

# Combination of traditional and computer based tools in mathematics education

Tünde Berta, Budapest (Hungary)

**Abstract:** In this paper I analyzed the advantages of the combined use of traditional and computer-based demonstration, which are connected to the solution of a difficult space-geometry problem. The distinguished feature of the paper lies in the combined use of 3-dimensional animations, interactive worksheet and traditional models. The interactive worksheet contains 3-dimensional animations created in DGS. The traditional worksheet harmonizes the traditional edge-model and the interactive worksheet.

**Zusammenfassung:** In diesem Artikel wird anhand eines raumgeometrischen Problems der Vorteil des Zusammenwirkens von zwei verschiedenen Anschauungsmitteln, einem traditionellen und einem computerunterstützten, dargestellt. Dazu wurden für dieses Problem sowohl spezielle interaktive Computer-Arbeitsblätter als auch traditionelle Modelle entwickelt. Die interaktiven Arbeitsblätter wurden mit Hilfe eines DGS erstellt und ermöglichen auch dreidimensionale Animationen. Auf traditionellen Arbeitsblättern wird das Problem dargestellt, und es werden Hinweise zum Umgang mit dem Computer und den anderen Arbeitsmitteln gegeben. Dadurch wird das Zusammenspiel dieser beiden Arbeitsmittel gewährleistet. Natürlich können die Schüler auf diesem Arbeitsblatt auch ihr Vorgehen protokollieren.

**ZDM classification:** B40, C30, D50, G40, U60

## 1. Didactical arguments

### 1.1 Different levels of representation

Some basic papers about mathematical representations play a considerable role in my research. For the topic of my actual presentation the work of Brunner (e.g., [1]) is highly important. In his sense we consider 3 levels of mathematical representations:

- **Enactive representation**  
Children acquire new knowledge through concrete manual activity and manipulations.
- **Iconic representation - demonstration**  
New knowledge will be obtained by figures or through imagined situations.
- **Symbolic representation**  
This means acquisition of knowledge through mathematical notions and symbols.

All of these 3 different levels of representations are present in any phase of the teaching process, but they play different roles.

These levels of mathematical representations cannot be separated during a teaching process in a classroom. The experiments in our research group show, that the simultaneous use of different levels is useful. The learning process of the children can be made more effective by a conscious change of these modes.

Computer-based tools in combination with the traditional teaching and visualization connect these 3 levels of mathematical representations. Computer-based tools make differentiation and individualization possible, which are very important in mathematics education.

### 1.2 Some aspects of teaching of geometry in Hungary

The teaching of geometry has a long tradition in the Hungarian mathematics teaching. Geometry is taught predominantly using traditional tools. It is a particular field of mathematics in which the combination of my three presented tools could be effectively utilized. I would like to draw attention to the importance of stereometry in the teaching of geometry.

One of the most important goals of teaching of geometry is the correct interpretation and reconstruction of 3-dimensional geometrical objects and their 2-dimensional figures. We can develop the geometrical space-conception of persons through learning and training, but it is influenced by other factors, cultura for example.

An emphasized aim in Hungarian mathematics teaching is to develop problem solving skills. Among problem solving strategy (e.g., [3]) heuristics (analogy, generalization, specialization, looking for a similar known problem, transfer the problem in a new form) occupies a central place. Analogy between 2- and 3-dimensional consideration has an outstanding role to play in the geometric problem solving. There are known problems, which can be solved with help of a planar section or projection. Some questions in the plane become easier through embedding it into 3-dimensional neighborhood (for example, center of homothety of tree circles, e.g., [6]).

The high-school geometry exercises book of Horvay K. and Reiman I (e.g., [2]). treats plane and space geometry in different chapters, but there are formulated the analog pairs too (for example straight line – plane, circle – sphere, triangle – tetrahedron, ...). These chapters include proof exercises, calculating exercises, exercises of intersection (for example, in the chapter on tetrahedra there are more than one hundred problems). The strict separation makes it more difficult for students to find the connection between similar types of problems. It is of high importance for teachers to provide students with examples highlighting the cross-connection between problems and their methods of solution.

### 1.3 The interactive worksheet as an instrument for integration

The main topic of my research is to investigate, how the education of geometry can be supported and linked by use of computers.

My hypothesis is that for the efficiency of the problem solving process it is useful to combine the “help” of the

three different levels of representation.

In this paper I show an example with combination of different visual tools, which enables students to develop space-conception and to solve a complicated problem. I choose a problem of two intersecting tetrahedra, whose solution is not simple.

**2. An example**

*Reflect a regular tetrahedron of edge a in the midpoint of one of its altitudes. Calculate the volume of the intersection of these tetrahedra.*

**2.1 Mathematical analysis of the problem**

According to Hungarian traditions I show a synthetic solution of this problem.

**First step** – find points lying in one plane:

Reflect vertex A in the midpoint of altitude starting from A (median at the same time). Vertex A\* obtained by reflection from A, is the center of the triangle BCD. Triangle B\*C\*D\* with center A is the reflection of the triangle BCD in O. The intersection of faces BCD and B\*C\*D\* is void.

**Second step** – describe the intersection:

The intersection is bounded by the planes of three faces meeting in A, and by those meeting in A\*. It follows from the properties of the reflection that the planes are pairwise parallel, consequently they bound a parallelepiped: A and A\* are opposite vertices and O is the center. The vertices are intersection points of the edges starting from A, respectively A\* of the tetrahedra with the corresponding face of the other tetrahedron.

AB and A\*B\* are images of each other under reflection, therefore they are parallel and lie in the A\*BA plane.

**Third step** – find points of intersection and the complete intersection:

The A\*BA plane intersects the triangle BCD in the median BA\* consequently it contains the midpoint F of edge CD. The intersection of the same plane with the triangle ACD is the median AF, the edge A\*B\* will consequently intersect the triangle ACD in a point X of the median AF. It is clear that A\*B\* ⊥ AB and A\* is the point dividing BF into three parts and it lies closer to F. Therefore edge A\*B\* will intersect the triangle ACD in its median. It follows from the above consideration that

$A^*X = \frac{a}{3}$ . The same is true in the case of all further edges starting A\* and A, respectively.

**Fourth step** – calculate the volume of this parallelepiped: The edges will intersect the corresponding faces of the other tetrahedron in the centers of gravity of the faces and one third of each edge will be contained in the tetrahedron. Each face of the intersection parallelepiped

will be a rhomb of edge length  $\frac{a}{3}$  and acute angle 60°.

The area of each rhomb is  $\frac{a^2\sqrt{3}}{18}$  (it can be subdivided

into two regular triangles of side length  $\frac{a}{3}$ ).

The altitude of the parallelepiped will coincide, for example, with the distance of A\* from the plane ACD.

The altitude of the parallelepiped is  $m = \frac{a}{3} \cdot \sqrt{\frac{2}{3}}$ ,

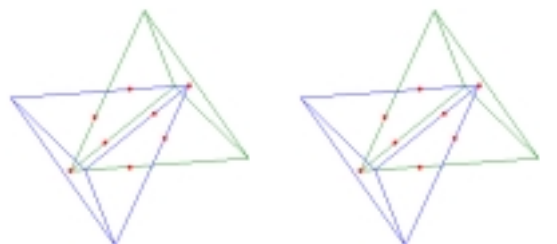
therefore the volume of it is  $V = \frac{a^3\sqrt{2}}{54}$ .

**2.2 Training of space-perception**

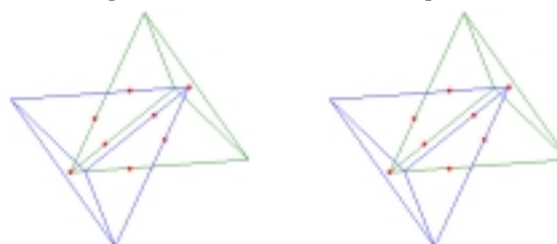
It is very difficult for the pupils to imagine this situation. Most of them find it difficult to imagine it in space. That's why I show as a first part of this worksheet a training of space-perception.

In this part the student has to draw into these figures according to visibility

a) union of tetrahedra      b) intersection of tetrahedra



c) one (green) of the tetrahedra without the intersection part      d) second (blue) tetrahedron without the intersection part.



As a help to students I show them on these figures intersecting points on their lines of intersection.

I tested it on university students, and I saw that to solve the training of space-perception correctly was a problem for them. I found many of their solutions incorrect. This exercise is easier to solve if I show a model of these tetrahedra and their intersection. This model also helps to solve the original problem of intersecting tetrahedra.

After I showed the model of tetrahedra and their intersection, every student could solve this task correctly. In this part we have a combination of two mathematical representations. The model is an enactive representation and these figures constitute the iconic representation.

### 2.3 An interactive worksheet


In the second part of the solution we apply a computer, an interactive worksheet, which has been prepared as a web page in html combined with animations made in Cabri.

The page consists of four parts:

#### First part - A problem (Fig.1)

If the student is able to solve the problem without any other help then the manual model, they can start to solve it in their exercise book.

#### Second part – Help (Fig.2)

If somebody needs more help then he can find it on this page by the click on the help button .

#### 2.3.1 A Cabri animation

As a help I show them a Cabri animation which gives them a hint to solve this problem (Fig. 3).

Éva Vásárhelyi created a macro for 3-dimensional constructions. With help of this macro we can construct an orthogonal projection of the Cartesian system of co-ordinates, which can be tilted and rotated – in order to represent any position relative to the screen. Using vector addition and the 3-dimensional co-ordinates of a point it is possible to combine the direct construction on the screen with the algebraic method.

The movement with help of this tool makes that the 2-dimensional construction we can see in 3-dimension.

This Cabri animation is based on the 4 steps of my presented solution.

#### Third part – Solution (Fig.4)

Finally I give one of the possible solutions on this web page.

#### Fourth part - The final result (Fig.5)

### 3. Didactical reflection

On this example we could see a combination of all 3 levels of mathematical representations, a combination of worksheet, manual model and animation. It was also an example of differentiation and individualization of studying, because the student could choose the way how they wanted to solve this problem. They could select different types of help and combinations of them (manual model, animation and full solution) according to their abilities, their wish.

With the use of concrete models my intention was to enable students to grasp the essence of the exercise as well as to enhance their spatial perception. The only drawback is that we discussed only one case.

Computer can easily handle the remaining possibilities. The advantage of the animation is its dynamism and flexibility, by means of which a better understanding of the problem is possible.

The two models are complementary: they summarize concrete experience, and visualize various cases and make further experimentation possible. The two models should be similar, otherwise they would be stored at two isolated places in the human brain. Both tools effect the short-range memory, the duration of which is about 15 minutes. The independent role of Cabri worksheet is important. The same applies to the tools of checking the solution of the problem. The combination of these tools will carry the new knowledge into the domain of long-range memory.

Experimentation in a group of Third-year university students (future teachers of mathematics) reconfirmed the practicality of the parallel application of the tools enumerated above. Simultaneous application of the manual model and animation made in a significant manner of the problem easier for them. At the same time the simultaneous application provided sufficient hints and suggestions to the solution.

We hope that the participants of this experiment will apply these teaching methods when they take up the teaching profession.

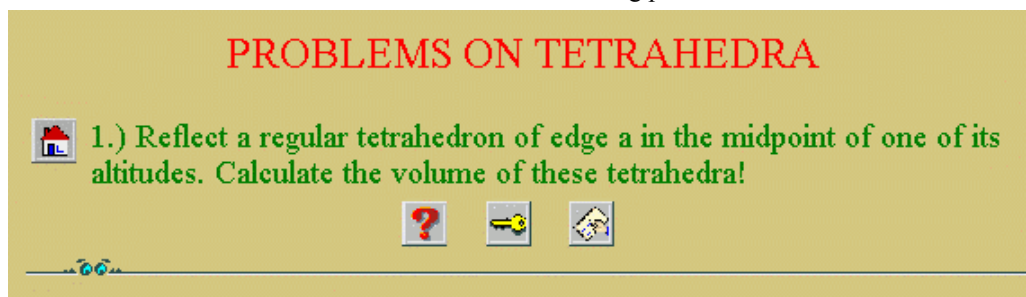


Figure 1. Problems on tetrahedra

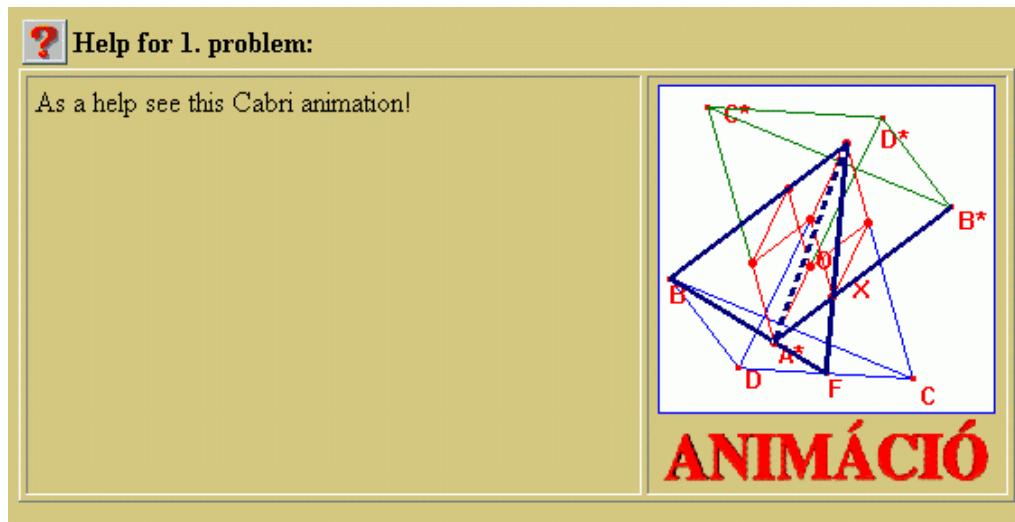
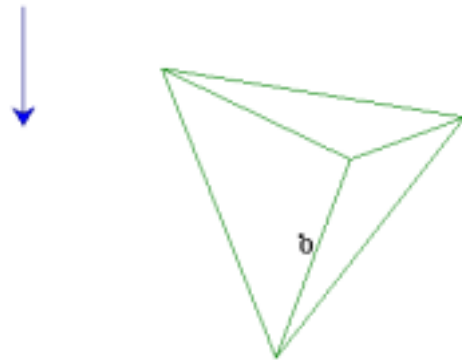
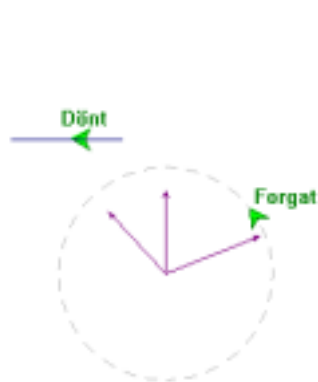
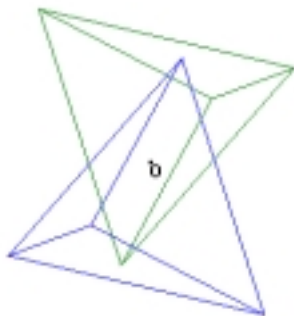


Figure 2. Help

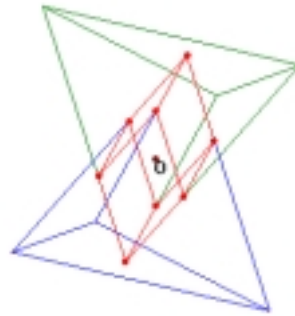
- What properties of a tetrahedron will be preserved and which of them will change under projections?
- How can one construct the altitude and its midpoint?



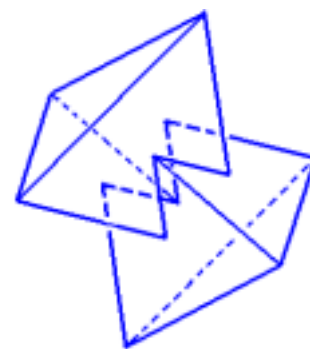
Reflection of a regular tetrahedron in the midpoint of one of its altitude.



The projection of tetrahedra



Points of intersection and interpenetration of the surface



The union of tetrahedra

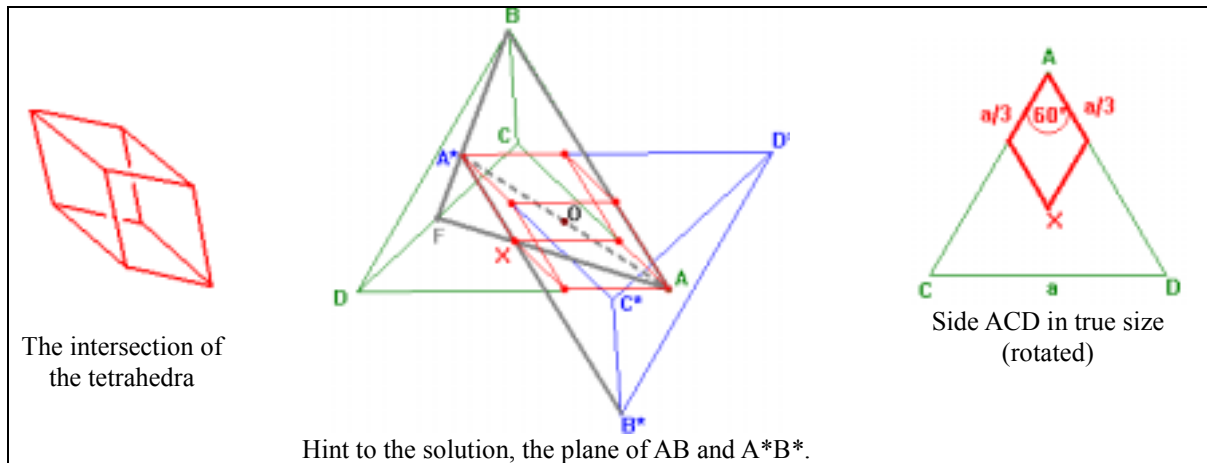


Figure 3. Steps of animation

**Key Solution of 1. problem:**

**Solution:** Reflect vertex A in the midpoint of altitude starting from A (median at the same time). Vertex A\* obtained by reflection from A, is the center of the triangle BCD. Triangle B\*C\*D\* with center A is the reflection of the triangle BCD in O. The intersection of faces BCD and B\*C\*D\* is void. The intersection is bounded by the planes of three faces meeting in A, and by those meeting in A\*. It follows from the properties of the reflection that the planes are pairwise parallel, consequently they bound a parallelepiped: A and A\* are opposite vertices and O

Figure 4. Solution

**The final result of 1. problem:** The altitude of the parallelepiped is  $\frac{a}{3}\sqrt{\frac{2}{3}}$ , therefore the volume of it is  $V = \frac{a^3 \cdot \sqrt{2}}{54}$ .

Figure 5. The final result

**References**

- [1] Bruner, J.S.: Der Prozess of der Erziehung. Schwann, Düsseldorf, 1970
- [2] Horvay K. and Reiman I.: Geometriai feladatok gyűjteménye I., Tankönyvkiadó, Budapest, 1987
- [3] Pólya, Gy.: Induction und Analogie in der Mathematic, Birkhauser verlag basel und Stutgart, 1962
- [4] Pólya, Gy.: A problémamegoldás iskolája I-II. Tankönyvkiadó, 1968
- [5] Vásárhelyi, É. (1999b): Combination of traditional and computer based tools as a strategy for problem solving. In: Creativity and Mathematics Education (Tagungsband). Münster, 163-166.
- [6] Vásárhelyi, É.: In der Ebene oder im Raum?, Shiftenreihe Didaktik der Mathematik Klagenfurt, 22, 1994

**Author**

Berta, Tünde, Department of Methodology of Teaching of Mathematics, Eötvös Lóránd University, Pázmány Péter sétány 1/C, 1117 Budapest, Hungary  
 e-mail: [btunde@cs.elte.hu](mailto:btunde@cs.elte.hu)