

## The teaching of proof at the lower secondary level – a video study<sup>1</sup>

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**Abstract:** Teaching mathematical proof is one of the most challenging topics for teachers. Several empirical studies revealed repeatedly different kinds of students' problems in this area. The results give support that students' abilities in proving are significantly influenced by their specific mathematics classrooms. In this paper we will present a method for evaluating proof instruction and some results of a video study that describe proving processes in mathematics classrooms at the lower secondary level from a mathematical perspective.

**Kurzreferat:** Der Beweis im Mathematikunterricht ist eine der größten Herausforderungen für Mathematiklehrer. Empirische Studien haben wiederholt verschiedene Schülerprobleme in diesem Bereich aufgezeigt und lassen annehmen, dass die Schülerfähigkeiten im Beweisen signifikant durch den spezifischen Unterricht beeinflusst werden. In diesem Beitrag präsentieren wir eine Methode zur Evaluation von Unterrichtsbeweisen sowie Ergebnisse einer Videostudie zum Beweisen im Mathematikunterricht der Sekundarstufe I.

**ZDM-Classification:** C73, D43, D53, E53

### 1. Introduction

Teachers' experiences in the mathematics classroom suggest that it is difficult for students to learn how to proof. Moreover, there are several studies on proof and argumentation providing empirical evidence of students' difficulties with proofs. Our own research indicates that students' views on proofs and their abilities in proving are significantly influenced by the specific form of mathematics instruction. However, the reasons for these differences in the students' performance remain unclear. Accordingly, we conducted a video study in order to analyse proof instruction in Germany. In this article we will present a method for evaluating instruction and will give some results that describe proving processes in the mathematics classroom at the lower secondary level from a mathematical perspective. Our research is integrated in the context of the international studies TIMSS and PISA. It aims at identifying cognitive and noncognitive factors playing a role in the development of proof competencies of students

### 2. The role of proofs in mathematics and in mathematics education

Concerning the role of proof in mathematics and in the mathematics classroom we would like to emphasize three aspects of mathematical proofs, namely proof as a social construct, the different functions of a proof, and the

distinction between the process and the result of proving. Since a detailed discussion of these three topics would go beyond the scope of this article, we will restrict ourselves to the main ideas outlining the framework of our research.

There is an extensive discussion about the nature of proof in mathematics (e.g. Hanna and Jahnke, 1996). Though mathematics is regarded as a strict and exact scientific discipline there are no precise definitions for basic notions like "proof". Mathematicians will probably be able to give some necessary conditions for a mathematical proof, but a proof is accepted by more or less implicit rules of the mathematic community: "A proof becomes a proof after the social act of 'accepting it as a proof'" (Manin, 1977, p. 48). This acceptance of a proof by the mathematics community depends on various factors. First of all, the proof has to validate whether a conjecture is true, though a proof has to satisfy various other functions. As stressed by Hanna and Jahnke (1996), de Villiers (1990) and others, proving in mathematics is more than validation. Hanna and Jahnke (1996) describe eight different functions of a proof (like explanation, systematization etc.).

Mathematicians know through their own work, that they must distinguish between the proving process and the proof as an outcome of this process. Sometimes the process of proving a theorem may take years and may include various approaches which may (or may not) lead to a success. In general, none of these efforts can be seen in the final product, the formal written proof. Consequently, the teaching and learning of proof should not be restricted to presenting a correct proof. It is more important to stress the process of proving rather than to give the outcome of this process. In order to differentiate between the process and the outcome of proving, Boero (1999) described an expert model of the process. It is divided into different stages and gives insight into the combination of explorative empirical-inductive and hypothetical-deductive steps during the generation of a proof. We refer to Boero (1999) for the original description of this model; an adapted version for the analysis of proving processes in the mathematics classroom is given in section 4.3.

### 3. Students' proof competencies – empirical results

Several empirical studies on proof and argumentation provide an overview about students' mathematical competencies in different countries. These studies give evidence that comparatively few students are able to deal with proof items or to reason mathematically (e.g. Healy & Hoyles, 1998; Lin, 2000; Reiss, Klieme & Heinze, 2001).

In a study with 669 students in grade 7 and 8 of the German Gymnasium (high attaining students) we focus on the question how cognitive and noncognitive factors influence the students' abilities to perform geometrical proofs (cf. Reiss, Hellmich & Reiss, 2002). We were able to identify three levels of competency: (I) basic competency, (II) argumentative competency (one-step-argumentation) and (III) argumentative competency (combining several steps of argumentation). Low-achieving students

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were not able to solve any items on level III whereas high-achieving students performed well on level I and level II items and satisfactorily on level III tasks (cf. Reiss, Hellmich & Reiss, 2002). A deeper analysis of students' responses to the test items and additional interviews (Heinze, in preparation) indicate that even high-achieving students have difficulties combining arguments to a proof and generating a proof idea whereas low-achieving students are hardly able to deal with proofs and have substantial deficits in their declarative and methodological knowledge.

The results indicate significant differences with respect to the achievement of distinct classrooms (range of mean scores in grade 7 between 22 and 68% of the number of possible points respectively between 12 and 59% in grade 8). A two-level analysis for the achievement test (grade 8) revealed that 42.4% of the variance with respect to the individual achievement could be explained by the specific classroom. It is reasonable to assume that an essential portion of this influence is based on instruction, since there are hardly any differences in respect to other factors like the number of students per class, their social background etc.

There are only few studies analyzing the specifics of the mathematics classroom. One of these studies (and presumably the most important study) is the TIMS video study comparing classrooms in Germany, Japan, and the United States (Stiegler et al., 1999). The results characterized the typical German teaching style as guiding students through the development of a procedure by asking them to orally fill in relevant information. The teacher generally presents the problem at the blackboard and thereby eliciting ideas and procedures from the class as work on the problem progresses (cf. Stiegler et al., 1999, p. 133). In the German sample of the TIMS video study proofs can be found only in a few lessons. Until now there are no results of video studies focussing on the teaching of proofs in Germany<sup>2</sup>.

#### 4. Research questions and design

##### 4.1. Research questions

The described theoretical framework and the empirical results mentioned above show that the students' difficulties in proving are influenced by different components of their proof competencies and the complex interplay in the proving process. This indicates that a successful teaching of proof in the mathematics classroom has to cover the whole process of proving, starting from the first experiments to generate an idea for a hypotheses up to the final step of writing down the complete proof line by line. Moreover, the students should be motivated to participate actively in this process. Observing and reproducing proofs may not be sufficiently beneficial for the learning process. Consequently, we have to focus on the situation in the mathematics classroom and, in particular, on the lessons of proof instruction. This is the starting point for

the present video study concerned with the process of proving in the mathematics classroom. The study is guided by the following research questions:

1. How are proofs taught in the mathematics classroom in Germany?
2. What are the stages emphasized by teachers in the proving process?
3. Are there gaps in the proving process or underemphasized stages?

##### 4.2 Sample

The video study presented here is based on 20 videotaped mathematics lessons on lower secondary level. The lessons have been selected from eight different classes of grade 8 in four schools (German Gymnasium, i.e. high attaining students). For each classroom between two and four consecutive lessons were videotaped. All lessons focussed on reasoning and proof in geometry, in particular, congruence geometry and the application of the congruence theorems for geometrical proof problems. Typical examples for these problems are the congruence of the opposite sides (or angles) of a parallelogram or the fact that a triangle is split up into four congruent smaller triangles when connecting the midpoints of the three sides.

The participating teachers explicitly agreed on being videotaped. They were asked to give (in their view typical) lessons on argumentation and proof. After every lesson the students were presented a questionnaire asking for specifics of the lesson. It turned out that the vast majority of the students characterized the videotaped instruction as the instruction they were used to.

The mathematics lessons were videotaped with two cameras. There was one camera fixed in the front of the classroom videotaping the whole class, and there was one camera in the back following the "place of action" (this was mostly the teacher). Recording of the sound was supported by two directional microphones and a radio microphone for the teacher. Afterwards, the two tapes were combined into a single tape, and finally the session was transcribed. The following analysis is based on these videotapes and transcripts.

##### 4.3 Design of the study

In order to analyze and evaluate the proving processes in the mathematics lessons we need criteria or an accepted model of necessary stages in this process. For this purpose we use Boero's model described above (cf. section 2). Though it gives a detailed description of the mathematical proving process, it is not possible to adopt this model without some necessary changes. Boero's model refers to experts' proofs in scientific mathematics and these differ in some aspects from the proofs in the mathematics classroom. Therefore, we adapted and operationalized Boero's stages to the following coding categories for the evaluation of proving processes in the mathematics classroom.

**Stage 1:** Like in Boero's expert model the first stages consists of the exploration of the problem situation, the generation of a possible conjecture and the identification of different types of arguments that give support for the

<sup>2</sup> In addition to our study there is one other ongoing video project on proofs concerning the teaching of the Pythagorean theorem in German classrooms: [www.dipf.de/projekte/qualitaetssicherung\\_pythagoras.htm](http://www.dipf.de/projekte/qualitaetssicherung_pythagoras.htm).

plausibility of this conjecture. In the case of the geometry lessons this means that the investigation of a drawing plays a role. We denote this stage as

- “treated well”, if all elements are performed substantially by the students;
- “treated”, if some of these elements are given by the teacher or if the stage is very short;
- “treated badly”, if the teacher performs the first stage;
- “not treated” in other cases.

**Stage 2:** The second stage consists of the precise formulation of the conjecture according to the shared textual conventions. This stage is denoted as

- “treated well”, if the students give a formulation of the conjecture (that may be corrected by the teacher);
- “treated”, if only the teacher gives a correct formulation (verbally or on a worksheet);
- “treated badly”, if there are mistakes in the formulation, and
- “not treated”, if there is no formulation of the statement, that is proved afterwards.

**Stage 3:** Again we have a stage of exploration that focuses on the conjecture that has been formulated precisely before. The aim of this third stage is the identification of appropriate arguments for the validation of the conjecture and a rough planning of a possible proof strategy. For the analysis of the proof processes we divide this stage into four subcategories: (1) the reference to the assumptions, (2) the investigation of the assumptions, (3) the collection of further information, and (4) the generation of a proof idea. We denote this third stage as

- “treated well”, if at least three of these subcategories are observed in this stage;
- “treated”, if two of these subcategories are observed;
- “treated badly”, if there is only one of these subcategories, and
- “not treated” in all other cases.

In some lessons one can observe only a very fragmentary third stage that leads to students’ difficulties in the fourth stage. The consequence is a repetition of the third stage for the collection of necessary information. These repetitions are counted separately since they give some extra information about the elaborateness of the third stage.

**Stage 4:** In this stage the elements of the previous stage are combined. The third stage ends with the identification of arguments and a rough planning of the proof resp. the proof idea. In the fourth stage it follows the combination of these arguments into a deductive chain that more or less constitutes a sketch of the final proof. Thus, the rough plan of the proof is ensured by mathematical arguments. This stage can be performed pure verbally or together with some written remarks; it is denoted as

- “treated well”, if both, students and the teacher, give substantial contributions;
- “treated”, if it is presented by the teacher primarily ;
- “treated badly”, if there are gaps or other deficits in the deductive chain, and
- “not treated” in all other cases.

**Stage 5:** This is the last stage for the proving process in school mathematics. Here the chain of arguments of Stage 4 is written down according to the mathematical

standards given in the respective mathematics classroom. In particular, it is important that this stage also gives a retrospective overview about the proof process. It is denoted as

- “treated well”, if all arguments resp. steps are repeated and written down;
- “treated”, if there is a retrospective summary of the proof process and/or the most important arguments are written on the blackboard;
- “treated badly”, if there is only a list of arguments at the blackboard, but no retrospective or summarizing comments according to Stage 5;
- “not treated” in all other cases.

**Stage 6:** The sixth stage, the approach to a formal proof, does not occur at all for mathematics lessons.

For our study we applied this operationalized model of the proving process to all proofs we identified in twenty videotaped lessons. We analyzed the proofs in two different ways: we measured the time spent for each stage and we determined the quality of each stage due to the categories described above. Hence, we got a characterization of the proving process with respect to qualitative and quantitative criterions.

## 5. Results

According to the research questions (cf. section 4.1) our aim is to describe the teaching of proof in mathematics classrooms and, particularly, which steps in the proving processes are emphasized or underemphasized. In the 20 videotaped mathematics lessons we found 22 proofs. For the evaluation of the proof instruction we used the adapted Boero model as described in the previous section. Each proof process was evaluated separately. In addition, we combined the results of all proofs in each classroom, and for an overall result we combined the results of all 22 proofs performed in the classrooms. In the following text the eight classrooms are labelled with capitals from A to H.

### 5.1. Time-based analysis

Diagram 1 gives an overview about how much time each stage took in the proof instruction (mean percentages).

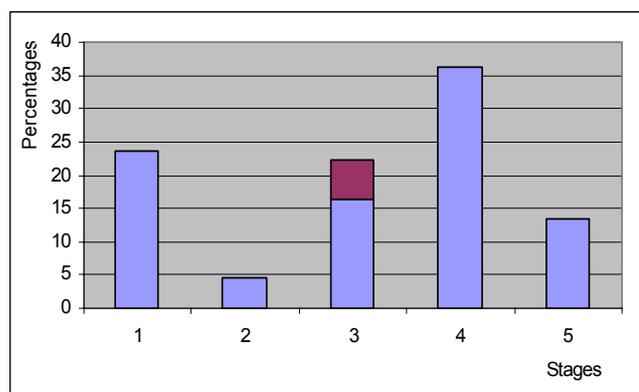


Diagram 1: Percentage of time used per stage over all proofs

One can see that most of the time is spent for the fourth stage (about 36%). The first stage as an experimental stage which mainly consists of drawing and meas-

urements takes about a quarter of the time. The important third stage, in which the exploration of the conjecture and the identification of arguments take place results in only 22% of the proof time, whereas a quarter of this time is concerned with a repetition of stage 3 since this stage was covered insufficiently during the first time.

This diagram reflects the typical process of dealing with proof problems in the videotaped lessons. First of all, the students have to draw a geometrical figure and they are asked to report about their observations, which might be interesting from the perspective of geometry. In general, this is a purely experimental stage; the students should measure and compare angles or lines. After that, a conjecture is discussed and formulated (Stage 2). Then the third stage starts and the teacher asks for ideas to prove this conjecture. Here the students have the possibility to propose some ideas. If the students are not able to generate the correct proof idea (i.e. the idea the teacher has in his/her mind), then the teacher gives some hints. After that the proof is organized step by step on the chalkboard. This takes place in a very special form of classroom discourse, during which the teacher leads the students through the proof by specific questions. In other words, the students have to follow the proof the teacher has in his/her mind. This stage is often interrupted by going back to Stage 3 since in many proofs the third stage was covered insufficiently. The text on the chalkboard is frequently a collection of arguments like the one expected during Stage 4. Sometimes some parts are already very detailed as it is expected during Stage 5. Only in a few cases a “real” Stage 5 takes place, during which a retrospective summarization is provided.

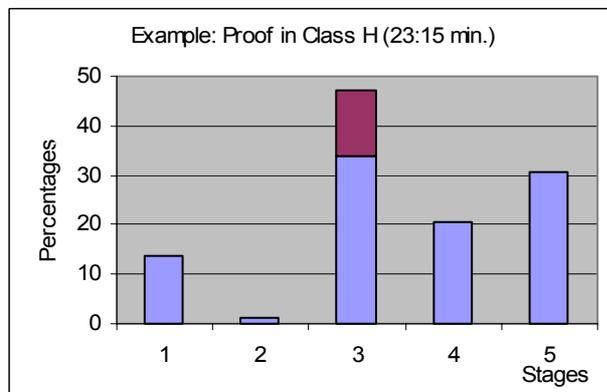
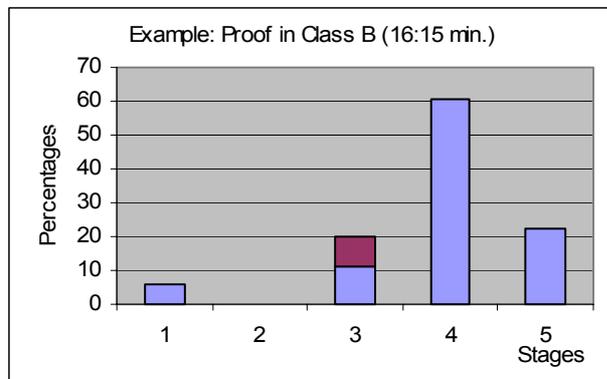


Diagram 2: Time-based analysis of examples for proofs

As mentioned above, Diagram 1 presents mean values of the classes. Sometimes the proof processes in a

specific classroom deviate substantially from this mean value. Diagram 2 describes how much time each of the five stages in two specific classrooms labelled as class B and class H took (in parentheses one can find the overall time of the proof process). Here we can observe that the third stage is handled quite differently in these classrooms.

We should mention that the Diagrams 1 and 2 provide only information about the accumulated portion of time for each stage in the proof process. In the diagrams, there is no information how this time is distributed over the whole process. In order to emphasize how different the proof instruction in the classes can be we give two examples of the distribution of the proof stages along the timeline (cf. Diagram 3). As one can see in the example of Class E, there is no third stage of exploration after the second stage. Only once during an interruption of Stage 4 Stage 3 occurs. In this example the congruence of the opposite sides of a parallelogram should be proved. The teacher elaborates the proof at the chalkboard by questioning the students. Hence, no exploration is necessary and most time of the proof process is spent for stage 4. There is no retrospective summarizing and no detailed formulation of the proof, and Stage 5 does not occur. The second example of Class C deals with a symmetric quadrilateral in which the diagonals should proved to be congruent. In nearly all parts of the lesson the proving process is determined by the students themselves. The Stages 1 to 4 are organized as expected by the Boero model. Stage 4 is interrupted by a step back to Stage 1 during which an additional drawing is made. Like in most of the proofs presented in the classrooms observed, Stage 5 is missing.

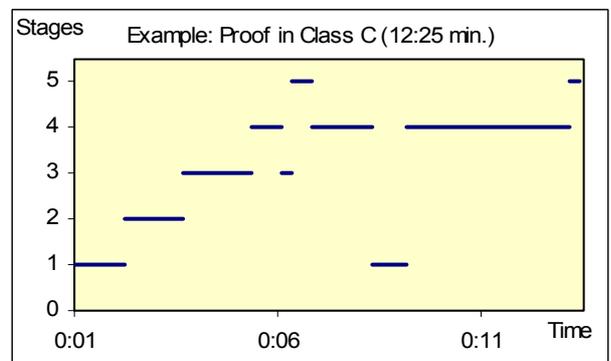
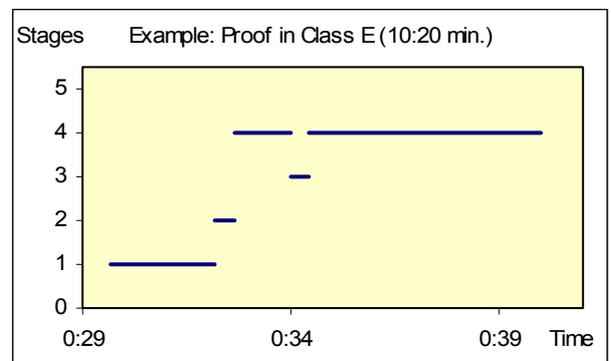


Diagram 3: Distribution of proof stages along the timeline

**5.2 Quality of proof instruction**

In section 5.1 we gave some results that specify the portion of time used for the different stages of the proof process in mathematics classroom. In addition to these results, we also collected data about the quality of the proof stages. The categories for the rating of the stages are described in section 4.3. In some cases it was not possible to rate the complete proving process, since some stages were partly done in a lesson before which was not videotaped or were part of homework. This problem occurred in our sample only during the Stage 1 and mainly the drawings were prepared at home. We did not evaluate this stage even when it occurred partly in the lesson, e.g. in the case of the examples of the classes B and C (compare the occurrence of Stage 1 in Diagrams 2 resp. 3 and in Diagram 5).

Diagram 4 gives evidence that the quality of Stage 2 and Stage 4 during the proof processes is high. Moreover, Stage 1 turns out to be treated satisfactory. Substantial deficits mainly occur during Stages 3 and 5. The requirements of these stages are not fulfilled in 9 resp. 12 out of the 22 proofs. Their quality in the remaining proofs is mostly rated as “treated” and only in few cases as “treated well”.

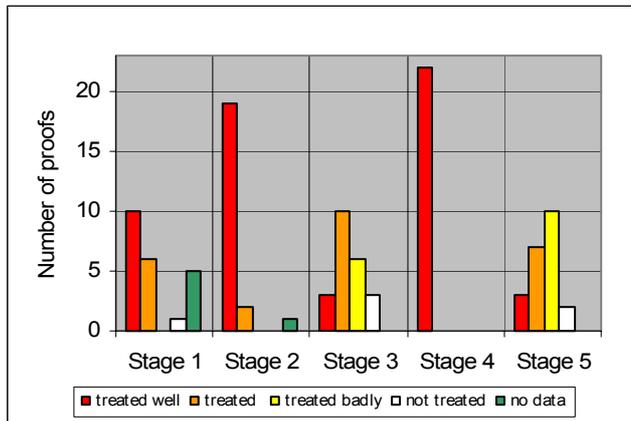


Diagram 4: Quality of the proving process

As described in section 5.1 there are differences between the 22 proofs. In the following we present a diagram for the quality of the proof stages for the different classes (diagram 5). Therefore, we map the categories of quality onto a percent scale (“treated well” = 100%, “treated” = 66%, “treated badly” = 33% and “not treated” = 0%) and provide the mean values for the quality of the proof processes in each class. However, it should be noted that we observed a different number of proofs in different classes (the number is given in parentheses) and that we did not evaluate Stage 1 of the proofs in the Classes B and H and of one proof in Class C (as explained before). Diagram 5 confirms the impression one gets from Diagram 4. Stages 1, 2 and 4 are treated on a satisfactory to high quality level, whereas Stages 3 and 5 are generally treated on a medium quality level. An exception is the proof in classroom H where all stages observed were rated as treated well.

If we combine the time-based results of Section 5.1 with the quality-based results of this Section 5.2 we can identify clear deficits in the proof processes for Stages 3 and 5. Both stages are important for the learning of

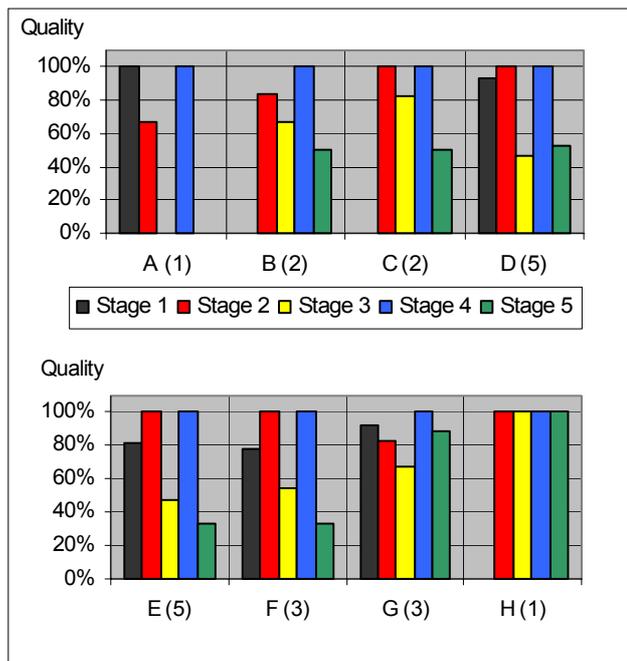


Diagram 5: Quality in different classrooms

mathematical proofs, however, Stage 3 as the stage of exploring of the conjecture, collecting additional information and generating a proof idea seems to be the most crucial stage for proof instruction on the lower secondary level. Accordingly, we take a closer look at this stage in the following paragraph.

**5.3 A detailed analysis of Stage 3**

In Section 4.3 we split the third stage into four different subcategories. These are (1) the reference to the assumptions, (2) the investigation of the assumptions, (3) the collection of further information and (4) the generation of a proof idea. In the videotaped lessons we analyzed which of these subcategories occurred in the proofs observed.

The results in Diagram 6 indicate that in general only one of the four components appears. In 17 of the overall 22 proofs the teacher and students refer to the assumptions given in the third stage. Only half of the proof processes

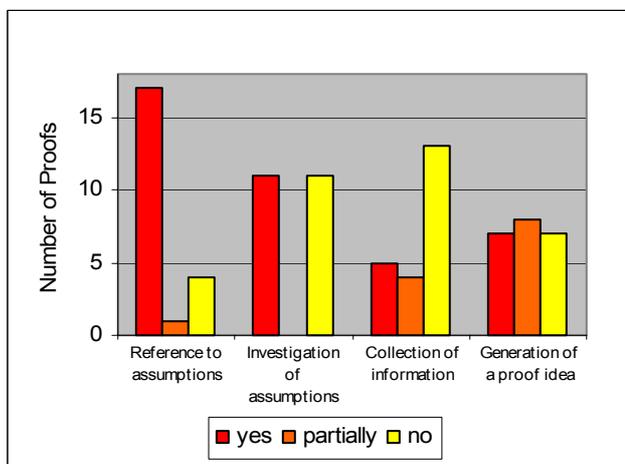


Diagram 6: Occurrence of components of Stage 3

we observed contain a subsequent investigation of these assumptions. The collection of additional information does not appear during thirteen proof processes and in four cases it is provided only partially. The generation of a proof idea is part of seven proofs only and it is provided partially during eight proofs.

Again we consider the different classrooms with respect to the third stage. In Diagram 7 the occurrence of the four components of Stage 3 is represented by percentages (occurrence = 100%, partial occurrence = 50%, no occurrence = 0%). For each classroom we present the mean values over the different proofs.

It is easy to read from the diagram that in nearly all classrooms these components are neglected (except for a reference to the assumption). There is only one classroom, namely class H, where the different aspects have been identified during instruction.

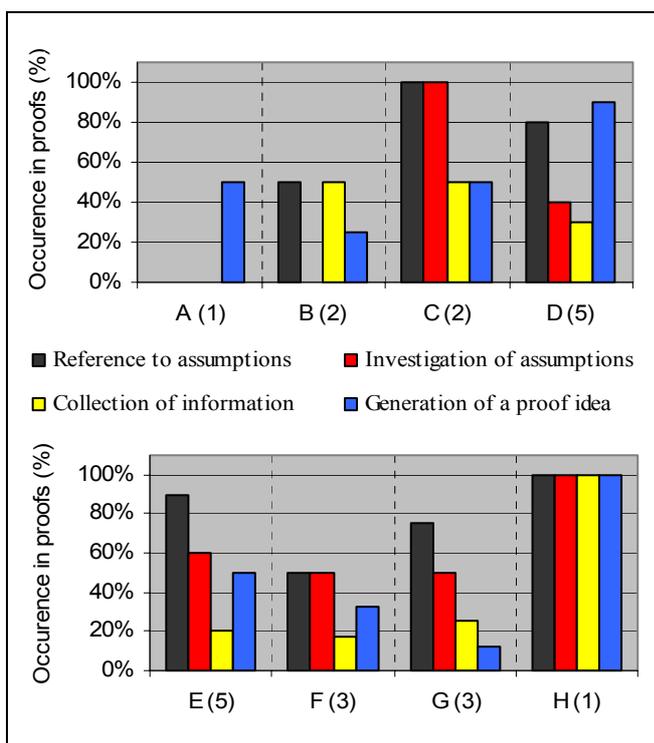


Diagram 7: Components of Stage 3 in different classes

## 6. Discussion

The results presented in Section 5 indicate that the teaching of mathematical proof in German classrooms has several deficits. Our investigation is based on Boero's model as a tool to describe the experts' proof process in mathematics. As explained in Section 4.3 we adapted this model for the investigation of the teaching of proof in mathematics classroom. We regard Boero's model as a valid description of necessary stages (or components) in the proof process. Consequently, we assume that all stages are of essential importance for the learning of mathematical proof.

Our analysis gives evidence that stages regarded to be essential for constituting a proving process are systematically neglected by teachers in their classroom instruction. As already mentioned in Section 5.1 the typical proof process in the mathematics classroom is planned and controlled by the teacher. This means that

the teacher leads the students through the "maze" of the proof situation. The role of the students is more or less to guess the direction their teacher has in his/her mind. This kind of instruction is caused by a style of teaching based on a sequence of short questions by the teacher and short answers by the students. Questions and answers aim at developing a specific piece of knowledge ("fragend-entwickelnder Unterricht"). This style of teaching is the most popular form of instruction in German secondary schools (cf. TIMSS video study Klieme, Schümer & Knoll, 2001). The problem with this kind of teaching is that there is no place for in-depth stages which are necessary in the proof process, e.g. for the exploration of the problem situation or the collection of additional information. In the videotaped lessons the first exploration stage mainly consists of making drawings and measuring the lines or angles. The second in-depth stage is the third stage. As shown in the diagrams in Section 5 the students have no time for a deeper investigation or exploration of the situation. As a consequence they get no real chance to solve the proof problems on their own. They have to follow the hints and questions of their teacher. As shown in Diagrams 1 and 2 most of the time in the proof processes is dedicated to Stages 4 and 5 (apart from the time for drawings in Stage 1). However, in general, these two stages do not coincide exactly with the description in the Boero's model. More or less we observed a mixture of the ideal Boero stages. The teacher elaborates the proof step by step at the chalkboard by asking question and giving hints (i.e. applying the "fragend-entwickelnde" style of teaching). This is more precise than it would be expected according to Boero's Stage 4 (organizing arguments in a deductive chain) but in many cases less precise than it would be expected during Stage 5 (formulation of proof according to the mathematical standards in school). From the students' perspective the proof can be divided into small pieces they have to deal with successively. There is hardly any doubt that many students will loose control of the proving process. Moreover, a retrospective summary of the proof is provided only in few cases.

The results of this video analysis give hints to answer the research question why the proof competencies of German students at the lower secondary level are poor compared to international standards (cf. Deutsches PISA-Konsortium, 2001; Baumert et al., 1997). As described in Section 3 several studies of our working group show in detail that there are different reasons for the students' poor performance on proof problems. We assume that the students' problems are significantly influenced by their proof instruction as the results of the video study indicate in this article. A more systematic investigation of this probable correlation is in process.

It should be emphasized that we analyzed the teaching of proof in the videotaped lessons from a mathematical perspective. As a consequence other questions and problems like the style of teaching, the combination of classwork and seatwork, the participation of students in the elaboration of the proof processes etc. are not discussed in this study. However, this mathematical point of view revealed important problems in proof instruction.

Finally, we want to stress that there are examples of

“good practice”. As shown in the Diagrams 5 and 7 the proof process observed in Class H satisfied all requirements of the adapted Boero model. Though the instruction in class H was also based on the “fragend-entwickelnden” style of teaching, Diagram 2 gives evidence that it is possible to provide sufficient time for instruction of the (necessary) in-depth Stage 3.

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