of Boolean algebra (or a previously proven result) as a means of justification.
IV. ***Use of Truth Tables*** We refer to [Gri94], pp. 163-165, where this method is called the use of membership tables. A justification using truth tables can be compared with a justification using Euler-Venn diagrams, establishing a natural bijection between the cells of the diagram and the rows of the truth table. In this way, we are getting very close to the real notion of an isomorphism between the two main models of Boolean algebra, although once again the use of the term isomorphism is not needed at this stage.

For historical accuracy it might be worthwhile to mention the difference between Euler and Venn diagrams. Euler introduced his famous circles in 1761 in order to visually illustrate classical rules of logic and syllogistic reasoning. In his diagrams with two circles you might find disjoint circles, properly intersecting circles, or one circle completely contained in the other, depending upon which form of reasoning you were attempting to illustrate. Over 100 years later, Venn revised Euler's method by using diagrams to "prove'' various identities in Boolean algebra. In Venn's approach the circles representing independent variables were always drawn as intersecting in the most general way. In particular, a Venn diagram with k circles has 2^k regions, and thus can be shaded in 2^{2^k} different ways. An interesting discussion of the pros and cons of using diagrams as a proof technique may be found in [Gar82]. For an excellent review of all methodological, historical, philosophical, and mathematical issues related to the use of Euler and Venn diagrams, we recommend the monograph [Shi94].

2.2 Principles of Elementary Combinatorics

The following principles of elementary combinatorics form a significant part of most syllabi in DM:

- Principle of multiplication;
- Principle of addition;
- Principle of bijection;
- Pigeonhole principle;
- Principle of inclusion and exclusion.

Most of these principles are quite simple, some might even say trivial. However, it is very important to convince students that when used together, they provide powerful tools. To be used effectively, students must perform certain steps to reduce an initial (perhaps quite sophisticated) problem to a number of very elementary standard problems.

In this paper, we pay special attention to just two of these principles: the Principle of Bijection and the Principle of Inclusion and Exclusion. The Principle of Bijection may be formulated as follows: For two sets A and B, we get equality of cardinalities if and only if there exists a bijection between the sets A and B.

Example 2.1 Count the subsets of the set {1,2,3}.

Initially, we solve this problem by what is called the "brute force approach." We list in a certain order all subsets, and then we list separately all binary sequences of length 3 as follows.

{}	000
{1}	100
{2}	010
{3}	001
{1,2}	110
{1,3}	101
{2,3}	011
{1,2,3}	111

After this, we discuss the existence of a natural bijection between the two listed sets which is obtained by means of characteristic functions for the subsets of the set {1,2,3}. Evidently, such a bijection can be established for an n-element set, where n is an arbitrary natural number. Now, using the Principle of Bijection in conjunction with the Principle of Multiplication, we obtain a formula for the cardinality of the power set of an n-element set, namely 2^n.

Example 2.2 Determine the number of natural divisors of 5! = 120.

We note that $120 = 2^3 \cdot 3 \cdot 5$. Therefore, for each natural number d, we see that d divides 120 if and only if d factors as $2^\alpha * 3^\beta * 5^\delta$ with

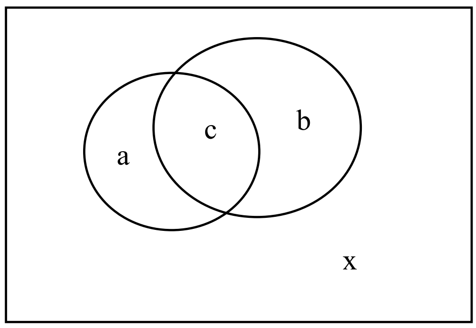
$$0 \le \alpha \le 3,$$
$$0 \le \beta \le 1,$$
$$0 \le \delta \le 1.$$

Once again, a combination of the Bijection Principle and the Principle of Multiplication leads to the desired result: $(1+3)(1+1)(1+1) = 4\cdot2\cdot2 = 16$.

We now discuss the Principle of Inclusion and Exclusion (PIE). This principle should be introduced after a careful consideration of all other principles of elementary combinatorics, so that students are able to immediately see its applications, ranging from those that are trivial to those that are more sophisticated. This provides one more opportunity to bring together topics from both logic and set theory, and to examine them from a combinatorial point of view. To give students an intuitive feeling for this principle, it is usually a good idea to begin with a simple example, such as the following.

Example 2.3 Consider a group of 30 students, 18 of whom have a driver's license, 20 of whom are native English speakers, and 7 of whom both have a driver's license and are native English speakers. Prove that there is an inconsistency in this data.

We may model this by a Venn diagram with two (intersecting) circles, representing the native English speakers and students with driver's licenses, respectively. We let x,a,b,c denote the cardinalities of the four regions in this Venn diagram, as indicated



below.

Then one easily obtains the following linear system of equations:

$$\begin{array}{rrrrl} x + & a + & b + & c & = 30 \\ & a & + & c & = 20 \\ & & b + & c & = 18 \\ & & & c & = 7 \end{array}$$

Solving these equations simultaneously shows that x = -1, an obvious contradiction. After PIE is introduced, one can revisit this example and obtain the contradiction immediately from the computation

$$x = 30 - (20 + 18) + 7 = -1.$$

Of course, it is important to quickly move on to more challenging problems where the power of PIE becomes evident.

There are many natural links between this topic and elementary number theory. In particular, at this stage in the course, one may introduce the Sieve of Eratosthenes and Euler's totient function, φ(n). Depending on the level of the audience, these links may be considered in a rudimentary form with the aid of a few simple examples, or in a more rigorous fashion with all the theorems and proofs.

2.3 Counting Methods in Elementary Combinatorics

Elementary combinatorics deals with such simple combinatorial objects as arrangements, combinations, and permutations. In order to work with a concrete combinatorial object, one has to consider certain kinds of representations. In many cases, the same object has more than one representation. For example, {{a,b}, {c,e}, {d,f}} = {{b,a}, {f,d}, {c,e}} = {{d,f}, {e,c}, {b,a}} = … are just a few of the many (in fact, 48) distinct representations for the same partition of the set {a,b,c,d,e,f}. Thus, it is extremely important to define a canonical representation for an object. In the partition example just given, we can agree to start each subset in the partition with the element appearing earliest in the alphabet, and then order the subsets in the partition using dictionary order. For instance, in the example above, the first representation for the given partition is canonical in this sense. It would be difficult to overstate the importance of canonical representations in discrete mathematics. In order to be able to manipulate canonical representations, a student should have a thorough understanding of partially ordered sets, Hasse diagrams, lexicographic order, and significant exposure to numerous examples of ordered sets in various applications.

Understanding how, in principle, to define a canonical representation and how to conduct a brute force search for all distinct objects is a crucial algorithmic skill. Work with canonical representations also provides a natural opportunity to consider equivalence classes and to observe quotient sets in "real mathematical life". This is nothing other than the art of enumeration of simple combinatorial objects. According to the pedagogical traditions of the former Soviet school, we distinguish constructive and analytic enumeration (the term "constructive enumeration" was coined by I.A. Faradzev in [Far78]). By constructive enumeration we mean the creation of a complete list of all objects of a desired type. Analytic enumeration, in contrast, is simply determining the cardinality of such a list (see [KliLP96] for a detailed discussion of this topic).

Classical objects of elementary combinatorics appear through the consideration of such typical dichotomies as ordered and unordered objects, and objects with or without repetition (ordered and unordered sets/multi-sets in another terminology). One rigorous way to define all such objects is through the use of functions, f: X→Y. For each set X (and Y) there are two options for the elements: distinguishable or indistinguishable. Furthermore, the function f typically belongs to one of three possible (not mutually exclusive) classes: injective, surjective or arbitrary. Thus, consideration of these types of problems provides 2·2·3 = 12 classical combinatorial problems which form the kernel of elementary combinatorics.

Such a formal understanding of the essence of enumeration is surely an achievement of mathematical didactics. Its roots originated in the 1960's and 1970's. Surprisingly, this strict formulation first appeared, to the best of our knowledge, in the literature quite recently (see [Sta86], where in section 1.4 this set of problems is referred to as the "Twelve-fold Way").

Example 2.4 Determine the number of different 4-letter "words" which can be constructed from the letters in MISSISSIPPI.

In an advanced class, one should explain that this problem can be solved by using exponential generating functions. However, even in such a class, we believe that an initial brute force approach is preferable. This approach begins with a clever decomposition, briefly presented in Table 1 below.

Table 1: Decomposition

Type	Symbol	Selection of Letters	Ordering	Total Amount
A1	x^4	2	1	2
A2	x^3y	2·3	4	24
A3	x^2y^2	3	6	18
A4	x^2yz	3·3	12	108
A5	xyzw	1	24	24
				176

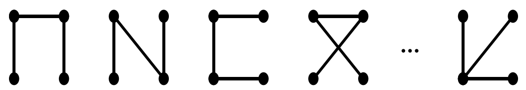
Using Table 1, it is possible to provide students with a visual explanation which utilizes a combination of the Principle of Addition and the Principle of Multiplication. We also naturally consider ordered and unordered objects, examine the significance of objects with repetition, and encounter once again the formulas for the number of objects of a prescribed type (although in this example such numbers may be obtained through a routine listing). Being able to solve this problem shows a fundamental understanding of the principles of elementary combinatorics.

2.4 Isomorphism of Graphs

One of the definitive achievements in the evolution of teaching strategies in DM during the last few decades is the understanding that the notion of graph isomorphism is a crucial ingredient in the course. This concept should be discussed as soon as possible, and definitely should not be postponed until the end of the course. Consider the following small selection of modern textbooks in DM: [Gri94], [Big89], [GooP02], and [Tuc95]. While they differ on a number of important features, each of these textbooks considers isomorphism at the beginning of the chapter(s) devoted to graphs. The reason is quite clear; an adequate understanding of graphs is practically impossible without an understanding of isomorphism.

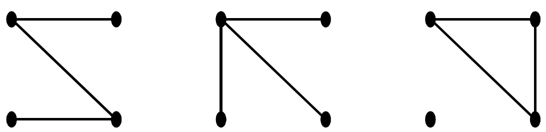
As a rule, when introducing graphs one begins with a rigorous set-theoretical definition of a graph Γ as a pair Γ = (V,E) consisting of the set V of vertices and the set E of edges (or arcs). The set E can be identified with a subset of Cartesian product V^2 (for directed graphs with loops) or with a subset of the set of two-element subsets of V (for simple graphs). Initially, it is probably a good idea to concentrate on the class of simple graphs; that is, undirected graphs without loops or multiple edges. One should stress that two graphs are equal by definition if they are equal as set-theoretical objects: that is, if they have the same vertex sets and the same edge sets. It is important to give examples of different representations for graphs, such as lists (of vertices and edges), diagrams, adjacency lists, adjacency matrices, and incidence matrices.

Constructively enumerating the set of all distinct graphs with a prescribed number n of vertices and a prescribed number m of edges turns out to be a nice exercise in elementary combinatorics, although not all students immediately realize the essence of the problem. Indeed, letting n = 4, m = 3, and fixing the vertex set as V = {1,2,3,4}, we must choose 3 of the 6 two-element subsets of V for our edge set. Thus, we have 20 distinct graphs. It is quite useful to depict (or have students attempt to generate) all such 20 graphs, such as (think of each graph as having the same labeling on the vertices):



One then raises the question: which of these graphs are essentially different and which are essentially the same? This leads to a rigorous definition of graph isomorphism. Returning to the example above, one obtains exactly three isomorphism classes of such graphs. In fact, we can arrange our list of all 20 graphs so that the isomorphism classes will be grouped together. This also provides a natural opportunity to illustrate canonical representations.

Another extremely important concept is that of an abstract graph, or graph without labels on the vertices. This is really an isomorphism class of labeled graphs (usually with a prescribed vertex set, although this is not obligatory). Some examples of abstract graphs are:



This topic provides an opportunity to demonstrate to students that a concrete pictorial view of the diagrams is not essential for isomorphic graphs (provided that we are not interested in topologically inequivalent drawings of the same abstract graph). This is analogous to the description of the grin of the Cheshire cat in Lewis Carroll's Alice in Wonderland. "'All right,' said the Cat; and this time it vanished quite slowly, beginning with the end of the tail, and ending with the grin, which remained some time after the rest of it had gone. 'Well! I've often seen a cat without a grin,' thought Alice; 'but a grin without a cat! It's the most curious thing I ever saw in all my life!'"

This naturally leads to a more serious investigation of the famous graph isomorphism problem. That is, we can discuss the ambiguity of whether a "good" solution exists for the problem of determining when two graphs are isomorphic. For many audiences, this idea may not be accessible. In advanced classes for mathematics and computer science majors, the concept of polynomial time algorithms may be briefly discussed (see [Luk93] for more details about the status of this problem). However, for most audiences, this problem will be addressed on a naïve heuristic level.

We refer to [KliPR88] and [KliRRT99] for more details concerning educational and computational aspects of the graph isomorphism problem.

3 Models for DM Courses

The union of the teaching experiences of the authors in the areas of DM includes several dozen courses delivered in at least six countries, including the former USSR, USA, Israel, Fiji, and Germany, although most long-standing activities took place in the first three countries mentioned. Based primarily on these experiences, but also taking into account information provided by our colleagues, this section is an attempt to represent a number of different models which may be used in the teaching of DM. We do not mean to imply that these models universally cover all possible options for teaching DM. Nevertheless, they represent quite typical situations at large universities in the USA, Israel, and other countries.

In our discussion we will refer to various educational institutions by the following abbreviations:

- UD University of Delaware, Newark, DE;
  BGU Ben-Gurion University of the Negev, Beer Sheva, Israel;
- WIS Weizmann Institute of Sciences, Rehovot, Israel;
- NTC Negev Technological College, Beer Sheva, Israel;
- USP The University of the South Pacific, Suva, Fiji;
- FU Free University, Moscow, USSR.

Numerous other institutions will be referred to anonymously, so as not to overload this presentation with too many details.

A number of courses that we taught, which definitely belong to the area of DM, were given to graduate students (for instance, topics courses on graph theory, combinatorics, algebraic graph theory, algebraic combinatorics, finite geometry, and so on [at UD, BGU, WIS]). Although these were interesting and exciting teaching experiences, the specific focus and content of these courses is beyond the scope of this paper. Nonetheless, it should be pointed out that most of the pedagogical principles shared by us are valid for all levels of the audience, from freshmen to final year Ph.D. students.

In this text, as previously mentioned, by teaching DM we mean teaching DM to undergraduate students. Several important factors are considered in our description of the various models:

(i) Is this a one-semester or two-semester course?
(ii) Are there pre-requisite courses that cover part of the material in DM?

(iii) How many contact hours are devoted to lectures and discussion/exercise classes?
(iv) What is the intended audience for the course (math, CS, chemistry, engineering, economics, etc.)?

Taking these factors into consideration, we list several possible models (not necessarily inclusive) for a course in DM.

A. **Math and CS Majors –** 4 hours of lectures and 2 hours of discussion (exercises), one semester, pre-requisite is a one-semester course of mathematical logic [BGU and FU].
B. **All Majors** – 3 hours of lectures and 1 hour of discussion (exercises), one semester, no pre-requisites. We call this course DM1 [UD and USP].
C. **Communication Engineering Majors** – 4 hours of lectures and 2 hours of discussion (exercises), one semester, pre-requisite is linear algebra [BGU].
D. **Math and CS Majors** – 3 hours of lectures and 1 hour of discussion (exercises), one semester, pre-requisite is DM1. We call this course DM2 [UD].
E. **Software Engineering Majors** – 3 hours of lectures and 1 hour of discussion (exercises), one semester, no pre-requisites [NTC].

We now very briefly discuss each of these models in more detail.

A. In this format, we have the greatest number of teaching possibilities. Students enter the class with a deep understanding of set theory, logic, relations, and natural numbers. They are familiar with proofs and the use of formal notation in proofs. The course itself covers elementary and enumerative combinatorics, recursion, ordinary and exponential generating functions, and elements of graph theory. Proofs are required, including rather sophisticated ones! Many of the exercises have features of Math Olympiad problems and require certain non-standard approaches.

B & D These two courses form a sequence for math and computer science majors. Taken together, DM1 and DM2 cover all the material in Model A (together with the pre-requisites). This was a typical model during the first decade of our teaching experience at UD. Unfortunately, during the last decade, DM2 was not offered every year, and thus the default DM course for most BS students became DM1. Another disadvantage of this model is that the same course is offered to different audiences, where the range of abilities and mathematical backgrounds may be substantial.

C. This model was created at BGU a few years ago when the Department of Communication Engineering realized that DM forms an important ingredient in the mathematical education of their students. There was a commitment that elements of set theory (on a naïve level) would be considered in Linear Algebra, thus ensuring that second year students would begin DM with a reasonable level of mathematical maturity. In fact, most of the syllabus for model A (as well as logic, relations, functions, and natural numbers) is included in the syllabus for model C. In such a course, although the level of mathematical rigor is clearly reduced, students still encounter formal proofs and many sophisticated techniques from enumerative combinatorics.

E. This course is quite similar to model C, although there are fewer contact hours and the audience is somewhat weaker. It should also be noted that DM was typically offered at NTC to this audience in the same semester as a computer software course which supported some logical languages. For the majority of the audience this combination of courses proved to be very beneficial.

These models will be referred to by name in subsequent sections as the need arises.

4 Vertical and Horizontal Educational Links

We borrow this term from the NSF program VIGRE ("Vertical Integration of Graduate Research and Education"), although our use of the term is somewhat different. Briefly, by a vertical educational link, we mean a prerequisite structure for taking undergraduate courses. This can be done formally, where certain courses must be taken before one is allowed to register for a given course, or informally with a "consent of instructor" prerequisite, where students registering for a course must first talk to the instructor to make sure they are indeed prepared for the material that will be presented in that course. One can also speak of "degenerate" vertical links within a course, where mastery of one topic is needed before certain other topics (or techniques) can be addressed.

By a horizontal educational link, we mean a reference to some other course, at about the same maturity level, that a student is taking, has taken, or is about to take. Such references are often made by instructors to recall certain ideas or to give students a preview of topics to come.

4.1 Vertical Links from High School to DM

In teaching DM at the university level, it would be extremely helpful to know that the elementary principles of set theory, logic, and basic counting were covered in the high school curriculum. At one point in time this was true, at least in a few countries. A good example of this is the "revolution" in mathematics education at the secondary schools in the former Soviet Union that took place in the 1960's (see [VilS74] for a discussion of this reform by one of its most important activists, N.Ya. Vilenkin).

In the United States, the reform movement in mathematics education began in the 1950's with the formation of the UICSM (University of Illinois Committee on School Mathematics) in 1951 and the NSF (National Science Foundation) in 1950. By 1958 after several revisions, courses for the four high school grades were implemented in a dozen pilot schools. These materials were divided into 11 units, which covered all the topics in the usual secondary school program. The two facets of understanding central to the development and the methodology of this curriculum project were the precision of language and the discovery of generalizations. The inclusion of DM topics, such as set terminology and notation, logic, and deductive structure and theory served the purpose of adding clarification and precision rather than of making the mathematics more rigorous. See [Hen63] for a more detailed discussion of this development.

Today many of these original 11 units form the core of courses in high school mathematics. However, by the late 1970's the classroom-piloted and revised materials had been replaced by commercial textbooks and much of DM had evolved from central organizing themes to self-contained chapters and topics in chapters. These topics might then be omitted by a teacher because students found them difficult and parents did not grasp their mathematical value. Today, some discrete topics continue to be addressed in the NCTM Curriculum and Evaluation Standards for School Mathematics, see [NCTM00]. However, an instructor of DM at the university level simply cannot assume that prospective undergraduate students have been exposed to these DM concepts in high school. This is in stark contrast to the experience of a calculus instructor at the university today, and is one of the reasons why teaching DM is often considered a more "challenging" assignment. Thus, for all intents and purposes, in the United States, the vertical link between high school mathematics and DM at the university is tenuous at best.

4.2 Vertical Links from Pre-Calculus and Calculus to DM

At many universities in the United States today, significant numbers of students take some sort of pre-calculus, college mathematics, or contemporary mathematics course. This is often a requirement for most students at the university. In a contemporary mathematics course one often finds an elementary discussion of some topics from DM. However, this course is not a prerequisite for subsequent courses.

Unfortunately, most students taking such a course do not go on and take additional mathematics courses at the university. Hence, once again, any vertical link between such courses and DM is tenuous at best, and probably should be ignored.

However, more and more students are taking calculus in high school, and most students in DM have had some exposure to this subject, either in high school or in a previous university mathematics course. Thus, while calculus is not a formal prerequisite for DM, most instructors in DM can safely assume that the vast majority of their students are familiar with this content area. This link can be useful in two ways. First, exposure to calculus should increase a student's mathematical maturity. Second, there are some calculus topics that often appear in DM, such as functions, sequences, and series. For example, sequences are typically defined by a formula in calculus, while they are most naturally defined recursively in DM. When series are presented in calculus, the fundamental issue is one of convergence. In DM, however, series are typically treated formally as tools for counting with no concern about convergence issues. Sometimes these links can effectively be used by attentive instructors of DM.

4.3 Vertical Links from DM to Other Courses

The previous sections indicate that currently there is at least one viable link from other courses to DM ("incoming arcs"). It should also be noted that one corollary of the relative absence of incoming vertical links to DM is that probably most universities should offer a number of different courses in DM, distinguished by the mathematical background and interests of the student audiences. However, this is basically a funding issue and will not be discussed further in this paper.

4.3.1 Vertical Links from DM to Probability

It is widely accepted that probability (and statistics) should play a significant role in any mathematics curriculum. There is an obvious vertical link from DM to probability; namely, counting techniques and basic enumerations are fundamental to any discussion of discrete probability. Either one of the two previously discussed prerequisite structures can be employed here. One can informally suggest to students that knowledge of DM will be extremely beneficial in the study of probability, but not formally require DM as a prerequisite. In this case, formal topics from probability theory would not necessarily appear in the DM syllabus, and historically one would use a book such as [Fel68] for the probability course. As this book is no longer in print, one can easily substitute [Gha99] for this task.

Alternatively, one can formally require DM as a prerequisite for probability. In this case, the syllabus for DM (see Section 2) should include topics such as the elementary rules of probability, Bernoulli trials, and expected value. A possible text for such a course in DM is [BakE99].

4.3.2 Vertical Links from DM to the Design of Algorithms

About two decades ago when DM was introduced into the university mathematics curricula and when computer science departments had a surplus of students, it was common practice was to include an introduction to the analysis of algorithms in the DM syllabus. At that time CS departments seemed content to let math departments play the role of introducing this topic to their students. Thus, program correctness and time complexity analysis often appeared in DM syllabi and textbooks at that time (see [BakE99], [Big89]).

Today, however, when CS departments have more faculty and fewer undergraduate majors than they did a decade ago, these algorithmic topics have typically moved back into the CS curriculum. Nonetheless, current DM courses still include "algorithmic approaches" to many problems. This not only prepares CS majors for a formal Design of Algorithms course to be taken later on, but also gives all students in DM exposure to constructive proofs and algorithmic thinking. Problems in DM where this is particularly appropriate include finding gcd's (Euclid's Algorithm), elementary primality testing (Sieve of Eratosthenes), 2-coloring bipartite graphs, and solving linear recurrence relations. It should be emphasized that the actual implementation of algorithms is best left to computer science courses.

4.3.3 Vertical Links from DM to Graph Theory

Graph Theory has become pervasive in most science and social science disciplines. Thus the link from DM to graph theory is a crucial one. How much can be included in the syllabus depends upon whether DM is a one-semester or two-semester course (see Section 3). In any case it is important to quickly get beyond the typical lengthy list of definitions at the start of graph theory to topics that are mathematically more significant. Since the follow-up courses in graph theory can be either theoretical or applied, the nature of this vertical link is two-pronged.

Graph algorithms, such as Kruskal or Prim for minimum cost spanning trees, Dijkstra for shortest paths, and the Hungarian Method for maximum matchings in bipartite graphs, form a significant ingredient in this vertical link. These algorithms will prepare students for many areas of applications. If time permits, a careful analysis of at least one of these algorithms, including a rigorous proof of correctness and a time complexity analysis, should be done.

4.4.4 Horizontal Links

Horizontal links are the references to other courses and walks of life that naturally relate to the topic being discussed in class. It is hoped that all experienced teachers of DM regularly do this to keep the material alive and relevant. For instance, an opportunity for this activity occurs when changing the index of summation in various enumeration problems. Students should be reminded that this is analogous to the method of substitution when computing integrals in a calculus course.

At some institutions collaboration between the mathematics department and various "customer departments" sometimes enables the instructor of DM to tailor the syllabus to the specific needs of the audience. This occurred at FU in 1990-92, where the audience consisted primarily of chemistry majors, and at BGU, where a version of DM was tailored to the needs of students studying communication engineering (using the textbook [Big89]).

5 Further Philosophical and Logistical Facets of DM Didactics

In this section we consider teaching DM from a global perspective. We discuss the overall goals of a DM course, the skills we want students to achieve by taking DM, the nature of proof and its relation to DM, the use of exercise classes and discussion sessions, the selection of textbooks, and the evaluation of students.

5.1 Key Goals of a Course in DM

First we elaborate the overall goals of DM, at least in our opinion. In particular, we indicate how these goals might differ when viewed from the perspective of a CS department as opposed to a math department.

5.1.1 Paramount Objectives

DM as an independent educational subject was defined within the last 40-50 years. The year that we might attribute to its birth would be 1957, the year in which the first sputnik was launched. This historical event created great challenges for principally new applications of mathematics in what had suddenly becoming a growing technological era (see the discussion in Section 5.4 below). From its early beginnings, DM was intentionally contrasted with continuous mathematics, and was once called a "grab bag of methods" by the eminent mathematician Saunders MacLane in response to an article promoting DM written by Anthony Ralston [Ral84]. This new scientific area had at its kernel combinatorics and graph theory, which in turn was linked with set theory, logic, algebra, geometry, algorithms, and optimization. Within the arena of the mathematical education of students, we suggest that DM has particular educational goals (perhaps, even a special mission) which previously had not been addressed in the teaching of mathematics.

Some of these meta-goals were formulated 20-25 years ago in the framework of the ACM standards for a curriculum in undergraduate computer science in the United States (see [Ber87] and the references in that article). Below are stated six features, the first two which are quotes from [Ber87], that we believe are essential to any course in DM.

- "DM is a terse and precise language of mathematical communication."
- "A primary use of DM is to make difficult problems tractable."
- Learning proof techniques is an essential feature of DM.
- DM provides the mathematical background for algorithmic thinking.
- DM provides an alternate paradigm to continuous mathematics in mathematical modeling.
- DM is considered by many instructors to be the ideal breeding ground for creative mathematical thought. (Perhaps this point of view is best driven home in [CheK92], where problems borrowed from various International Mathematical Olympiads form the backbone of this textbook.)

5.1.2 Math and CS Perspectives

In most universities there is a joint course in DM for both CS and Math majors. As a rule, the syllabus for such a course is a compromise between the interests expressed by these two departments. On the one hand, a course in Discrete Structures (rather than DM) probably would best serve the algorithmic interests of most beginning CS students. Topics in such a course would include Boolean algebra, Turing machines, formal logic, searching and sorting, and various programming applications (see [Lev80] for a typical textbook). On the other hand, DM as a subject in pure mathematics is a more sophisticated object. In this context DM may be regarded as a first step in the mathematical education of someone who will eventually work in some area of algebra, geometry, combinatorics, graph theory, probability theory, and so on. Emphasizing proofs, introducing the notions of recursion and generating functions, and providing challenging exercises for the students are essential features in this context.

Thus the interests and goals of the CS community and the mathematics community are quite different, and anyone teaching either Model A or Model B & D of a course in DM must learn the art of compromise.

5.1.3 Achievement Skills

In this section we describe the skills we believe all students taking DM should achieve. These skills are more or less independent of the model for DM being offered (see Section 3), although obviously the course syllabus will have some impact on what is possible to achieve.

- Students should finish a course in DM with a fluency in combinatorial reasoning. That is, they should be able to use the basic principles of elementary combinatorics, they should be able to decompose combinatorial problems into a sequence of simpler problems, and they should be able to manipulate various representations of the standard combinatorial objects.
- Students in DM should learn algorithmic thinking. In particular, students should have enough common sense to try brute force approaches as a first attempt, should understand divide and conquer strategies, and should understand how to represent combinatorial objects on a computer.
- Students should understand the art of combinatorial counting. This is especially true for math majors, and includes such skills as the effective use of ordinary and exponential generating functions as well as the use of recursion.
- Students should be able to use graphs to model a wide variety of problems. Of course, time limitations become a serious concern in developing this skill, especially in one semester courses. Thus we see once again the importance of introducing graphs as early as possible in DM.
- Students should learn the essence of classification. This includes an understanding of equivalence relations and isomorphisms. Perhaps this is most important for future applications both inside and outside mathematics. That is, students need to learn how to avoid needless mess and chaos, and how to create order out of disorder.

5.2 Proofs

Although the role of proof was previously discussed (see Section 2.1), in this section the perspective taken is more global. Clearly, establishing the validity of specific results is the heart of any mathematical activity. Many of the proofs in DM are required for the future development of the subject matter. For example, proofs are required to establish whether a given relation is an equivalence relation, to confirm the correctness of an elaborate definition, to establish a practical criterion for the fulfillment of some necessity condition, and to demonstrate that an algorithm will terminate after a finite number of steps. Moreover, in DM many proofs are constructive in nature, and thus are useful algorithms in their own right.

Typically there is not enough time to provide all possible proofs in a course of DM. One should choose proofs that potentially increase the students' understanding of the mathematics involved. In many cases, a complete proof may be replaced with an outline of the proof, with a proof of a particular case, or with a particularly striking example.

When is a proof to be considered "beautiful" or "from the book"? There is no reasonable way to answer this question; it is simply a matter of taste. A good example is a combinatorial proof of a binomial identity, where each side of the identity is shown to represent the number of elements in the same set of combinatorial objects (counted in two different ways). The stronger students in a class often appreciate the beauty in such an argument, while the weaker ones typically are befuddled by the argument.

Finally, we mention the role of experimentation. Mathematical experimentation, followed by conjecture, and then eventually proof is the standard mode of operation for most practicing mathematicians. It is always nice to introduce this practice in the classroom, time permitting. As the use of computers and symbolic manipulation packages becomes more and more prevalent, this mode of teaching will undoubtedly become more common. Instructors will have to change their behavior to allow for more cooperative efforts with the students inside the classroom. There are many parts of DM where such experimentation seems particularly appropriate, elementary number theory being a prime example.

5.3 Exercises and Discussion Sections

As in other mathematics course, lectures in DM should be followed by exercises or discussion sections. There are two main forms for this activity. In most European (and Israeli) universities, small groups of students will meet each week for two hours with a teaching assistant (TA, for short), who may be a graduate student or a faculty member. These meetings are called exercise classes. The TA plays an active role in these meetings, illustrating the lectures of the previous week through some "warm up" exercises, followed by some standard exercises, and then some more challenging exercises. The majority of students taking DM at these institutions are quite serious, and both the lecturer and the TA have office hours that are fully utilized by the students.

In most American universities the situation is somewhat different. Again small groups of students meet with a TA once per week. But here the meeting time is usually one hour, and the TA is definitely a graduate student. Moreover, the role of the TA is much more passive. That is to say, the TA spends most of his time answering student questions, serving as a mediator between the lecturer and the students, and helping students overcome various difficulties facing them (sometimes, psychological ones). Office hours for the TA tend to be more heavily attended than for the lecturer, partly because the TA again is viewed as an advocate.

5.4 Textbooks

Writing a textbook is undoubtedly the clearest possible way of expressing one's teaching philosophy and methodology concerning a given subject area. We have supported this thesis in practice, writing two textbooks in the area of DM ([BakE99] and [KliPR88]).

When discussing textbooks in DM, one needs to start again in the year 1957 with the pioneering book Introduction to Finite Mathematics by Kemeny, Snell, and Thompson [KemST57], whose last edition appeared in 1974. This was an innovative textbook, not only by its content but also by its style and method of presenting material. It is important from a historical point of view to note that the title calls the subject area "finite mathematics", not "discrete mathematics". Finite mathematics still exists today, at least in the USA, as a separate course from DM. It is taught primarily to business students (and sometimes biology students), and emphasizes matrix arithmetic, linear programming, a little finite probability, and perhaps an introduction to Markov chains. There is little overlap with DM, at least as defined in this article.

Restricting now to DM as defined here, there are a few textbooks that we feel deserve special credit due to their essential input in the development of DM as an educational subject. In particular, we mention [Vil69], which lead to the establishment of the Russian tradition of teaching elementary combinatorics as part of DM, and the comprehensive book [Liu68], which was the principle root of all serious teaching literature in DM. We consider these the forerunners of modern textbooks in DM.

Of the modern textbooks in DM that we have tested, we feel the following are among the best: [Bal91], [Big89], [Bru04], [CheK92], [GooP02], [Gri94], [Rob84], and [Tuc95]. We also mention a few classics; these are books not necessarily intended for undergraduate students, but rather are encyclopedic sources of information for instructors of DM. These include [GraKP94] and [Sta86]. Assuming a homogeneous audience, we believe that currently there is no problem selecting a good textbook (or textbooks) for a course in DM. For example, we used [Tuc95] for Model B, [Big89] and [Gri94] for Model C, and [Bal91] and [Gri94] for Model E. Each time the audience seemed quite happy with the text selection. Moreover, it is worth mentioning that the supply of textbooks in DM is exploding with each passing year. For example, depending upon one's taste and the nature of the (homogeneous) audience, anyone of the recent texts [Ros03], [LovPV03], [PemS03], [Wal03], [Epp04], or [WheB05] could be effectively used.

However, for the advanced students in Model A or for a non-homogeneous audience in Model B, there is still a serious problem. There does not seem to be any available text written in a multi-level fashion that can simultaneously be used by a range of students, varying from fairly weak to exceptionally strong. This is especially important for large American universities with diverse audiences, where tradition and economics dictate that only one textbook be assigned. One of the goals of this paper is to inspire others to write such a definitive textbook.

5.5 Grading and Exams

Strong differences in the culture, traditions, and types of courses offered at various universities around the world make it impossible to make any universally accepted statements concerning the giving of exams and the awarding of grades in DM. However, we list below a few models with which we have some experience, and then describe some of salient features of each.

5.5.1 The Soviet Pattern

Most of the exams in the former Soviet Union (and many other European countries, as well) were oral. Typically, a lecturer prepares a few dozen "tickets", each of which contains one or two theoretical questions and one or two exercises. The theoretical questions usually involve the formulation and proof of some standard proposition. Students then randomly select a ticket and are given 30-60 minutes to prepare an answer, after which they orally present their solution in 15-20 minutes before the examiner (instructor and/or TA). Students might be given extra time to clarify some claims or provide an answer to some supplementary question. The exam is then graded on a basis of 5 to 2 (in the former Soviet Union style), which is roughly equivalent to the A to D basis in the U.S. Sometimes plus/minus grading is used.

The main advantage of such a system is the fact that it allows for almost all of the presented material to be fair game for the exam, including most of the proofs given in the course. This partly explains why the level of mathematical rigor among university students was so high in the former Soviet Union. Namely, this exam pattern pushed students to become comfortable with the theoretical mode of thinking in all their courses, including calculus. As a byproduct, DM did not have to concentrate on introducing proof and abstraction; this was already accomplished before students took DM. One disadvantage of this system is the obvious element of subjectivity in the awarding of marks.

5.5.2 The Israeli Pattern

In Israel course grades are primarily determined by the final exam, which lasts at least three hours. Sometimes there is one midterm exam, which may account for 20-25% of the final grade. Homework assignments may account for an additional 5-10% of the course grade. Nonetheless, the final exam is by far the dominant factor in grading. Students are allowed to take the final exam twice. If a second attempt is made, the results from the first attempt are automatically cancelled, even if the first score is higher. This gamble often causes great anxiety among the students. Each instructor finds his own way of preparing students for the final.

Psychological support is needed, encouraging students to do their very best on the first attempt at the final, and yet preparing them for the possibility of failure. Test-taking skills should be discussed, such as telling each student to find the easiest problem and do it first. This tends to calm the nerves and also maximize the total score. Which problem is easiest depends upon the relative strengths of the student. Those who like to think algorithmically and are good at computation will choose one kind of problem, while those with strong creative abilities and a distaste for messy computations will undoubtedly make a very different choice.

It should be mentioned that sometimes choice is allowed on the final exam, say by allowing students to choose 5 of 7 problems. This eliminates some of the pressure, and allows the instructor to better find out what the students know.

5.5.3 The American Pattern

The typical American pattern for awarding grades puts less emphasis on the final exam and places more emphasis on homework, quizzes, and (several) intermediate exams. Most courses have a total number of points, say 600, that can be awarded to each student. The final letter grade for the course (A through F, with possible plus/minus grading) is determined by the number of points earned. This correspondence can vary dramatically from one course to another, and even from one instructor to another within a given course. However, in multi-section courses at some universities, there is a course coordinator and sometimes common exams (at least, a common final) are given. In such cases, the correspondence between accumulated points and final letter grades is often spelled out by the coordinator, with only minor variations available to individual instructors.

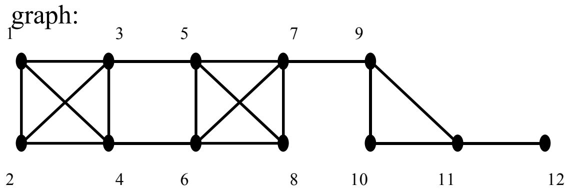
In the American system the predicted final grade becomes clearer and clearer to individual students with each passing stage of the course. The final exam primarily serves as a confirmation that the anticipated grade is indeed the correct one. Student complaints about final grades are relatively rare in this system, at least in comparison to the Israeli system. Students do not feel a great deal of pressure during final exams, and pretty much know what grade they will receive in each of their courses. This certainly can be viewed as an advantage of the system.

The main disadvantage is the relatively short time period for each exam. Midterm exams typically last only 50 minutes. Final exams are most often two hours in length. In a course such as DM this is not enough time to adequately attack complicated theoretical or computational problems, especially on the midterms. Hence, exams tend to test student knowledge somewhat systematically, but not very deeply.

5.5.4 A Sample of Problems

Below we present a handful of problems which are taken from actual DM exams given by us and which are somewhat different than the standard ones found in most textbooks. Of course, the audience and type of course determines whether or not such problems are appropriate.

- Let A={1,2,3,4,5,6,7,8} and let R be an equivalence relation defined on A. Suppose we know that
  a) (1,2), (2,3), and (7,8) are in R, while (1,5) and (1,6) are not in R;
  b) |R|= 22 (that is, R consists of 22 pairs).
  Determine all possibilities for the quotient set A/R.
- In how many ways is it possible to select three distinct numbers from {1, 2, 3,…, 99} whose sum is congruent to 0 modulo 3?
- Find a closed formula for the sum from k=0 to k=n of k(k+1)(k-1).
- How many different 7-digit (decimal) numbers contain exactly 4 distinct digits?
- Find the number of spanning trees in the following graph:



6 Concluding Remarks

Discrete and continuous mathematics are objectively distinct parts of mathematics, and this fact is now well recognized by the mathematical community. Nonetheless, the role of DM in many math departments is still undergoing change. This should not be done in a revolutionary manner. Changes should occur in stages, taking into account the existing traditions and goals in making the change.

The success of any model for a course in DM depends heavily upon the syllabus and the textbook. Teaching DM as a two-semester sequence is definitely preferable, but not always feasible. When teaching only a one- semester course, it is better to decrease the amount of new material covered and spend the extra time making sure the audience has the required background knowledge for the course. In any case, even when the audience is quite mathematically mature, completing a course in DM will likely leave the students with a broad spectrum of emotional experiences, from inspirational highs achieved through steady progress in combinatorially and algorithmic thinking to the depressing lows caused by the necessity of learning this new language of mathematics.

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References

[BakE99] Baker, R.D., Ebert, G.L. (1999). Discrete mathematics. Dubuque, Iowa: Kendall/Hunt.

[Bal91] Balakrishnan V.K. (1991). Introductory discrete mathematics. Mineola, NY: Dover.

[Ber87] Berztiss A. (1987). A mathematically focused curriculum for computer science. Communications of the ACM, 30 (5), 356-365.
[Big89] Biggs N.L. (1989). Discrete mathematics, 2nd edition. New York, NY: Oxford University Press.
[Bru04] Brualdi R.A. (2004). Introductory combinatorics, 4th edition. Upper Saddle, NJ: Prentice Hall.
[CheK92] Chen, C.C., Koh, K.M. (1992). Principles and techniques in combinatorics. River Edge, NJ: World Scientific.
[Cuo03] Cuoco, A. (2003). Teaching mathematics in the United States. Notices of the American Mathematical Society, 50(7), 777-787.
[Epp04] Epp, S. (2004). Discrete mathematics with applications. Belmont, CA: Brooks/Cole.
[Far78] Faradzev, I. A. (1978). Constructive enumeration of combinatorial objects. In: Problèmes combinatoires et thèorie des graphes (Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976), Colloq. Internat. CNRS, 260, 131-135.
[Fel68] Feller, W. (1968). An introduction to probability theory and its applications, Volume 1 (3rd edition). New York-London-Sydney: John Wiley & Sons.
[Gar82] Gardner, M. (1982). Logic machines and diagrams. Chicago, IL: University of Chicago Press.
[Gha99] Ghahramani, S. (1999). Fundamentals of probability, 2nd edition. Upper Saddle River, NJ: Prentice Hall.
[GooP02] Goodaire, E.G., Parmenter, M.M. (2002). Discrete mathematics with graph theory (2nd edition). Upper Saddle River, NJ: Prentice Hall.
[GraKP94] Graham, R.L., Knuth, D. E., Patashnik, O. (1994). Concrete mathematics: A foundation for computer science. Reading, MA: Addison Wesley.
[Gri94] Grimaldi, R.P. (1994). Discrete and combinatorial mathematics: An applied introduction (3rd edition). Reading, PA: Addison-Wesley.
[Gru84a] Grunbaum, B. (1984). The construction of Venn diagrams. *College Mathematics Journal 15(3)*, 238-247.
[Gru84b] Grunbaum, B. (1984). On Venn diagrams and counting of regions. *College Mathematics Journal 15(5)*, 433-435.
[Hen63] Henderson, K.B. (1963). Mathematics. In *Using Current Curriculum Developments*, Washington, D.C.: Association for Supervision and Curriculum Development, NEA.
[KemST57] Kemeny, J.G., Snell, J.L., Thompson, G.L. (1957). Introduction to finite mathematics. Englewood Cliffs, N.J.: Prentice-Hall.
[KliLP96] Klin, M., Liskovets, V., Pöschel, R. (1996). Analytical enumeration of circulant graphs with prime-squared number of vertices. Sèm. Lothar. Combin., 36, Art. B36d, approx. 36 pp. (electronic).
[KliPR88] Klin, M.Ch., Pöschel, R., Rosenbaum, K. (1988). Angewandte Algebra für Mathematiker und Informatiker. (German) [Applied algebra for mathematicians and information scientists] Einführung in gruppentheoretisch-kombinatorische Methoden. [Introduction to group-theoretical combinatorial methods] Braunschweig: Friedr. Vieweg & Sohn.
[KliRRT99] Klin, M., Rücker, Ch., Rücker, G., Tinhofer, G. (1999). Algebraic combinatorics in mathematical chemistry. Methods and algorithms. I. Permutation groups and coherent (cellular) algebras. Match, 40, 7-138.
[Lev 80] Levy, L.S. Discrete structures of computer science. (1980). New York-Chichester-Brisbane: John Wiley & Sons.
[Liu68] Liu, C.L. (1968). Introduction to combinatorial mathematics. New York-Toronto: McGraw-Hill Book Co.
[LovPV03] Lovasz, L., Pelikan, J., Vesztergombi, K. (2003). Discrete mathematics: elementary and beyond. New York, NY: Springer.
[Luk93] Luks, E.M. (1993). Permutation groups and polynomial-time computation. In: Groups and computation (New Brunswick, NJ, 1991), DIMACS Ser. Discrete Math. Theoret. Comput. Sci., 11, (pp. 139-175). Providence, RI: Amer. Math. Soc.
[NCTM70] National Council of Teachers of Mathematics (ed.) (1970). A History of Mathematics Education in the United States and Canada. Reston, VA: NCTM.
[NCTM00] National Council of Teachers of Mathematics (ed.) (2000). *Principles and Standards for School Mathematics*. Reston, VA: NCTM.
[PemS03] Pemmaraju, S., Skiena, S. (2003). Computational discrete mathematics. Cambridge, England: Cambridge University Press.
[Ral84] Ralston, A. (1984). Will discrete mathematics surpass calculus in importance? College Math. J. 15 (5), 371-382.
[Rob84] Roberts, F.S. (1984). Applied combinatorics. Englewood Cliffs,NJ: Prentice Hall.
[Ros03] Rosen, K. (2003). Discrete mathematics and its applications. New York, NY: McGraw-Hill.
[Shi94] Shin, S.-J. (1994). The logical status of diagrams. Cambridge: Cambridge University Press.
[SieK98] Mathematics education as a research domain: a search for identity. Book 1, 2. An ICMI study. Edited by Anna Sierpinska and Jeremy Kilpatrick. (1998). New ICMI Studies Series, 4. Dordrecht: Kluwer Academic Publishers.
[Sta86] Stanley, R.P. (1997). Enumerative combinatorics. Vol. 1. With a foreword by Gian-Carlo Rota. Corrected reprint of the 1986 original. Cambridge Studies in Advanced Mathematics, 49. Cambridge: Cambridge University Press.
[Tuc95] Tucker, A. (1995). Applied combinatorics. 3rd edition. New York: John Wiley & Sons, Inc.
[Vil69] Vilenkin N.Ja. (1969). Комбинаторика. (Russian) [Combinatorics] Moscow: Izdat. “Nauka". Vilenkin N.Ya. Combinatorics. Translated from the Russian by A. Shenitzer and S. Shenitzer (1971). New York-London: Academic Press.
[VilS74] Vilenkin, N.Ja., Šre□der Ju.A. (1974). The concepts of mathematics and the objects of science. (Russian) Voprosy Filos., no. 2, 16-26.
[Wal03] Wallis, W.D. (2003). A beginner's guide to discrete mathematics. Boston, MA: Birkhauser.
[WheB05] Wheeler, E., Brawner, J. (2005). Discrete mathematics for teachers. New York, NY: Houghton Mifflin.

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