Adult learners' criteria for explanations

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Abstract
This article explores adult learners' preferences for explanations of mathematical statements in terms of kinds of reasoning and formats of presentation. Based on data from questionnaires and interviews it is concluded that *familiarity* and *clarity* influenced students' preferences more than the format or reasoning used. A contrast between the factors influencing students' choices and those of instructors is also reported. Implications for teaching and research are drawn from the study.

Kurzreferat
In dem vorliegenden Artikel wird untersucht, welche Arten von Erklärungen mathematischer Aussagen erwachsene Lernende bevorzugen, und zwar in Hinsicht auf die Art logischen Denkens und die Form der Darstellung. Basierend auf empirischen Daten aus Fragebögen und Interviews wird geschlossen, dass Lernende in ihren Präferenzen mehr durch Vertrautheit und Klarheit der Argumente beeinflusst werden als durch die Form oder logische Schlussweise der Argumente. Auch werden Unterschiede dargestellt zwischen den Faktoren, die Lernende und Lehrpersonen in ihrer Wahl beeinflussen. Implikationen der Studie für Unterricht und Forschung werden aufgezeigt.

ZDM Classification: B68, C28, C78, E38, E58

Research into students' reasoning and acceptance of arguments has considered the influence of the kind of reasoning employed (e.g., inductive versus deductive) as well as the format of presentation used (e.g., symbolic versus visual). While it seems clear that students do prefer some arguments over others, their preferences cannot be entirely explained in terms of kinds of reasoning or formats of presentation. This article explores adult learners' preferences for explanations of mathematical statements in terms of kinds of reasoning and formats of presentation, but also presents data from interviews that suggest other factors influencing adult learners' preferences.

Related Research

Martin and Harel (1989) present the results of a study in which undergraduate students enrolled in mathematics courses for prospective elementary school teachers were asked to rate "proofs" of two statements, one familiar and one new to them. The proofs are described as being either inductive or deductive. There were four types of inductive arguments: two small numerical examples, one large numerical example, an example contrasted with a non-example, and a pattern of twelve related examples. There were three types of deductive arguments: a correct general symbolic proof, an incorrect symbolic "ritualistic" argument, and a "particular proof" in which the structure of the general symbolic proof was followed, but specific numbers were used instead of variables. This type of argument, for which Balacheff (1987) uses the term "generic example", is of special interest to us.

Generic examples illustrate the distinction between the kind of reasoning and the format of presentation of an argument. A generic example can be read in two ways, with different kinds of reasoning being used. If the numbers being used are seen as specific then a generic example is only an example: the reasoning is inductive. If the numbers are seen as standing for any numbers, then the example is truly generic, and the reasoning is deductive. In both cases the format of presentation is the same, only the kind of reasoning used to interpret it differs. Research that attempts to analyse students preferences for some arguments over others in terms of kinds of reasoning must deal with the difficulty that the kind of reasoning an argument suggests to a student is not always clear, as in the case of generic examples.

Martin and Harel found that most students rated inductive arguments highly. The "pattern" argument was rated significantly higher than the other inductive arguments. It included twelve examples compared to one or two examples in the other inductive arguments, but an argument of the pattern type was not included for the unfamiliar statement, so this result can only be said to suggest the plausible conclusion that multiple examples are more convincing than one or two examples. Most students also rated the deductive proofs highly, except in the case of the incorrect symbolic argument for the unfamiliar statement. While Martin and Harel do not make this point, this outcome could be interpreted to suggest that students are more careful in their analysis of a proof of a statement they do not already know to be true. Many students (46%) simultaneously rated both deductive and inductive arguments highly, showing that acceptance of one kind of reasoning does not preclude acceptance of another kind of reasoning. This leaves open, however, the question of what kind of reasoning students would prefer if they had to choose.

Hoyles (1996, 1997) engaged in a comprehensive study of students' views of proof, including what characteristics of a proof are valued by the students and what characteristics of a proof the students perceive as valued by their teachers. She surveyed 15 year old, high achieving students in the UK and found that the students were influenced by the format of the proof; they rated proofs with a formal presentation as likely to receive high marks from their teachers, regardless of whether the proof was correct or not.

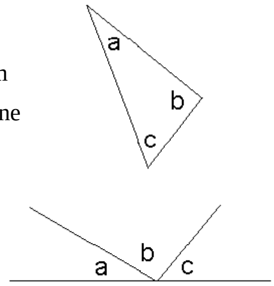
Hoyles' analysis categorised the formats of proofs as empirical, enactive, visual, narrative, symbolic and formal. Using an example to show that a statement is true is characteristic of an empirical proof. In an enactive proof, the student engages in or imagines an activity to show the statement is true. A visual proof uses a representation to show that a statement is true. A narrative proof is expressed in natural language in paragraph format. A symbolic proof makes use of mathematical notation. A formal proof is a chain of statements, each of which is either an axiom or derived from previous statements, usually expressed in two columns, one of statements and the other of justifications for those statements. These categories are not necessarily exclusive.

Reid (1995) investigated the kinds of reasoning

Amanda:

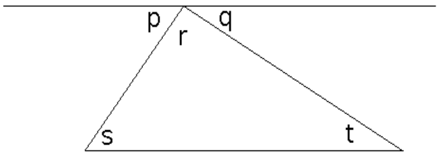
I tore the angles up and put them together. It came to a straight line which is 180°.

I tried for an equilateral and an isosceles as well and the same thing happened.



Cynthia:

I drew a line parallel to the base of the triangle



<u>Statements</u>	<u>Reasons</u>
p = s	Alternate angles between two parallel lines are equal
q = t	Alternate angles between two parallel lines are equal
p + q + r = 180°	Angles on a straight line

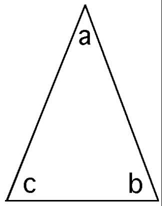
therefore, s + t + r = 180°

Barry:

I drew an isosceles triangle, with c equal to 65°.

<u>Statements</u>	<u>Reasons</u>
a = 180° – 2c	Base angles in isosceles triangle equal
a = 50°	180° – 130°
b = 65°	180° – (a + c)
c = b	Base angles in isosceles triangle equal

therefore, a + b + c = 180°



Dylan:

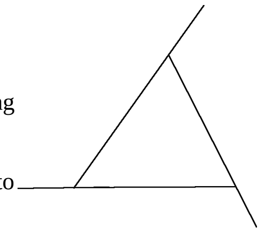
I measured the angles of all sorts of triangles accurately and made a table.

They all added up to 180°

a	b	c	Total
110	34	36	180
95	43	42	180
35	72	73	180
10	27	143	180

Ewan:

If you walk all the way around the edge of the triangle, you end up facing the way you begun. You must have turned a total of 360°. You can see that each exterior angle when added to the interior angle must give 180° because they make a straight line.



This makes a total of 540°. 540° – 360° = 180°

Figures 1a-e:
Responses to Question 1: “Why does the sum of the interior angles of any triangle equal 180° ?”

undergraduate and secondary school students used in problem solving. He grouped reasoning into three broad categories: Deductive, analogical and inductive. Deductive reasoning proceeds from agreed upon premises to conclusions, using logical arguments. Reid makes a distinction between deductive reasoning that is *unformulated* and deductive reasoning that is *formulated*. A person engaged in unformulated reasoning is not aware of their own reasoning, hence their record of their reasoning (in speech or writing) will include gaps. Reasoning by analogy depends on similarities between two cases. In inductive reasoning the conclusion is based on several specific cases. It should be remembered that the borderline between inductive and deductive arguments is not perfectly clear. Recall that arguments based on generic examples (Balacheff 1987) refer only to specific cases, but those cases are chosen arbitrarily, which allows the arguments to be interpreted as general and deductive.

The type of argument Martin and Harel (1989) describe as "ritualistic" is based on what Reid (1993) calls "formulaic" reasoning, which superficially resembles deductive reasoning, but which is characterised by a lack of any underlying logical structure. Formulaic reasoning occurs when someone tries to produce a proof to satisfy a teacher or some other authority figure, without understanding what kind of reasoning is involved. Instead they focus on the outward appearance of proofs they have seen. The finding by Hoyles (1997) that students thought that their teachers would prefer formal proofs, even if those proofs were flawed, suggests that the students might make use of formulaic reasoning in writing proofs for their teachers.

It should be clear that types of reasoning and formats of presentation are not completely independent. Not every format is possible for every type of reasoning and in some cases the type of reasoning implies a certain format. For example, the formal format is only possible for deductive or formulaic reasoning.

Reid (1995) found that the reasoning used by students was related to the need they had for reasoning. Students used inductive and deductive reasoning to verify statements they were not sure of. They used reasoning by analogy and deductive reasoning to explain statements they had previously verified. They used all three kinds of reasoning to explore, to make new statements about a situation.

Both Harel and Martin, and Hoyles, use the word "proof" for the arguments they offered students. They mean proof in the everyday sense of an argument which convinces, not in the sense of mathematical proof. Reid, however, uses "proof" only to refer to arguments based on deductive reasoning. In the following we will use the word "explanation" to refer to the arguments offered to the students, both to avoid confusion about the word "proof," and because the students were responding to questions of the form "Why?" which suggests an explanatory answer. Of course, we can not know if the students actually experienced the answers they chose as explanatory, only that they chose them in contexts that we would expect to give rise to a need for explanation.

Andy:
Let one odd number be (2n+1) and the another odd number (2m+1), then (2n+1) +(2m+1) = 2(n+m) +2

Bill:
::::::::. + ::::::. =
:::::::::::::::.. = :::::::::::::::

Cora:

13 + 45 = 58	7 + 9 = 16
113 + 335 = 448	1077 + 517 = 1594

Drake:
An odd number plus an odd number equals and even number because of the same principle which says a negative number times a negative number is a positive.

Figures 2a-d: Responses to Question 2: “Why is the sum of two odd numbers even?”

The Study

Participants

82 students attending the College of the North Atlantic, in Happy Valley-Goose Bay, Labrador, Canada participated. These students were enrolled in a range of programs including technical programs, Early Childhood Education, and Adult Basic Education (ABE). The mathematics courses required in these programs (if any) did not include any emphasis on proof. This choice of participants allowed the comparison of our results with those of Harel and Martin, Hoyles, and Reid, but with a contrasting sample, as their studies involved high achieving secondary schools students and university students.

Methods

The study employed a mixture of methods. Naturalistic observations in classrooms, qualitative analysis of interviews and quantitative analysis of data from questionnaires were combined to provide a rich picture of the features of explanations that were valued by the participants. Participant-observation occurred on a daily basis over a five week period in the ABE mathematics class in which 12 of the participants were enrolled. Observing the kinds of explanation offered by students and instructors in the classroom setting provided data that helped to determine what kinds of explanations students prefer, and whether the same kinds of explanations were preferred by their instructors. Results from the observational phase of the study are not included here; see Roberts (1999).

All 82 participants completed a questionnaire, based on that of Hoyles (1996, 1997). The questionnaire is structured around three mathematical questions, one geometric and two arithmetic/algebraic:

- Why does the sum of the interior angles of any triangle equal 180° ?
- Why is the sum of two odd numbers even?
- Why do perfect trinomial squares have the form $x^2 + 2bx + b^2$?

The content of questions was chosen in such a way as to be appropriate for the mathematical backgrounds of the students, based on consideration of the school and college mathematics curricula, and pilot testing with other students not involved in the study. In each case a question was asked and four or five possible explanations were listed (see Figures 1-3).

The explanations can be classified both by format and by the kind of reasoning used (see Tables 1-3). The formats are the same as were used by Hoyles: enactive, visual, formal, empirical, narrative and symbolic. The kinds of reasoning include deductive (both formulated and unformulated), inductive (both single and multiple examples) and analogical types, and also one generic example. Not all types and formats were included in the explanations given for each question, both because the number of explanations for the students to evaluate would have been excessive, and because plausible explanations in every combination of format and kind of reasoning are not possible.

For each question the students were asked to select the response they preferred. The questionnaire was administered in their regular classrooms with their instructor present, and the students were allowed as much time to complete it as they wished.

Responses to the first question were those used by Hoyles (1996, 1997). Responses to the second question were adapted from Hoyles, except for Drake's response which was based on a response to this question given by a student interviewed by Reid (1995). Responses to the third question were generated through interviews with students not participating in the research.

After the questionnaires were collected, person to person interviews were conducted with eight students in the Adult Basic Education class where the observations had occurred, as well as with two mathematics instructors who taught in that program. The students were selected

from among those that volunteered, based on the variety of responses to explanations they displayed in their class. The interviews allowed for a deeper understanding of why the students preferred one explanation over another. In addition, comparison with the instructor interviews allowed an opportunity to see whether students mirror their teachers' explanations outside the classroom setting. The questions from the questionnaire were discussed as well as other questions that will not be considered here (for more details see Roberts 1999). The responses to the survey and interviews are shown in Tables 1-3. Our analysis of the survey results here is based in part on a simple classification of the number of students who chose an explanation by comparison with the number one would expect based on random selection. So for questions 1 and 3, explanations that were chosen by about 20% of the participants are considered to be "neutral" in terms of the students' preferences. Those that were chosen by much less or much more than 20% were taken to indicate an overall dislike or preference for those explanations. For question 2 explanations that were chosen by about 25% of the participants are considered to be "neutral" in terms of the students' preferences. Those that were chosen by much less or much more that 25% were taken to indicate an overall dislike or preference for those explanations. For more statistical analysis see Roberts (1999).

Julia:

If you multiply two same binomials such as (x+b)(x+b) using the FOIL method, then the first two terms of the two binomials will multiply to $x * x = x^2$; the two outside terms will be x times b = xb ; the two inside terms will be b times x = bx ; and the last two terms of each of the two binomials multiplied tog ether will be b^2. Combining like terms, the xb and the bx will equal 2bx. Thus, (x+b)(x+b) will always multiply into the form $x^2 + 2bx + b^2$.

Jody:

$(x+2)(x+2) = x^2 +2x + 2x +4 = x^2 + 4x + 4$
$(x+3)(x+3) = x^2 +3x + 3x +9 = x^2 + 6x +9$
$(x – 5)(x – 5) = x^2 – 5x – 5x + 25 = x^2 – 10x + 25$
$(3x+4)(3x+4) = 9x^2 + 12x + 12x +16 = 9x^2 + 24x + 16$
$(2x – 3)(2x – 3) = 4x^2 – 6x – 6x + 9 = 4x^2 – 12x + 9$

Therefore perfect squares will always have the form $x^2 + 2bx + b^2$.

Lisa:

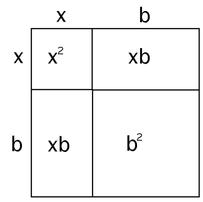
If you take the number 144,
then 144 is equal to $10^2 + 2(10)(2) + 2^2$.
Likewise, $169 = 13^2$ is $10^2 + 2(10)(3) + 3^2$.
Finally, $81 = 9^2$ is $8^2 + 2(8)(1) + 1^2$

Dena:

Using the distributive law:
(x+b)(x+b)
$(x+b)x = x^2 + bx$
$(x+b)b = xb + b^2$
$(x+b)(x+b) = x^2 + bx + xb + b^2$
(The “2” comes because “xb” occurs in both distributions).

Cheryl:

(x+b) represents a line segments of length (x+b).



Figures 3a-e: Responses to Question 3: “Why do perfect trinomial squares have the form $x^2 + 2bx + b^2$” ?

Survey results and discussion

The presentation and discussion of the results from the survey will be structured by the following two questions:

- Did the students surveyed choose explanations with particular formats?
- Did the students surveyed choose explanations based on particular kinds of reasoning?

Did the students surveyed choose explanations with particular formats?

Empirical and *narrative* formats seem to have been preferred by the students surveyed, although other factors must also have influenced their choices. The most popular choices overall, Cora's (43%), Drake's (40%), and Julia's (39%) were either empirical (Drake's) or narrative (Cora's and Julia's). On the other hand, one narrative explanation (Ewan's) and one empirical explanation (Lisa's) were among the least popular. Ewan's narrative enactive explanation for the triangle sum was chosen by only 6 participants (7%), and Lisa's empirical explanation for the trinomial form was chosen by only 8 (10%) of the participants. These explanations have some unusual features, however. Ewan's combines an indirect deductive argument with a geometric approach (based on LOGO programming) that was unfamiliar to the participants. Lisa's provides numerical examples for an algebraic proposition, which requires an understanding of the way the algebra represents number, and it was not the only empirical explanation available, the other possibility being Jody's, which uses algebraic examples and was chosen by 17 participants (21%).

Visual formats seem to have been disliked by the students surveyed. Bill's explanation of the odd number sum using dots and Cheryl's explanation of the trinomial form using a diagram were among the least popular. Only 3 participants (4%) chose Bill's and only 9 (11%) chose Cheryl's. The other visual explanation on the survey was Amanda's which was chosen by 16 (20%) of the participants, but note that it could also be classified as enactive (see below).

The participants' preference for narrative and empirical formats is consistent with Hoyles' findings (Healy & Hoyles 1998), but their dislike of visual formats is in contrast to Hoyles' results, in which visual formats were as popular as empirical formats. Two of the three visual explanations were identical to items used in Hoyles' study. This suggests that, while there are similarities, there are also important differences between the criteria participants in this study (low achieving adults in Canada) use to select explanations and the criteria used by participants in Hoyles' study (high achieving 15 year olds in the UK). These differences supports Hoyles' (1997, p.8) contention that curricula can have impacts on students' thinking different from those that are intended by curriculum designers.

Enactive, *formal* and *symbolic* explanations seem also to have been disliked, but this result is due to the influence of two explanations with other features that could have influenced the students' choices. We have already discussed the small number of participants who chose Ewan's enactive explanation and some possible reasons for their dislike of it. The only other enactive explanation on the survey was Amanda's which was chosen by 16 (20%) of the participants. Discounting Ewan's explanation would lead to the conclusion that the participants' attitude toward enactive explanation was neutral, but such a conclusion must be considered highly tentative as it is based on only one case. Most of the formal and/or symbolic explanations (Barry's, Cynthia's, Jody's, Dena's) received neutral responses. One, however, (Andy's) was chosen by only 11 participants (13%) in spite of being one of only four choices. It is not immediately obvious what special features of Andy's explanation might have led to this result.

Did the students surveyed choose explanations based on particular kinds of reasoning?

The students seem to have preferred explanations based on induction from multiple examples, or reasoning by analogy.

There were five explanations based on *multiple examples* on the survey. One of these, Cora's, was among

Table 1: Classification of responses and results for Question 1: “Why does the sum of the interior angles of any triangle equal 180° ?”

Response	Form	Reasoning	Survey	Interview	Inst #1	Inst. #2
Amanda:	Enactive, visual	Inductive: multiple examples	16 (20%)	4 (S1, S2, S3, S4) (50%)	–	–
Barry:	Formal	Formulaic	21 (26%)	1 (S5) (13%)		
Cynthia:	Formal	Deductive, formulated	18 (22%)	2 (S6, S7) (25%)	*	*
Dylan:	Empirical	Inductive: Multiple example	21 (26%)	1 (S8) (13%)	+	
Ewan:	Narrative, enactive	Deductive, formulated	6 (7%)	0	–	–

“S*n*” refers to Student #*n*;
“*” indicates an instructor’s first choice;
“+” indicates other choices acceptable to an instructor;
“–” indicates a choice explicitly rejected by an instructor.

the most popular explanations, being chosen by 35 (43%, from among four choices) of the participants. Another, Dylan's, was chosen by 21 participants (26%, from among five choices, slightly more than the 20% one would expect on the basis of chance). Two others received neutral responses (Amanda's, 20% and Jody's, 21%). One, Lisa's, was disliked overall, being chosen by only 8 (10%) or the participants. As was noted above, the possibility of choosing Jody's explanation instead might account for the small number of participants who chose Lisa's.

The popularity of explanations based on multiple examples contrasts with Reid's (1995) finding that inductive reasoning was not used by students when attempting to explain in problem solving situations. It may be that the participants in this study did not expect the explanation they selected to actually be *explanatory* for them, either because questions of the form "Why ...?" do not call forth a need for explanation in them, or because they have no expectation that explanations presented as such in mathematical contexts should actually be explanatory. The questions of what contexts create a need for explanation in students and what expectations they have of "explanations" presented to them by teachers requires further study.

Only one explanation based on reasoning by *analogy* was included on the survey (Drake's), but it was very popular, being chosen by 33 (40% from among four choices) of the participants. As this is a single case we will not try to generalise much from it, but the fact that reasoning by analogy (even falsely as in this case) was preferred by so many of the participants suggests that further research is needed on students' understanding of the use of analogy in mathematics.

The use of *deductive reasoning* in an explanation seems not to have been an significant factor in the students' choices. Deductive explanations were among the most popular (Julia's) and the least popular (Ewan's, Andy's, Cheryl's). Cynthia's correct deductive explanation for Question 1 was slightly less popular than Barry's formulaic explanation, suggesting that while some of the participants may have been sensitive to the logical structure of the explanations, just as many or more were operating on the basis of surface features.

The only explanation based on inductive reasoning from a *single example*, Bill's, could also have been interpreted as deductive reasoning based on a *generic example*. It was the least popular explanation of all, being chosen by only 3 participants, and also was rejected by both instructors. This suggests that most students and the instructors did not see Bill's explanation as a generic example. Instead they saw the explanation as specific to the number of dots shown, which meant that in addition to there being only one example, it involved painstaking counting of dots to understand. Healy and Hoyles (1998) report on one student, Susie, who preferred a proof based on multiple examples (like Cora's) to a visual proof like Bill's for this reason. This exposes a danger of explanations based on generic examples; if the example is not seen as generic then the explanation is very weak, becoming an induction from a single case.

Summary

From the survey data it is clear that there are other factors involved in the participants' choices of explanations other than the format of the explanation or the reasoning the explanation is based on. While empirical and narrative explanations were often chosen, other explanations with these formats were unpopular. Similarly, while explanations based on analogy or multiple examples were among the most popular, other explanations based on multiple examples were among the least popular, and the reception of deductive explanations was even more mixed. We will now turn to the interview data which provides additional insight into the students' criteria for choosing explanations.

Interview results and discussion

The presentation and discussion of the results from the interviews will be presented in four parts: results related to participants' preferences for particular formats of explanations, results related to participants' preferences for particular types of reasoning in explanations, results related to other factors influencing students' preferences, and results related to the need or purpose the explanation may have had for the participants. In this section key words in the participants' comments are indicated in *italic*. This does not indicate that the participants placed any emphasis on those words themselves when speaking.

Students' preferences for particular formats of explanations

As can be seen in Tables 1-3 the adult learners who were interviewed are not a representative sample of the group

Table 2: Classification of responses and results for Question 2: “Why is the sum of two odd numbers even?”

Response	Form	Reasoning	Survey	Interview	Inst. #1	Inst #2
Andy:	Formal, Symbolic	Deductive: formulated	11 (13%)	1 (S5) (13%)	*	*
Bill:	Visual	Inductive: single example or Generic example	3 (4%)	1 (S7) (13%)	–	–
Cora:	Empirical	Inductive: multiple examples	35 (43%)	3 (S2, S3, S4) (38%)	–	
Drake:	Narrative	Analogy	33 (40%)	3 (S1, S6, S8) (38%)	–	

“S*n*” refers to Student #*n*; “*” indicates an instructor’s first choice;
“+” indicates other choices acceptable to an instructor; “–” indicates a choice explicitly rejected by an instructor.

surveyed. They did not choose particular explanations as often as one would expect based on the survey responses (for example, Amanda's explanation was relatively unpopular with the survey group, but the most popular with the interview group). The two groups were similar, however, in preferring empirical and narrative formats (Cora and Drake's for Question 2, and Lisa, Julia, and Jody's for Question 3), and in preferring explanations based on multiple examples (Students #1, #2, #3, #4 and #8 all chose an explanation based on multiple examples for two of the three questions, although not always the same two). The reasons they gave for their choices, and the reasons the instructors gave, provide additional insights into what it is about such formats that influenced their choices.

Enactive Format

Two students used phrases that suggest that the enactive format of Amanda's explanation for Question 1 may have influenced their choosing it: "She *cut the angles and made a straight line.* Like a circle is 360° but, ah, but if you *cut it in half,* then you get a straight line 180°" (Student #1), "Amanda's answer because she *tore up* the angles" (Student #2). No other participants made comments related to the enactive format of this or any other explanation and these two students also gave additional reasons for their choice. This suggests that while the enactive format of an explanation may influence students choosing that explanation, other factors are as important or more important.

Narrative format

Narrative format was not mentioned as a factor in choosing an explanation by most of those interviewed. Only Student #6 expressed a preference for words over numbers as a reason to choose Drake's explanation for Question 2: "because it's *written out* not using numbers." On the other hand, Student #2 rejected the same explanation because of its narrative format: "This one here [Cora's] is a lot easier to see. The numbers make it easier. ... Drake's is *like a word problem.*" This result suggests that there may be individual differences in the influence that a narrative format has on students' preference for or dislike of an explanation.

Visual format

Student #7 chose Bill's explanation because "I'm better with *visuals* sometimes. It depends on what I'm doing. If I can *see* things." Student #2 rejected Bill's explanation because of its visual format: "The numbers make it easier. You don't have to count the dots." The instructors also rejected Bill's visual explanation: "hated the dots" (Instructor #2), "I find the dots confusing, but a *visual learner* might like it" (Instructor #1). Student #7 and Instructor #1 appear to believe that a preference for visual formats is a trait some people have and others do not. Both instructors commented on the presumed appeal for such people of Cheryl's visual explanation for Question 3. Instructor #1 commented "A *visual learner* would probably like that way [Cheryl's] sort of better, but it is not as good as the other [Julia's]." Instructor #2 preferred Cheryl's explanation over all and said, "Cheryl's shows through the use of an application the process and should be easier to see for students as *area*. Dena's is not bad. She shows where the two comes from, but Cheryl's provides a *diagram*." Unfortunately, Student #7, who seems to have been the only "visual learner" interviewed, was not asked about Cheryl's explanation for reasons of time.

Empirical format: Use of Examples

Of the formats identified for study, the empirical format, making use of examples, was mentioned most often by the participants as influencing their choices. For the students it was given as a reason for choosing an explanation. For the instructors it was a reason to reject an explanation. Three students chose Cora's explanation for Question 2 and gave the use of examples in the explanation as a reason for their choice. Student # 2 said "The *numbers* make it easier," and Student #3 mentioned the importance of multiple examples, "Not only does she give *more than one example* ... it's clear." Student #4 may have been trying to contrast examples with a narrative approach: "Given bunch of *examples*, right, which I think would be easier to do than just trying to explain something," but all we can be sure of from this comment is that examples are seen favourably.

While students chose explanations because they used examples, the instructors rejected explanations for the same reason. For Question 1, Instructor #2 rejected Amanda's explanation for this reason: "Amanda's not proving for all cases; she shows it for *one instance only*" and Instructor #1 rejected Dylan's for its empirical approach: "Dylan's is *trial and error* where he measured angles." Instructor #1 gave the same reasons and referred back to Dylan's explanation when rejecting Cora's explanation for Question 2, "Cora's is the same as Dylan's but it is *trial and error.*" Both instructors rejected Lisa and Jody's explanations for Question 3 because of the use of examples: "Not a whole lot of *cases* to support"

Table 3: Classification of responses and results for Question 3: "Why do perfect trinomial squares have the form $x^2 + 2bx + b^2$" ?

Response	Form	Reasoning	Survey	Interview	Inst. #1	Inst. #2
Lisa:	Empirical	Inductive, multiple example	8 (10%)	1 (S1) (20%)	–	
Julia:	Narrative	Deductive, formulated	32 (39%)	3 (S2, S3, S4) (60%)	*	
Jody:	Empirical, symbolic	Inductive, multiple example.	17 (21%)	1 (S8) (20%)	–	–
Dena:	Formal, symbolic	Deductive, formulated	16 (20%)	0	+	+
Cheryl:	Visual	Deductive, unformulated	9 (11%)	0	+	*

"S*n*" refers to Student #*n*; "*" indicates an instructor's first choice;
"+" indicates other choices acceptable to an instructor; "–" indicates a choice explicitly rejected by an instructor.

(Instructor #1 commenting on Lisa's), "Jody's is *trial and error*" (Instructor #1), "Jody's doesn't prove anything. He only shows it for these *particular cases*, not for all cases" (Instructor #2).

The students' and instructors' reactions to empirical explanations is a sharp contrast, but not a surprising one. This difference in attitude is consistent with the results reported by Hoyles (1997), Fischbein (1982), Martin and Harel (1989), and many other researchers. Part of this difference may be related to the need or purpose the explanation had for the participants; the instructors' comments suggest that they rejected examples as not being enough to verify the statements, while the students' comments suggest that other needs might have been addressed by the examples. This point will be considered in more detail below.

Formal format: Statements and Reasons

Student #5 mentioned formal aspects of the explanations chosen for Questions 1 and 2: "Barry's because first he showed what he did - why in *statement and reasons* and then a *formula* at the end," "Andy's - he is saying what one number is and another in *formula* and then he went on to say why he did it - then the *formula*." Student #5's mention of "statements and reasons" was echoed by Student #6, who chose a different explanation for Question 1 for the same reason: "Cynthia's answer because she is using *statements and reasons*." Student #2 also mentioned "statements and reasons" as an influence, but a negative one: "Amanda's answer because ... *It doesn't involve statements and reasons*." The instructors chose Cynthia's explanation (Student #6's choice) for Question 1 and Andy's explanation (Students #5's choice) for Question 2. Their reasons also included mention of formal aspects of the explanations, although they used phrases that suggest they may have been thinking of the underlying deductive structure of the explanations rather than its surface form: "Cynthia, she used *postulates and theorems, proven statements*," "Andy's makes more sense because it's *algebraically laid out*" (Instructor #1), "Andy's shows some thought to *mathematical aspects and factoring*" (Instructor #2).

Different ways

One student mentioned an aspect of the format of explanations that we had not anticipated, the use of two different ways to arrive at the result. Student #1 chose Lisa's explanation for Question 3 because it showed two ways of arriving at the squares; "because she shows *two different ways*." While this reason seems to us to be a weak one in this case, an explanation that presents two independent ways to arrive at the statement could be expected to be more convincing than one that only uses one approach. This possibility is worth additional study.

Students' preferences for particular types of reasoning in explanations

Deductive reasoning

The interview participants were similar to the students surveyed in preferring a few explanations based on deductive reasoning (especially Julia's for Question 3). Very few of them, however, made comments suggesting that the reasoning used in the explanation influenced their choices. Student #7 was the only exception to this pattern. For Question #1 this student commented "Cynthia's - she justifies using *logical arguments* - straight line and equivalent angles ... Kind of like Ewan's too because that one's using *reasoning* - all of them would be the same angle right - not going to change degrees." Not only did Student #7 refer to logic and reasoning, but there is also a recognition that Ewan's explanation is logical, a fact that eluded the instructors. As noted above, the instructors also made comments that suggest that the use of deductive reasoning influenced their choices (e.g. "postulates and theorems, proven statements"), but whether the reasoning or the form of its presentation was most significant is unclear.

Generic examples

Student #7 was also unusual in preferring Bill's visual explanation for the evenness of the sum of two odd numbers. The reason given for this choice is unclear, but suggestive:

> I'm better with visuals sometimes. It depends on what I am doing. If I can see things. Not that I would dispute that [Cora's] or that [Drake's]. I know there is something better than that - just the same. I don't know if it is one of those tease testers - some kind of book - some explanation. You know like all kinds of game type of things. I know there was a whole bunch of stuff similar to that, but there's some many different things - so many numbers. Andy's is alright, but like right now I can't think odd numbers. I'm trying to think of the algebra stuff - the numbers - the equations. It is logical to see where it worked.

The two explanations that are not dismissed entirely, but still seen as deficient in some way (Cora's and Drake's) were the two most popular in the survey. Student #7's choice, Bill's explanation, was the least popular. It seems likely to us that Student #7 recognised that Bill's explanation can be read as a generic example, and was much more influenced by the logical structure of the explanations than by their format. The reference to puzzle books suggests that this is a person who has spent some amount of time engaged in reasoning tasks recreationally. The link between such puzzle solving and attitudes towards mathematical reasoning deserves further research.

Inductive reasoning

Most of the comments reported above concerning the use of examples, indicating a preference for an empirical format, could apply also to inductive reasoning. To reiterate, most of the students mentioned the use of examples as a positive influence in their choices of explanations and the instructors mentioned examples as a negative influence.

Reasoning by analogy

While three of the students interviewed chose Drake's analogy for Question 2, only Student #1 made a comment that indicates an understanding of the structure of the analogy: "Drake's answer because it goes along with a negative times a negative gives you a positive. So an odd plus an odd is even. Yeah, so an odd number is like a negative number and an even number is like a positive." Student #8 gave no reason for choosing Drake's answer, and Student #6 seems to have chosen it because of its narrative format rather than its analogical structure. This suggests that we be cautious in ascribing the popularity of this explanation on the survey to a preference for analogical reasoning.

Other factors influencing students' preferences

Hanna (1983) lists a number of criteria she claims mathematics use to determine whether a proof is acceptable. Two of these are closely related to some criteria the students interviewed mentioned for accepting explanations:

> 1. *They understand* the theorem, *the concepts embodied in it*, its logical antecedents, and its implications. There is nothing to suggest that it is not true. ...
> 5. There is a convincing argument for it (rigorous or otherwise), *of a type they have encountered before.*
>
> (p. 70, our italics)

The student interviews reveal that, like mathematicians, they preferred explanations that are clear and straightforward and that make use of concepts with which the students were familiar. In other words, they preferred explanations in which they understood the concepts used. We have grouped the data related to this observation under the two headings, *familiarity* and *clarity*.

Familiarity

Four students indicated that familiarity with the terms or methods used in the explanation influenced their choice. Familiarity was explicitly mentioned by Student #2 as a possible reason for choosing Julia's explanation for Question 3 "because she is using the FOIL method — maybe because it is *familiar*." (FOIL is used in North America as a mnemonic for an algorithm for multiplying binomials: "Multiply together the First terms, the Outside terms, the Inside terms and the Last terms.") Two other students mentioned the FOIL method in their reasons for choosing Julia's explanation: "She is using the FOIL method of multiplication" (Student #3) and "because she is using the FOIL method" (Student #4) and these students may also have been influenced by their familiarity with this technique. Student #8 preference for Jody's explanation "because she uses the FOIL method" seems also to be based on familiarity. For Student #1 familiarity with the measure of the straight angle being 180° was a factor in choosing Amanda's explanation for Question 1: "She cut the angles and made a straight line. Like a circle is 360° but, ah, if you cut it in half, then you get *a straight line, 180°*."

Lack of familiarity with some aspect of an explanation was an important factor in rejecting explanations. This is especially clear in the instructors' comments on Ewan's explanation of the triangle sum. Instructor #1's first reaction was "I don't understand it at all." After studying it she said it "explains it in a roundabout manner" and that it was "No better than the others." Instructor #2 said "Ewan's uses no mathematical concepts." Given that the reasoning in Ewan's explanation is not especially roundabout, and that it does make use of some important mathematical concepts, it would seem that the instructors' rejection of it reflects their unfamiliarity with the approach to geometry used in the explanation.

Student #3 mentioned on two occasions how a specific lack of familiarity with part of an explanation could lead to rejection of it. Question 1: "I found some of the other's answers - they're big equations and stuff like alternate angles between two parallel lines are equal. *How do you know this? You don't*." Question 2: "If you read Drake's, he goes into the same principle which says a negative number times a negative number is positive. Well, *if you didn't know that* or *if you weren't versed in algebra*, you wouldn't know that; whereas this [Cora's] is basic addition." Given Student #3's point here, we can interpret Students #1 and #8's reasons for choosing Drake's answer as indicating that familiarity with multiplication of negative numbers was an influence for them: "Drake's answer because it goes along with a negative times a negative gives you a positive" (Student #1). "Drake's because negative times a negative number is positive" (Student #8).

Clarity

Clarity, ease or straightforwardness of an explanation, was given many times as a reason for choosing particular explanations. Three students gave such reasons for choosing Amanda's explanation for Question 1. They commented "It is *easy*. It doesn't involve statements and reasons" (Student #2), "It's *clear* ... It is very *clean*. It's not cluttered .... This says everything. It answers the question fully without going into great big long details." (Student #3) and "It's *obvious*." (Student #4). Student #8 also gave straightforwardness as a reason for choosing Dylan's answer for the same question: "Dylan's — pretty *straightforward* — you take any angle and make the measure equal 180°."

Clarity and ease were also given as reasons for choosing Cora's explanation for Question 2: "This one here is a lot *easier* to see. The numbers make it *easier*." (Student #2), "Not only does she give more than one example ... it's clear" (Student #3) and "Given bunch of examples, right, which I think would be *easier* to do than just trying explain to something" (Student #4).

Student #4 chose Julia's explanation for Question 3 "because she is using the FOIL method, right, and in my opinion, it is easier for students to understand." The simplicity of Julia's explanation also impressed Student #3: "It's *simple — simplicity* itself. This here is very *simple straightforward* instructions. If you had different numbers and you were going to do this, you could almost follow like a recipe which she has here and learn and teach yourself how to do something like that."

It is perhaps worth noting that three students (#2, #3 and #4) were responsible for most of the comments concerning ease, simplicity, clarity, or straightforwardness of explanations. In only one case did an instructor mention clarity, and then it was as a critique: "Dena' is a little *unclear*, because she starts with one equation, then splits it in two and then makes it one or reverts back to one. I can see what she's done, but someone else might be confused by that."

Needs or purposes for explanations

We have referred to the answers offered to the participants as "explanations" but as we noted above, it is possible that their explanatory power was not essential to their being chosen. The answers may have satisfied (or failed to satisfy) other needs felt by the participants, and so have been chosen or rejected on other grounds. The comments of the interview participants reveal three needs or purposes for the answers that seem to have influenced choices: explaining how, explaining why and verifying that.

Explaining How

Student #3 chose Julia's answer for Question 3 because it satisfied a need to know *how* to do something: "Julia's answer for this one because it's — she is *explaining what she is doing*. She is using the FOIL method of multiplication. Not only that, *she'll go through every step of the FOIL method each time*. ... if you were just learning how to do this and I saw this — Jody's answer you wouldn't know what to make of it. It would be very difficult to follow ... whereas, this here is very simple, straightforward, *instructions*. If you had different numbers and you were going to do this, you could almost follow like a *recipe* which she has here and learn and teach yourself *how to do something* like that."

Instructor #1 may have had something similar in mind: "They're both similar, except Julia's gives an explanation. In fact, Julia's is probably better. ... Julia *explains step by step what you are doing*."

Explaining Why

Some students seemed to interpret the answers as explanations in the sense we intended. Student #5 chose Andy's answer to Question 2 because : "he is saying what one number is and another in formula and then he went on to say *why* he did it." Student #4 also seemed to want an answer to be explanatory in the sense of helping a person to understand, but used the verb "explain" in a negative sense (referring to Drake's analogy): "examples, ... which I think would be easier to do than just trying to *explain* something. Okay, like a negative times together would give you a positive; whereas, if you were given an example, then I would say students learn better, would *understand* better." This claim that examples can be explanatory contradicts a claim sometimes made in the research literature (e.g., Reid 1995, Hanna 1989). Further research is needed in this area.

Verification

Student #3 contrasted some answers as being explanatory with others that only verify, but chose a non-explanatory answer to Question 2: "Well, while Cora's answer *doesn't explain why* it is, she does show *that it is.* Okay. ... It may not be *explained to a level of understanding*, but it's *taken as a given* by the way she explains it. She says look at it — no matter how many times you do it, it works out that way. Therefore, *it's got to be true*."

Student #2 was also clear that Cora's answer verified the claim sufficiently:

Interviewer: So would you be *convinced* given a few examples that the sum of two odd numbers will always be even?
Student #2: *Sure*!

The instructors, on the other hand, rejected answers that they felt did not verify the statements. This is interesting not only because of the contrast between the instructors' treatment of examples as verifications and explanations, but also in that the instructors, more than the students, treated the answers as verifications more than explanations.

The instructors' reason for rejecting answers based on examples or empirical tests was lack of accuracy, which is much more of an issue for a verification than an explanation. "Tearing up paper is not *accurate*, in my opinion" (Instructor #1 referring to Amanda's answer for Question 1). "Amanda's is not proving for all cases; she shows it for *one instance only* — no *accuracy*" (Instructor #2). "Cora's is the same as Dylan's but it is trial and error and has only *four, not enough to substantiate*. It's only four, *not a large sample size*" (Instructor #1, Question 2). "Not a whole lot of *cases to support*" (Instructor #1, referring to Lisa's answer for Question 3). "He only shows that it is for those *particular cases, but not for all cases*" (Instructor #2, referring to Jody's answer for Question 3). It is interesting to note that it was not the use of examples *per se* that led the instructors to reject answers based on examples. Instead it was the small number of examples given. This suggests that they are evaluating the answers based on their usefulness in verifying the statements, but employing criteria more appropriate to science than mathematics when doing so.

Conclusion

The data as a whole suggests that the reasoning used in an explanation and the format of the explanation are less important to students than clarity and familiarity. There are important differences between students in the criteria they apply, and especially between the students and the instructors interviewed.

The survey data supports a number of claims previously made in the research literature. The participants in this study, like those in Martin and Harel's study, accepted inductive and deductive explanations as well as formulaic explanations. Inductive explanations with multiple examples were more popular than those with only a single example. Like the students in Reid's study, the participants also accepted explanations based on reasoning by analogy. The most popular formats, empirical and narrative, were also popular with the students in Hoyles' study. On the other hand, the visual format was popular with the students in Hoyles' study, but disliked by most of the participants in this study and inductive reasoning was considered to be explanatory, while Reid found that inductive reasoning was not explanatory for the university and secondary school students he studied.

In the interview data there is a contrast between the factors influencing the students' choice of explanations and those of the instructors. The students interviewed preferred explanations because they were clear and straightforward and because they made use of concepts with which the students were familiar. In other words, they preferred explanations in which they understood the concepts used. Their choices seem to have been based only partially, if at all, on the kind of reasoning used in the explanations. The instructors, on the other hand, seemed to take the reasoning more into account. Almost all the explanations accepted by the instructors, and all their preferred explanations, are based on deductive reasoning. This raises an issue for teaching. When providing explanations teachers are likely to base their choices on their own criteria for explanations, but these criteria are not necessarily the same as the students' criteria. In the long run part of learning mathematics must be learning to share the same criteria for explanations, but in the short run that teacher may be faced with the dilemma of choosing a mathematically acceptable explanation that is not an explanation for the students, or choosing an explanation acceptable to the students that is not mathematically acceptable.

It should be clear that there are a number of limitations to this study. The participants were adult students at one institution in one region of Canada. There is no guarantee that students elsewhere would respond to the questions in the same way, although there is no obvious reason for these students to be atypical. The use of some questionnaire items that are comparable to those used by Hoyles allows for the degree of typicality of the responses to be judged. The students' "preferred" explanations on the survey were limited to those offered them in contrast with the others offered them. Limitations of time, space, and creativity in coming up with plausible explanations meant that the range of explanations the students could chose from did not include every format or type of explanation in every context. This study can not be said to have definitively addressed the question of what factors influence students' acceptance of or preference for an explanation. It does however suggest some important factors that have not previously been taken into consideration in such studies, which offers guidance for future research in this area.

This study also raises a number of questions that require further research. The circumstances under which a generic example is understood as a generic example, versus a single example, need clarification. The idea that a set of examples can be considered explanatory should be explored in more depth. Contexts which give rise to a need for explanation in mathematics need to be explored and critical characteristics of those contexts identified. The degree to which students expect statements in mathematics to have explanations, and how that could be measured needs to be investigated further. The link between recreational puzzle solving and mathematical reasoning deserves attention. And further research is needed into the role of reasoning by analogy in mathematics.

References

Balacheff, N. (1987): Processus de preuve et situations de validation. In: Educational Studies in Mathematics. 18, p. 147-176

Fischbein, E. (1982): Intuition and Proof. - In: For the Learning of Mathematics, 3(2), p. 9-18

Hanna, G. (1983): Rigorous proof in mathematics education - Toronto: OISE Press.

Hanna, G. (1989): Proofs that prove and proofs that explain. - In: Proceedings of the Thirteenth International Conference on the Psychology of Mathematics Education. Paris, p. 45-51

Hanna, G. (1990): Some pedagogical aspects of proof. - In: Interchange, 21(1), p. 6-13

Healy, L.; Hoyles, C. (1998): Students Performance in Proving: Competence or Curriculum? - In: Schwank, I. (Ed.) European Research in Mathematics Education: Proceedings of First Conference of the European Society for Research in Mathematics Education August 1998, Osnabruck, Germany. (http://www.fmd.uni-osnabrueck.de/ebooks/erme/cerme1-proceedings/cerme1-proceedings.html)

Hoyles, C. (1997): The curricular shaping of students' approaches to proof. - In: For the Learning of Mathematics 17(1), p. 7-16

Hoyles, C. (1996): Beyond the classroom: The curriculum as a key factor in students' approaches to proof. - In: Y. Pothier, (Ed.), Proceedings of the Annual Meeting of the Canadian Mathematics Education Study Group, Halifax (Canada), p. 7-25

Martin, G.; Harel, G. (1989): Proof Frames of Preservice Elementary Teachers. Journal for Research in Mathematics Education, 20(1), p. 41-51

Reid, D.A. (1993): Pre-formal, formal, and formulaic proving. - In: Proceedings of the 1993 Annual Meeting of the Canadian Mathematics Education Study Group, Toronto.

Reid, D.A. (1995): The need to prove. - Edmonton, Canada, University of Alberta, Doctoral Diss.

Roberts, R. (1999): What makes an explanation a good explanation? Adult learners' criteria for acceptance of a good explanation. - St. John's Canada, Memorial University of Newfoundland, Master's thesis.

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