

## Visualization and art in the mathematics classroom<sup>1</sup>

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**Abstract:** In this paper we summarize our concepts and practice on computer-aided mathematical experimentation, and illustrate them by *Mathematica* projects that we have developed for our research and the courses "Computer-aided mathematical modelling" and "Computer Algebra I-II" held for students of life sciences at University of Szeged and computational engineering at TFH Berlin, University of Applied Sciences.

**Zusammenfassung:** In diesem Artikel fassen wir unser Konzept und unsere Erfahrungen zu computergestützten mathematischen Experimenten zusammen. Wir illustrieren sie mit in *Mathematica* realisierten Beispielen. Diese wurden sowohl im Rahmen unserer Forschung als auch für die Kurse „Computergestützte mathematische Modellbildung“ und "Computer Algebra I-II" entwickelt, die sich an Studenten der Universität Szeged bzw. an Studenten im Studiengang Computational Engineering der TFH Berlin richten.

**ZDM classification:** D40, I10, M50, M60, M80, N80

### 1. Introduction

Computer visualization and modelling tools on high powered desktop computers with realistic graphical capabilities have become an organic part of scientific research and education, and their influence is still increasing (for example, see [2,3,4,6,8,10,13]). Such techniques are widely used in almost every field of scientific research and engineering, even in the most abstract disciplines such as mathematics. They help to understand and illustrate problems since the phenomena in applied fields can be described by quite complicated mathematical models. In particular, they can increase the effectiveness of the mathematics courses for life science and engineering students.

Realizing this fact, several years ago we introduced computer tools in our traditional classes, introduced new ones on computing methods, visualization and modelling. Although engineering and life sciences seem to be far from each other, it turned out that they strengthen each other. Hence, our joint research and developments are fruitful for both fields.

In this paper we summarize the cornerstones of our computer-aided teaching and some of the results of our practice. Then, we illustrate the ideas with some of our experimental projects prepared in *Mathematica*. We believe that they can be applied for other courses.

### 2. Teaching concepts

The fact that mathematics is "only" a tool for engineering and applied sciences can be generally accepted. Students work with mathematical models and often do hard calculations without a very deep study of the mathematical theory behind. Explorations, experiments, and visualization can fasten the understanding since they are often more convincing than a rigorous mathematical proof. Then, computer programs can substitute tiresome manual calculations. In addition, we note that mathematical programming and computer implementations of numerical methods are of basic importance for engineering students. Hence, combining the traditional treatment and manual drill with experimental studies can result in a deeper and more applicable mathematical knowledge, and it improves the creative thinking.

According to the above statements, our teaching has three essentially different but more or less overlapping phases:

- The theoretical mathematical courses essentially follow the traditional lines, where the computer applications as demonstrational materials help to understand the theoretical concepts and methods.
- In the computer-based courses (Computer-aided mathematical modelling [10,11,12], Computer-algebra I-II [9]) the students learn special mathematical packages or general computer-algebra systems, learn how to use them to perform mathematical algorithms.
- Finally, they solve practical problems during their whole study with the help of computerized mathematical procedures and methods.

We need to take into account in our curriculum development that the main interest of our students is directed toward their specific fields in engineering, physics, biology, chemistry, medicine, pharmacy, etc.

What concerns the theoretical courses, we emphasize that computer is not used instead of blackboard and chalk. We do use computer only if the traditional tools are not useful enough. For the computer-based courses, the main points are as follows:

- the students would like to gain maximal benefit with "minimal" mathematical and computer-technical effort;
- we emphasize, and prefer, computer-aided experimentation and visual arguments to the formal mathematical methods;
- consequently, a large set of user-friendly modelling "skeletons" (recipes) and case studies are prepared and used;
- in particular, the programming tools which are fundamental for the engineering students are learned via applications;
- a teacher and an instructor hold the class together: the teacher gives the main line and the instructor helps to solve the arising individual problems;
- the teacher and the students work on computers simultaneously;

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- students use and improve the schemes to problems according to their particular interest.

We need to mention that the synchronization of courses has a significant influence on the success since the technical tools should be used parallel to their learning. This problem has no general resolution. Intensive courses cannot be usual held in any regular university teaching system, but they are real solutions in some special cases. For "computational engineering" postgraduate students at TFH Berlin, University of Applied Sciences, and "mathematics" Ph.D. students at University of Szeged we started to organize intensive Computer Algebra courses. Such a course consists of two one-week long parts (beginning and advanced) at the beginning of the semesters. These students own the necessary mathematical knowledge, and they can immediately apply the learned methods and tools in their applied studies.

### 3. On the project development

There are a great number of books and computer developments published on computer algebra applications (see the references). Most of them either deal with the details of the technical tools, or use computer-algebra systems to help in the understanding of the mathematical theory. Some deal with modelling problems, giving quite deep knowledge of the considered problems, but do not give easily applicable "recipe-collections". Hence, using the experiences with the published works, we have developed *Mathematica* projects that fit completely the stated concepts of our courses. The projects can roughly be classified into the following groups:

- illustrations to mathematical theories;
- introduction to the technical tools including the power of graphics;
- computer implementation of mathematical concepts and procedures, the modelling steps;
- projects to investigate elementary continuous and discrete models;
- projects to investigate advanced models;
- mathematical programming;
- tools and projects for interactive modelling.

The structure and the style are uniform in each project. During the development we follow the following concepts:

- hyper-media structure;
- short mathematical and technical summaries;
- user-friendly computer implementation of theoretical methods;
- graphical explanations and arguments emphasizing the role of visualization methods such as animation, colouring, and interactive methods;
- experimentation: constructive methods are preferred;
- numerical solution of mathematical systems is preferred; attention is paid to the theoretical and numerical limitations;
- dangers of trusting only experiments are treated;
- well-prepared statements, programming examples, avoiding typing and syntactical difficulties;

- simple and complex modelling schemes (skeletons), case-studies are presented.

These projects can help to understand the qualitative as well as the quantitative properties of the considered phenomena. We believe that our projects improve the students' creativity and scientific thinking.

## 4. Examples

Let us consider now some examples from the simplest up to the very advanced ones. It is not possible to present the illustrations and projects in details, and the colouring and animations cannot be shown either. Hence, we only summarize the main objects and the features.

### 4.1 Experiments with computer plots

Doing the simplest but essential experiments, the students learn the basic plotting features, as well as the use and the dangers of the computer plots. For example, changing the number of sample points can cause a dramatic change on the quality of the plots (Fig. 1). Pay attention to possible singularities, watch the extremely large growth rate, error messages (zero divide, for example). Note that some singularities cannot appear on the computer plots.

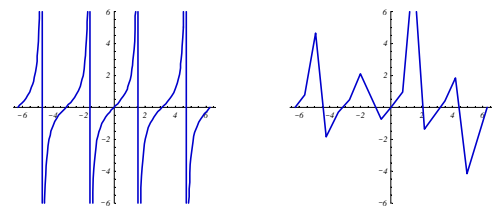


Figure 1. High and low quality plots of  $\tan(x)$

### 4.2 Zooming in and out

It is a well-known method in video technique. "Zooming out" gives us a global plot of functions, while "zooming in" visualizes the invisible (Fig. 2) and can help to understand local properties such as continuity and linear approximation at a given point (differentiation).

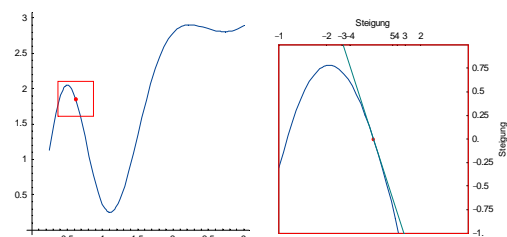


Figure 2. "Zooming in" illustrating the differentiability

### 4.3 Computer-aided investigation of function

After doing some graphical experiments on the function considered, the students use analytic and/or numeric methods to describe the properties and find specific points (zeros, maxima, minima, etc.). The importance of the initial guess for these points is emphasized.

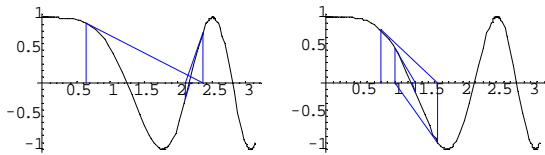


Figure 3. Different initial points for the Newton iteration

**4.4 Experiments with series expansions**

Taylor approximation around a point or Fourier series expansion of periodic technical functions (square and triangle signals, etc.) are essential. We developed projects to estimate iteratively the coefficients by graphical information. Other projects visualize the series expansion and the spectrum by animations (Fig. 4). They can give an intuitive and visual understand in these difficult mathematical constructions.

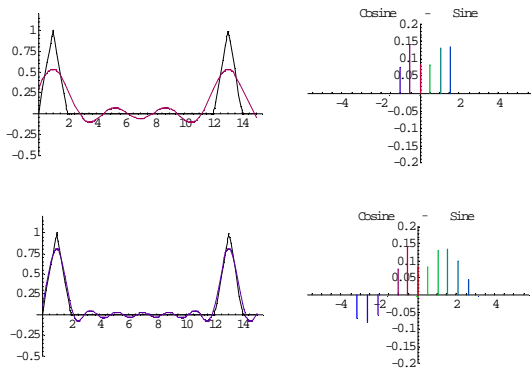


Figure 4. Two frames of a spectrum animation of a triangular wave

**4.5 Experiments with wavelets**

Basic ideas of wavelet theory can be taught at an undergraduate level by use of computer experiments. This is an algorithmic approach. The Daubechies wavelet can be easily constructed by the recursion formulae for the scaling function  $\varphi(t) = \sqrt{2} \sum_{k=0}^3 \alpha_k \varphi(2t-k)$  with the coefficients  $\alpha_0 = \frac{1+\sqrt{3}}{4\sqrt{2}}$ ,  $\alpha_1 = \frac{3+\sqrt{3}}{4\sqrt{2}}$ ,  $\alpha_2 = \frac{3-\sqrt{3}}{4\sqrt{2}}$ ,  $\alpha_3 = \frac{1-\sqrt{3}}{4\sqrt{2}}$ . This formulae can be evaluated iteratively at dyadic points  $k/2^j$ ,  $k, j \in \mathbb{N}$  starting with  $\varphi(0) = 0$ ,  $\varphi(1) = \frac{1+\sqrt{3}}{2}$ ,  $\varphi(2) = \frac{1-\sqrt{3}}{2}$ ,  $\varphi(3) = 0$ ,  $\varphi(t) = 0$  for  $t \notin [0, 3]$ . Using this scaling function the wavelet  $\psi$  can be given by the formulae:  $\psi(t) = \sqrt{2} \sum_{k=0}^3 \beta_k \varphi(2t-k)$  with  $\beta_k = (-1)^{4-k} \alpha_{3-k}$ . Both functions are continuous everywhere and differentiable almost everywhere. However at the dyadic points they are only differentiable from the left. This fact can be visualized by zooming techniques [14,15].

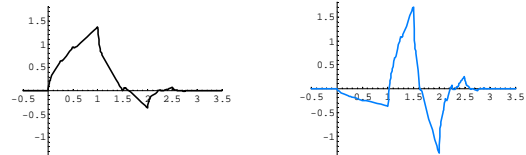


Figure 5. Scaling function and wavelet

**4.6 Some mathematical graphics: functions in different coordinate systems**

Plotting different surfaces can sometimes be an active relaxation among serious modelling problems. Meanwhile our students can understand the different coordinate systems and transformations. For example, plotting a function in Cartesian, cylindrical, polar, and toroid systems can result in interesting surfaces (Fig. 6). We can enjoy the results. In addition, our students can improve their visual thinking.

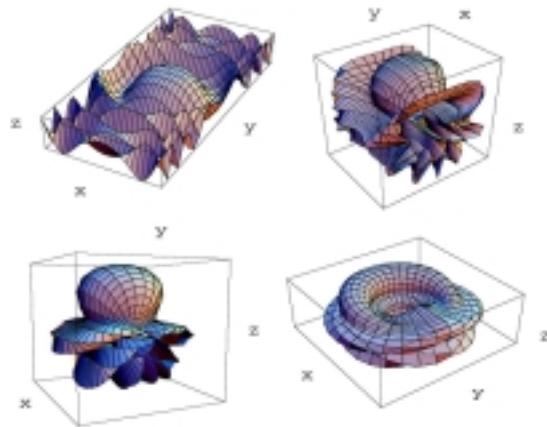


Figure 6. Waves on different surfaces

**4.7 Visualizing trajectories**

This simple but useful project is completed by the students on the third class. Essentially, it contains the plot of a parametric curve and the tangent vectors. The students learn how to visualize a fourth dimension (the time) in a 3D plot by using colouring, animations and stroboscopic plots (Fig. 7). Later on, replacing the given curve by a solution of a differential equation, this project is used for experiments for real modelling problems.

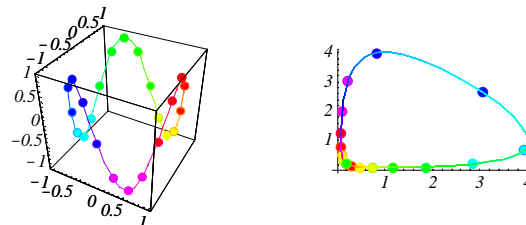


Figure 7. Stroboscopic plots

**4.8 Curve fitting projects**

The students graphically and analytically investigate datasets (Fig. 8), use transformations and filters if needed

(they also learn these methods here), solve curve fitting problems, and try to improve the fitting by finding better fitting families.

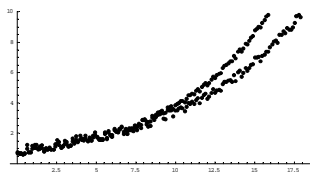


Figure 8. A non-homogeneous dataset

**4.9 Critical reflection of hand made and computer made results**

The students know how to solve an ordinary differential equation by separation of variables, and they can verify the result with the help of a computer algebra system. Now, they plot the slope field and the solution (Fig. 10), and then they can recognize that the obtained solution is not a solution on the whole domain of the function and that computer solutions should be checked carefully. The students check whether the validity of their calculation is limited by indirectly given conditions. More critical reflections of the use of computers in teaching areas can be found in [1].

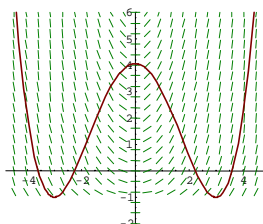


Figure 9. Slope field of  $y' = x\sqrt{y+1}$  with one solution

**4.10 1D differential equations**

The students learn the basic steps together with the corresponding computing tools of the investigation of 1D ODE's concerning direction field, equilibriums, and their stability properties without solving the equation; then the properties of the solutions and their dependence on initial conditions.

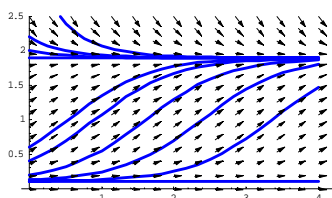


Figure 10. The direction field and solutions of the equation  $x' = x(2-x) - 0.2$

**4.11 Models described by planar differential equations**

The students investigate 2D vector fields, the trajectories in 2D and the integral curves in 3D. They can improve now the project prepared for 2D and 3D parametric curves (see Example 4.5). To investigate the properties of the equilibria, they learn the method of Liapunov

functions. Finally, the students get acquainted with the linearization methods.

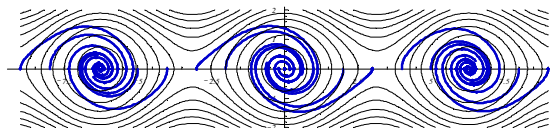


Figure 11. The trajectories of  $x'' + 0.3x' + \sin(x) = 0$  and the contour lines of the energy

**4.12 Experiments with impulsive effects**

Since their formal description and behaviour is sometimes quite complicated, impulsive systems are rarely considered in details in graduate courses [13]. Our projects deal with some simple but important models, and help to understand the nature of such systems. One of them considers the problem of the repeated drug dosage. Here the students investigate the relation between the times elapsed between the doses, the importance of the first dose (Fig. 13). They have to find an optimal setting to avoid either the under- and overdosing.

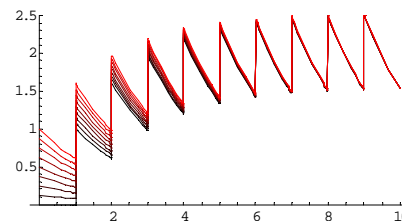


Figure 12. The dependence on the first dose

**4.13 Iterative problems, difference equations, cellular automata**

It is well known that iteration and recursion are basic mathematical algorithms. Hence, our students have to get acquainted with the computer tools to investigate and solve iterative problems from the simplest to the very advanced ones (see Examples 4.3 and 4.5). With a little effort they are able to solve and visualize difference equations. Figure 14 shows some solutions (“integral curves”) and the phase maps of an initial set of a planar difference system that shows some kind of pulsation effect.

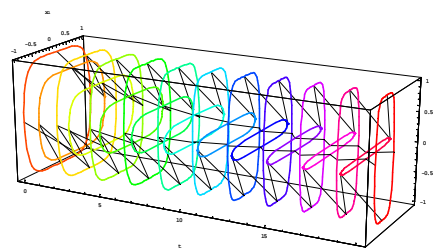


Figure 13. Solutions and phase maps of a pulsative difference system

Engineering students and PhD students in any applied science often need to do spatial modelling whereas they are not skilled to tract partial differential equations. By

discretising partial differential equations we obtain partial difference equations or cellular automata. In addition, they themselves are very important models of several real phenomena (such as reaction-diffusion problems in ecology) or engineering problems. By defining the discrete local rules, the behaviour of such a system can be easily simulated. Note that S. Wolfram needed a software package for his cellular automata research, and hence he developed *Mathematica* [16].

Among the introductory projects in this field, Figure 15 shows some states of a discrete wave.

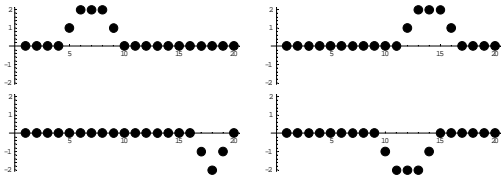


Figure 14. Discrete 1D wave propagation and reflection

The study of space-time systems is even more complicated in higher dimensions both theoretically and using computer simulations. To illustrate such cases, Figures 13 and 14 shows the evolution of a cellular automaton that describes the behavior of an ecological system of two competing species.

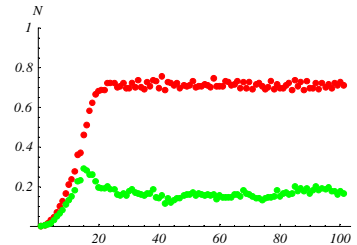


Figure 15. Evolution of the frequencies of the two species in time

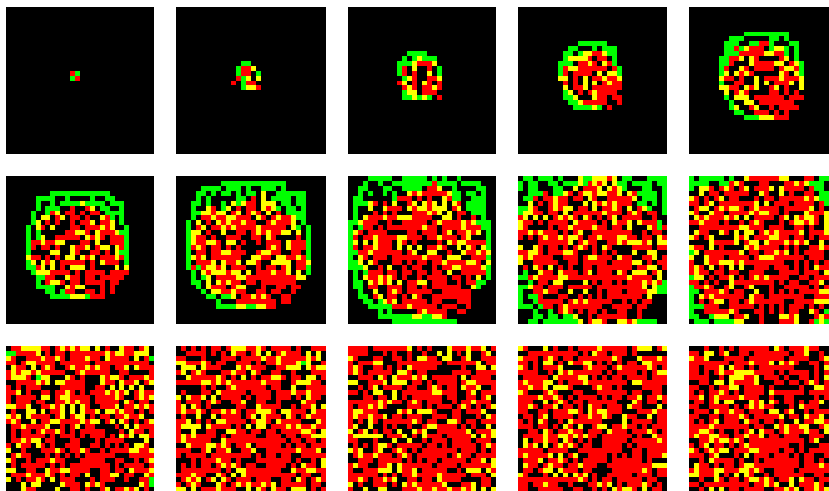


Figure 16. A CA simulation of an ecological model of two competing species

**5. Math-Art in the classroom**

G. H. Hardy [7] said, the beauty is the main criterion in mathematics. Ugly mathematics is useless. The beauty of mathematics appears in the theories, but also in the form of artworks. For common people mathematical art (different from computer art!) is kept almost identical to iterative forms and fractals. In addition to their scientific importance, their beauty is obvious. But math-art is much wider. We can find beauty in almost every, even in the simplest mathematical construction. For illustration, see our examples, especially Example 4.6.

Artistic issues are very important in the mathematics classroom at all levels. Students can be motivated to learn deeper theory, to get practice in mathematical

experimentation by letting them create artworks. The necessity of this motivation can be underlined by the fact that a big number of students have some inhibitions about mathematics. We hope that after such math-art experience our students will certainly be able to recognize mathematical structures in the real world and hence the break between the abstract sciences and the real life can become much narrower than it is now. Beyond lessons, successful exhibitions also demonstrated that math-art approaches people to mathematics (see the web page <http://silver.szote.u-szeged.hu>). Instead of more explanations we close our report with the surface described by the formula

$$\left\{ \left( \left| \sin \left( \frac{7u}{2} \right) \right|^{0.5} + 0.4 \right) \cos(u), \left( \left| \sin \left( \frac{7u}{2} \right) \right|^{0.5} + 0.4 \right) \sin(u), 4(0.02 \sin(29u) + 1) \sin^{0.4}(v) \right\}$$



Figure 17. A flower as a parametric surface

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