

Computer aided treatment of 3d-problems in analytic geometry

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Abstract: In this contribution a method for the treatment of analytic geometrical problems is introduced which integrates three-dimensional computer graphics and computer algebra. At this a new computer graphics tool is used which has been developed for the visualization of the corresponding spatial configurations and for the graphical solution of spatial analytic problems. (The virtual sphere device is used for flexible and individual visualization.) The tool is suitable both for the demonstration by the teacher and for the interactive work of the students. The computer algebraic treatment is motivated by giving an explanation for the computer graphical solution. Firstly the general algebraic problem is solved and graphically illustrated, after this the general solution is numerically specified. The procedure of analytic solution can be illustrated also computer-graphically.

Kurzreferat: In diesem Beitrag wird eine Methode für das Behandlung dreidimensionaler analytisch-geometrischer Aufgaben vorgestellt, welche dreidimensionale Computergrafik und Computeralgebra integriert. Dabei wird ein neues Grafikwerkzeug verwendet, das für die Visualisierung und die grafische Lösung solcher Aufgaben entwickelt worden ist. (In diesem Werkzeug dient das Virtual-Sphere-Device der flexiblen und individuellen Visualisierung.) Dieses Werkzeug eignet sich sowohl für die Demonstration durch den Lehrer als auch für das interaktive Arbeiten der Schüler. Die computeralgebraische Behandlung der Aufgabe ist motiviert durch die Frage, was unter anderem berechnet werden muss, um die grafische Lösung auf der Oberfläche des Grafiksystems zu erhalten. Deshalb wird zuerst die betreffende allgemeine Aufgabe gelöst, um durch Eingabe konkreter Daten die Lösung einer speziellen Aufgabe zu erhalten. Der analytische Lösungsweg lässt sich auch computergrafisch veranschaulichen.

ZDM-Classification: B40, C30, D50, G40, U60

1. Introduction

Experience shows that many students of secondary schools have difficulties to imagine spatially the analytic-geometric task in IE^3 , the way of solving and the solution itself.

Suitable menu-controlled 3D graphics systems like "DreiDGeo" (Rechenberger et al. 2000) may support the development and practise of a corresponding spatial ability. We assume the "pictorialistic thesis", that imagination of spatial configuration and its variation of position and dimensioning in "student-heads" can just be developed by visualized and observed material or pictorial representations.

Blackboard drawings or handmade transparencies, mainly of questionable quality of perception, are no basis for developing an adequate spatial-geometric understanding in working on tasks of spatial analytic geometry – which doesn't exclude that students can solve those tasks algorithmically without spatial understanding.

Besides the deficit which conventional medias have, occurs another one: In general those tasks of analytic geometry are specific and posed with "nice" numbers; there are no general tasks (with variables) because they require a great amount of algebraic calculation.

The components of computer algebra and computer numeric of corresponding mathematical tools, such as DERIVE, allow, independent of the complexity of the task, to calculate general solutions.

The general solution can be specified with given data of specific tasks. In that way you rule all kinds of equivalent tasks, but you have to discuss the solvability of the general task, which make these kinds of tasks – beyond mere numerical handling – more interesting from the mathematical point of view.

A connection between computer-graphical and computer-algebraic handling of a task consists in the computer-graphical solution with the appropriate specified general algebraic solution, which lies under the surface of the graphic tool.

2. Computer-aided treatment

We proceed like that: First the special task is computer-graphically visualized and solved. After that the computer-algebraic handling of the general task belonging to it follows.

The understanding for the (conceptual-abstract) general task and its solution is supported by the computer-graphical handling of the special task and its solution.

We select the following task for an exemplary treatment:

Given two lines g_1, g_2 .

g_1 :

$$\vec{x} = \vec{p} + \lambda \cdot \vec{a} \text{ with } \vec{p} = (-6, 6, 8), \vec{a} = (2, 3, -3),$$

g_2 :

$$\vec{y} = \vec{q} + \mu \cdot \vec{b} \text{ with } \vec{q} = (-3, -8, 3), \vec{b} = (5, -2, 3)$$

and a point $R = (-4, -5, 9)$.

Find a line g which is concurrent with g_1, g_2 und R .
What are the points respectively angles of intersection of g with g_1 und g_2 ?

2.1 Computer-graphical treatment

We outline a method of treatment, which is based on the visualization, construction and measurement tool “DreiDGeo”:

- (1) Numerical implementation of given objects belonging to a special task for creating its graphical representation. (Input)
- (2) Experimental examination of the existence of a graphical solution.
- (3) Construction of auxiliary objects out of given objects in graphical representation for developing a solution. (Solution strategy)
- (4) Construction of solution objects in graphical representation from given objects resp. auxiliary objects.
- (5) Reading of the object(s) to be found in numerical representation (solution)
- (6) Variation of data for given objects. (operative question: Who does the solution correspond to the given data?)

Annotation:

During the steps of solution there is an ongoing visualization of graphical configuration, e.g. through rotation of the outlined virtual sphere and an improvement of visualization though e.g. fading in resp. out of objects, variation of object transparency, zooming, design of the three-dimensional Cartesian coordinate system, ...

The saving of graphical configuration of solutions serves as extension of the repertoires of tasks; printing resp. export of an appropriate graphic file are used for documentation.

Solving the task with DreiDGeo we implement the straight lines g_1 , g_2 and then the point R in the corresponding masks (Fig. 1; you can design the graphical objects with colours by “Eigenschaften”). The objects are visualized automatically in a three dimensional Cartesian coordinate system. (Fig. 2.1)

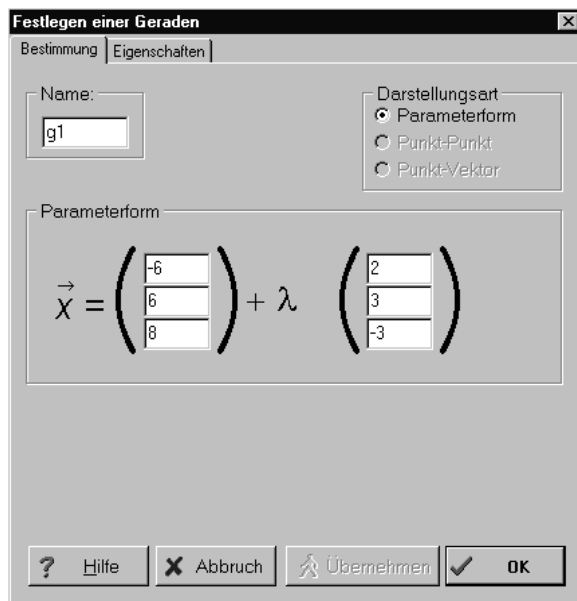


Fig. 1 (Input of a point)

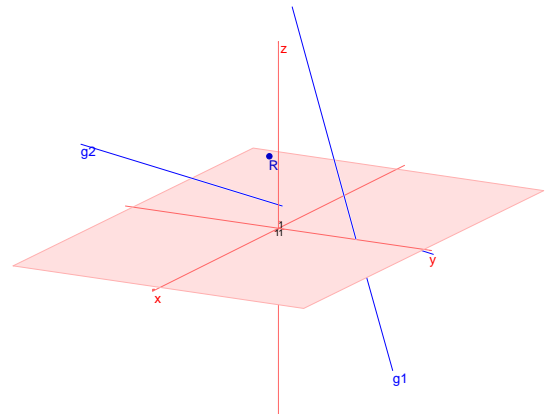


Fig. 2.1 (Visualization of the given objects)

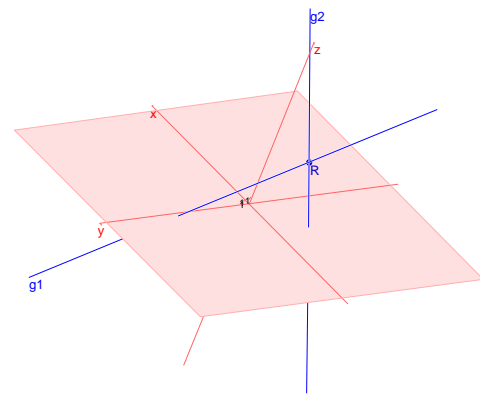


Fig. 2.2 (Experimental existence of a solution)

Through rotation we can see the position of the crooked lines and the point R. We can see on experimental way, that there has to be a straight line g which goes through R and is concurrent with the two straight lines, by rotating the configuration in a way that the line to be found has to appear as a perpendicular of the monitor-plane through R and being tangent to g_1 and g_2 .

We generate a plane e_1 through R and g_1 and the plane e_2 through R and g_2 as auxiliary objects, because the wanted straight line g has to be in these planes, as it goes through R and g_1 resp. through R and g_2 (fig. 3.1). We have a look at this configuration from the “behind” (fig. 3.2).

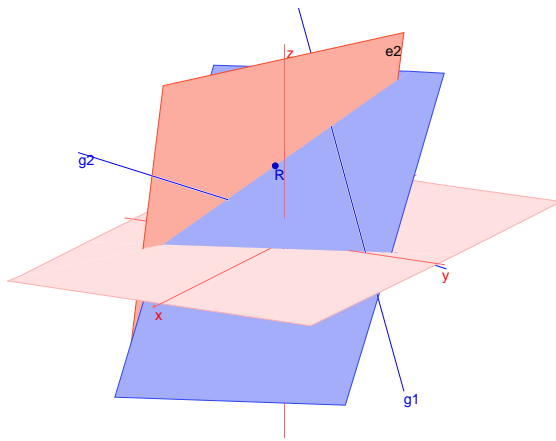


Fig. 3.1 (Planes as auxiliary objects)

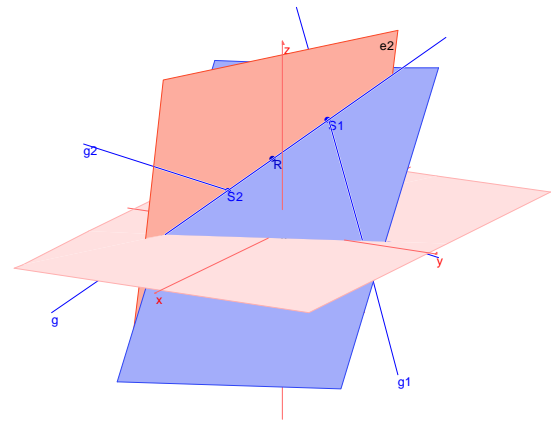


Fig. 4 (Visualization of the line to be found)

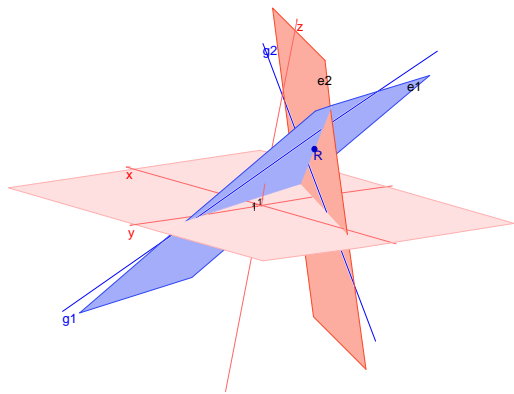


Fig. 3.2 (The scenery from behind)

g has to be the intersection line of these two planes; which is constructed by $e1$ and $e2$ (solution is visualized in fig. 4)

We can see g in its parametric form; it goes through the point $(-4,5,9)$ and has the direction vector $(-3.08, 2.93, 2.36)$. Corresponding to it we construct the intersection points $S1, S2$ (For example in fig. 5: output of $S1$ with exact coordinates). Finally we measure the intersection angle between $g1$ and g (Output fig. 6) resp. $g2$ and g . – In figure 6 you can see also the comfortable register for administration (recording, selection, composition and output) of graphical objects, in which you can switch on /off the visualization of objects resp. designation of objects in the column “s” resp. “B”.



Art	Name	s	B
<input checked="" type="checkbox"/>	R	<input checked="" type="checkbox"/>	<input checked="" type="checkbox"/>
<input checked="" type="checkbox"/>	g1	<input checked="" type="checkbox"/>	<input checked="" type="checkbox"/>
<input checked="" type="checkbox"/>	g2	<input checked="" type="checkbox"/>	<input checked="" type="checkbox"/>
<input checked="" type="checkbox"/>	e1	<input checked="" type="checkbox"/>	<input checked="" type="checkbox"/>
<input checked="" type="checkbox"/>	e2	<input checked="" type="checkbox"/>	<input checked="" type="checkbox"/>
<input checked="" type="checkbox"/>	g	<input checked="" type="checkbox"/>	<input checked="" type="checkbox"/>
<input checked="" type="checkbox"/>	S1	<input checked="" type="checkbox"/>	<input checked="" type="checkbox"/>
<input checked="" type="checkbox"/>	S2	<input checked="" type="checkbox"/>	<input checked="" type="checkbox"/>

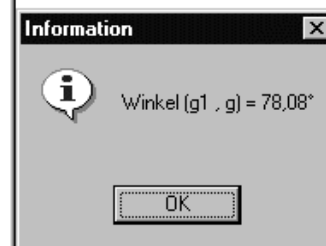


Fig. 5 (Output of a point of intersection)

Fig. 6 (Output of an intersectional angle)

Finally we change the lines $g1, g2$ in a way so that they e.g. intersect. Then the solution of the task is trivial: g is the connection line of the intersection point S with R

(Fig. 7). Configurations for (different) parallel straight lines g_1, g_2 : in case R is in the plane which is built by g_1 and g_2 , there are infinite straight lines g , otherwise there is no solution.

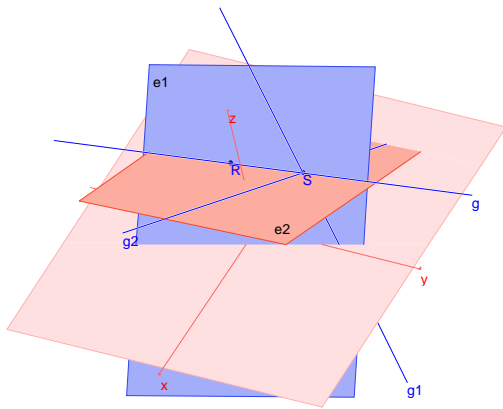


Fig. 7 (Variation of the task with its solution)

The following diagram illustrates the working with DreiDGeo. Now we have to clear up how to calculate the solution of our task using an appropriate computer tool.

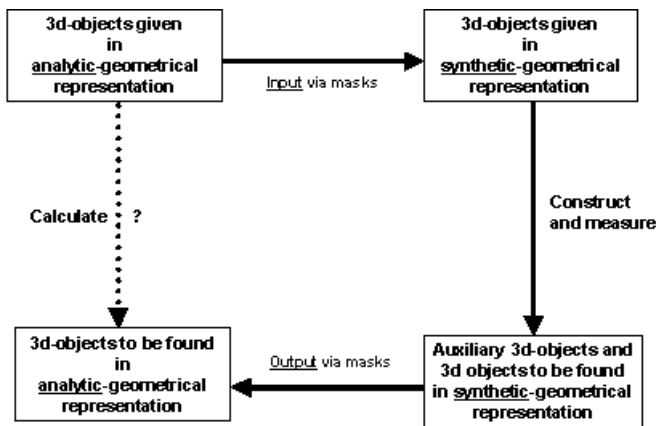


Diagram (Working with DreiDGeo)

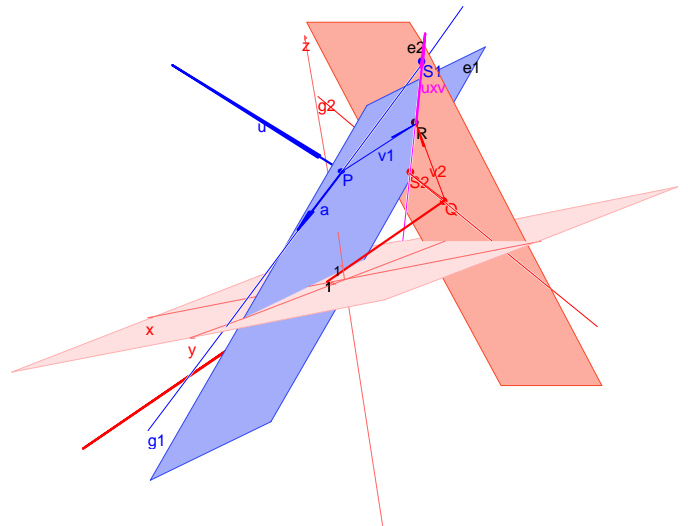
2.2 Computer algebraic treatment

We describe an appropriate handling method in DERIVE; which is carried out similar to other menu controlled assistant programmes as there is for example MATHCAD and which are able to be used with other kinds of general calculations:

- (I) Solving of the general task
- (II) Specification of the general task with given data of a special task
- (III) Discussion of solvability.

The solution of our task is demonstrated in a printout (printout 1.1/2) which we commentate like that:

The lines 1-22 contain the general solution. (The vectors are contextually symbolized by small latin letters; “CROSS” stands for “cross product”; the scalar product is built with an “.” and the absolute value with “ABS”.)



Objekte			
Art	Name	s	B
	R	<input checked="" type="checkbox"/>	<input checked="" type="checkbox"/>
	g1	<input checked="" type="checkbox"/>	<input checked="" type="checkbox"/>
	g2	<input checked="" type="checkbox"/>	<input checked="" type="checkbox"/>
	e1	<input checked="" type="checkbox"/>	<input checked="" type="checkbox"/>
	e2	<input checked="" type="checkbox"/>	<input checked="" type="checkbox"/>
	g	<input checked="" type="checkbox"/>	<input checked="" type="checkbox"/>
	S1	<input checked="" type="checkbox"/>	<input checked="" type="checkbox"/>
	S2	<input checked="" type="checkbox"/>	<input checked="" type="checkbox"/>
	P	<input checked="" type="checkbox"/>	<input checked="" type="checkbox"/>
	Q	<input checked="" type="checkbox"/>	<input checked="" type="checkbox"/>
	r-p(v1)	<input checked="" type="checkbox"/>	<input checked="" type="checkbox"/>
	v1	<input type="checkbox"/>	<input type="checkbox"/>
	r-q(v2)	<input checked="" type="checkbox"/>	<input checked="" type="checkbox"/>
	v2	<input type="checkbox"/>	<input type="checkbox"/>
	a	<input type="checkbox"/>	<input type="checkbox"/>
	b	<input type="checkbox"/>	<input type="checkbox"/>
	u	<input type="checkbox"/>	<input type="checkbox"/>
	v	<input type="checkbox"/>	<input type="checkbox"/>
	Pf1(a)	<input checked="" type="checkbox"/>	<input checked="" type="checkbox"/>
	Pf2(b)	<input checked="" type="checkbox"/>	<input checked="" type="checkbox"/>
	Pf3(u)	<input checked="" type="checkbox"/>	<input checked="" type="checkbox"/>
	Pf4(v)	<input checked="" type="checkbox"/>	<input checked="" type="checkbox"/>
	uxv	<input type="checkbox"/>	<input type="checkbox"/>
	Pf5(uxv)	<input checked="" type="checkbox"/>	<input checked="" type="checkbox"/>

Fig. 8 (Visualization of objects to be used for calculation) The idea of solving consists in the description of the wanted lines g and R and of an direction vector of g , which you get as cross product $\vec{u} \times \vec{v}$ of the normal vectors \vec{u}, \vec{v} of the auxiliary planes e_1, e_2 , which are put on with the point-normal-form in line 10/11.

The illustration resp. the developing of the algebraic way of solution can be done in DreiDGeo (fig.8) by extension of the graphical solution configuration of the special task (fig. 4)

In line 15 the point-direction form of the intersection line is built out of the normal vectors of the auxiliary planes. The resolution of these equations in line 18 resp. 19 to λ resp. μ with the automatic solution, gives the intersection points of the line to be found with the given ones.

The insertion of general coordinates and components from the lines 5,6 and 8 leads due to "simplification" to more extensive terms for the components of the vector \vec{s} , the parameter of the intersection point λ, μ , which we

```
#1: "Find a line which is concurrent with two lines and a point given."
#2: "Equations of lines g1, g2"
#3: x := p + λ·a
#4: y := q + μ·b
#5: [p := [p1, p2, p3], a := [a1, a2, a3]]
#6: [q := [q1, q2, q3], b := [b1, b2, b3]]
#7: "Point R"
#8: r := [r1, r2, r3]
#9: "Plane e1 determined by R and g1 und e2 by R and g2"
#10: (s - r)·u = 0
#11: (s - r)·v = 0
#12: "Normal vectors of planes e1 and e2"
#13: u := CROSS(r - p, a)
#14: v := CROSS(r - q, b)
#15: "Intersection of e1 and e2: equation of the line g to be found"
#16: s = r + σ·CROSS(u, v)
#17: "Calculating intersectional points of g with g1 and g with g2"
#18: ((p + λ·a) - r)·v = 0
#19: ((q + μ·b) - r)·u = 0
#20: "Calculating angle between g and g1 respective g2"
#21: w1 = ACOS( (CROSS(u, v)·a) / (|CROSS(u, v)|·|a|) )
#22: w2 = ACOS( (CROSS(u, v)·b) / (|CROSS(u, v)|·|b|) )
```

Printout 1.1 (Development of a general solution)

The lines 24-41 contain the solution of the specific task. In the lines 24-26 the numbers are assigned to the appropriate variables (the advantage of that is, that if you change those values, you can solve all of the specific tasks). \vec{s} is in normal style: $\vec{s} = (-4, -5, 9) + 2\sigma(-514, 489, 411)$. The values of the angles are approximated. All the solutions confirm the "computer-graphical" solution.

Altogether you can manage in DERIVE with the options "simplify", "substitute", "solve" and

just give the one of λ due to space-problems (printout 2), and the intersection angles w_1 and w_2 . If you use the determinants, you could represent those terms more clearly of course (we do not go into the possibility of definition of macros for calculations here).

The intersection angles between the lines are defined by the scalar product and the Arcuscosinus, whereby the angle are calculated as acute ones (lines 21/22).

"approximate" for determining the general or special solutions.

Examination of Solvability: if the intersection point of g with g_1 resp. g_2 doesn't exist, that means, if the equation in line 18 resp. 19 is reshaped to $(\vec{p} - \vec{r}) \cdot \vec{v} + \lambda \vec{a} \cdot \vec{v} = 0$ resp. $(\vec{q} - \vec{r}) \cdot \vec{u} + \mu \vec{b} \cdot \vec{u} = 0$, has no solution, then the task has no solution. That is the case for $\vec{a} \cdot \vec{v} = 0$ resp. $\vec{b} \cdot \vec{u} = 0$, that means, if the direction vector of the line g_1 resp. g_2 is parallel to e_2 resp. e_1 . If you choose appropriate values for coordinates resp. components those cases can be illustrated computer-graphically in DreiDGeo.

```

#23: "Data and results of a concrete task"
#24: [p1 := -6, p2 := 6, p3 := 8, a1 := 2, a2 := 3, a3 := -3]
#25: [q1 := -3, q2 := -8, q3 := 3, b1 := 5, b2 := -2, b3 := 3]
#26: [r1 := -4, r2 := -5, r3 := 9]
#27: "Line g to be found:"
#28: s = [- 1028 · σ - 4, 978 · σ - 5, 822 · σ + 9]
#29: "Intersectional point of g1 with g:"
#30: [λ = - 167 / 90]
#31: [λ = -1.855]
#32: [- 437 / 45, 13 / 30, 407 / 30]
#33: [-9.711, 0.4333, 13.56]
#34: "Intersectional point of g2 with g:"
#35: "Schnittpunkt von g2 mit g:"
#36: [μ = 81 / 109]
#37: [μ = 0.7431]
#38: [78 / 109, - 1034 / 109, 570 / 109]
#39: [0.7155, -9.486, 5.229]
#40: "Angle between g and g1 respective g2:"
#41: w1 = 101.9
#42: w2 = 117.2
    
```

Printout 1.2 (Specific solution)

$$\lambda = \frac{b_1 \cdot (p_2 \cdot (q_3 - r_3) + p_3 \cdot (r_2 - q_2) + q_2 \cdot r_3 - q_3 \cdot r_2) - b_2 \cdot (p_1 \cdot (q_3 - r_3) + p_3 \cdot (r_1 - q_1) + q_1 \cdot r_3 - q_3 \cdot r_1) + b_3 \cdot (p_1 \cdot (q_2 - r_2) + p_2 \cdot (r_1 - q_1) + q_1 \cdot r_2 - q_2 \cdot r_1)}{a_1 \cdot (b_2 \cdot (q_3 - r_3) + b_3 \cdot (r_2 - q_2)) - a_2 \cdot (b_1 \cdot (q_3 - r_3) + b_3 \cdot (r_1 - q_1)) + a_3 \cdot (b_1 \cdot (q_2 - r_2) + b_2 \cdot (r_1 - q_1))}$$

Printout 2 (Point of intersection S1: general term of parameter)

2.3 Integration of the "manual" treatment

Next to the conventional manual standard for handling analytic-geometrical tasks comes now another standard, which is represented by the computer-supported treatment of a task. – There is the question after the relation of these standards to each other, considering the missing possibility of computer usage with appropriate software by students in class. The teacher can at least computer-graphically demonstrate the specific task with its solution, according to this illustration the students carry out the solution manually. After that they work manually at the beginning of the general task. The teacher demonstrates the computer-algebraic solution and its specification.- and perhaps one day the computer room is vacant and there are enough computers so that the students can work in pairs with DreiDGeo. But what use is such a computerized mathematics lesson, if the common examinations do not offer adequate possibilities to make use of the acquired competence for solving mathematical problems with appropriate computer tools?

3. Final comments

Comment 1

The tool DreiDGeo is no dynamic geometrical system, because we can't produce the graphical basic object directly nor can we manipulated the produced ones directly, besides the rotation of a configuration. Such prototypical systems, which can also be used in analytical geometry in IE³, are just finished for Macintosh (3D-Geometer, Klemenz 1994/99) – with the fault of clear perception spatial objects in depth. MiniGeometer (<http://geosoft.ch>), derived from 3DGeometer, a Java-Applet for interactive constructing in space, contains no improvements of the mentioned problem, besides it has a more complicated user-surface. A promising Macintosh-development, the Cabri-Géomètre-3D, is not published yet and not available for Windows.

Comment 2

The usage of menu-controlled computer-tools, such as DreiDGeo and DERIVE, should be favoured to appropriate command-driven systems, because we know from experience that the knowledge of relative many commands with its syntax is a mnemotechnical

problem to the temporary user, which is harder to control than knowing the options of a menu-controlled tool which is designed for dialogs. It seems to be that the usage of commando-driven systems is rather an obstacle to the integration of “computer” into mathematic lessons, even in upper secondary education.

Comment 3

Even when students, nowadays missing an genuine integration of computers into the curriculum, can't use the possibilities of discovering and solving tasks with computer tools, at least the teacher might have the opportunity to use such tools for producing and designing tasks.

Comment 4

To show the variety of methods created by computer tools, the concerned tools should be known more or less. On one hand the actual curriculum can't provide those conditions because of time and on the other hand the learning of different user-surfaces can't be expected from the student. The visualization of 3d-tasks of analytic geometry in DERIVE is because of its pixel graphics not satisfying; therefore you can't do this topic without tools such as DreiDGeo. A modular, multifunctional and adaptive computer tool for the mathematics lessons, whose user-competence is learned in a long-term learning process with mathematical topics according to grades could induce a better integration in mathematics lessons. – But at that time such a tool is not available.

Comment 5

If we think that the main topic of analytic geometry, that is the algebraic description of geometric objects and the algebraic solution of geometrical problems, has a general educational character, we have to ensure that the main topic becomes more attractive and the solutions more

easier for students and teachers. – We hope that the above made computer-aided treatment of analytic-geometrical tasks in IE³ helps to improve the learning and teaching in this topic.

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