

Assessing and Understanding U.S. and Chinese Students' Mathematical Thinking: Some Insights from Cross-National Studies¹

Jinfa Cai, Newark, DE (USA)

Abstract: If the main goal of educational research and refinement of instructional program is to improve students' learning, it is necessary to assess students' emerging understandings and to see how they arise. The purpose of this paper is to address issues related to assessments of students' mathematical thinking in cross-national studies and then to discuss the lessons we may learn from these studies to assess and improve students' learning. In particular, the issues related to assessing U.S. and Chinese students' mathematical thinking were discussed. Then, this paper discussed the findings from two studies examining the impact of early algebra learning and teachers' beliefs on U.S. and Chinese students' mathematical thinking. Lastly, the issues related to interpreting and understanding the differences between U.S. and Chinese students' thinking were discussed.

Kurzreferat: Wenn Sinn und Zweck von Unterrichtsforschung und -veränderung eine Verbesserung des Lernen von Schülerinnen und Schülern ist, dann ist es notwendig zu untersuchen, wie Verständnis entsteht und wie es sich entwickelt. Ziel dieser Arbeit ist, Fragen im Zusammenhang mit der Erhebung mathematischen Denkens in internationalen Studien anzusprechen und zu diskutieren, wie diese Erkenntnisse zur Beurteilung und Verbesserung des Lernens beitragen können. Insbesondere werden diese Aspekte anhand der Untersuchung des mathematischen Denkens von nordamerikanischen und chinesischen Schülerinnen und Schüler diskutiert. Der Beitrag betrachtet die Ergebnisse zweier Studien über den Einfluss früher Algebra-Lernerfahrungen und der Einstellung von Lehrpersonen auf das mathematische Denken nordamerikanischer und chinesischer Kinder. Abschließend werden Fragen der Interpretation und des Verständnisses der Unterschiede im Denken nordamerikanischer und chinesischer Schülerinnen und Schüler erörtert.

ZDM-Classification: C30, D10, C70

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1. Introduction

Cross-national studies have provided a large body of knowledge about how children do mathematics in the context of the world's varied educational institutions, and about what cultural and educational factors might influence their learning of mathematics (Kaiser, Luna, Huntley, 1999; Ma, 1999; Robitaille & Garden, 1989; Stevenson et al., 1990; Stigler & Hiebert, 1999; U.S. Department of Education, 1996). However, because of the complexity of assessing and interpreting cross-national differences, we are just beginning to understand how we should use findings from cross-national studies to improve education around the world. Since the main goal of educational research and refinement of instructional program is to improve the learning opportunities for all students, it is necessary to assess students' emerging understandings and to see how they arise. The more information teachers obtain about what students know and think, the more opportunities they create for student success (Darling-Hammond, 1994).

The purpose of this paper is to address issues related to assessment methods and research findings from cross-national studies in mathematics, and then to discuss the lessons we may learn from these studies for the purpose of conducting classroom assessments and improving students' learning. First, the issues related to assessing U.S. and Chinese students' mathematical thinking were discussed. Based on an emerging understanding from a series of cross-national studies involving U.S. and Chinese students, this paper shows how a variety of assessment tasks and detailed analyses can help examine students' mathematical thinking and reasoning. Then, this paper reviewed the findings from two studies examining the impact of early algebra learning and teachers' beliefs on U.S. and Chinese students' mathematical thinking. Lastly, the issues related to interpreting and understanding the differences between U.S. and Chinese students' thinking were discussed. In the discussion of each issue, a great effort is made to use available research evidence to address the issue, and then research needed to address the issue more completely is suggested.

2. Assessing U.S. and Chinese students' mathematical thinking

There is no consensus on what mathematical thinking is, but it is widely accepted that there are many aspects of mathematical thinking where information could be gathered (Ginsburg, 1983; Schoenfeld, 1997; Sternberg & Ben-Zeev, 1996). Researchers identified characteristic features of mathematical thinking (Krutetskii, 1976; Sternberg & Ben-Zeev, 1996). For example, Krutetskii (1976) found that able and less able students used different styles of mathematical thinking in their problem solving. Able students are more likely than less able students to use generalizations in mathematical problem solving.

In cross-national studies, mathematical thinking has generally been assessed based on a set of tasks. Thus, the heart of international assessment comparisons is the set of tasks on which mathematical thinking is to be assessed

(Mislevy, 1995). It is well-documented that having routine problem solving skills does not necessarily mean having non-routine problem solving skills (Hatano, 1988; Steen, 1999; Sternberg, 1999). To better understand the international differences in students' learning and mathematical thinking, it is desirable to use various types of assessment tasks, thereby measuring different facets of mathematical thinking. It would be informative to include both tasks that may better measure students' high-level thinking skills and tasks that may better measure students' procedural knowledge and routine problem-solving skills.

In addition, information about how students approach the solution of a given problem is more important than whether or not they are able to recognize the correct solution. To understand students' mathematical thinking, it is important to examine critical cognitive aspects of problem solving, such as solution strategies, mathematical misconceptions/errors, mathematical justifications, and representations. The examination of solution strategies can provide qualitative aspects of information regarding students' mathematical thinking and reasoning (Sternberg, 1991). Learning can be viewed as a qualitative change in a student's way of experiencing and understanding of mathematics (Marton & Ramsden, 1988). The study of misconceptions provides information with respect to the characteristics and treatment of misconceptions and the extent to which students modify their misconceptions. The examination of solution justifications and representations reveals the ways in which students process a problem and reflects the ways in which students communicate their mathematical ideas and thinking processes.

In the past several years, Cai (e.g., 1995, 1998, 2000a, 2000b; Cai & Hwang, 2002) used a variety of assessment tasks and analyzed cognitive aspects of problem solving to examine U.S. and Chinese students' mathematical thinking beyond computation, correctness, and problem solving.

2.1 Assessments beyond computation

In particular, Cai has used both routine and non-routine problem-solving tasks to examine U.S. and Chinese students' mathematical thinking (e.g., Cai, 1995, 1998, 2000a). Routine problem-solving tasks were selected based on Mayer's model of solving word problems. In his model, Mayer (1987) identified four cognitive component processes of solving routine mathematical problems: translation, integration, planning, and execution. Non-routine problem-solving tasks were selected from a pool of open-ended problems. It should be indicated that these studies did not use national representative samples, instead used roughly comparable samples. Even though national representative samples were not used, but these studies allowed for the examination of U.S. and Chinese students' performance beyond computation and the trend of performance differences between the two nations is interesting and significant.

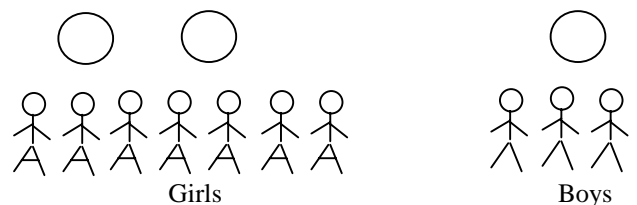
For example, in one of the studies (Cai, 2000a), a sample of 310 sixth-grade Chinese students and a comparable sample of 232 sixth-grade U.S. students were

assessed using a set of 31 multiple-choice items for the component processes of solving routine problems: 6 items for the translation component, 6 items for the integration component, 6 items for the planning component, and 13 items for the execution component. In order to cover different operations (addition, subtraction, multiplication, and division) on different types of numbers (whole numbers, decimals, and fractions), the execution component required more items than each of the other three components.

In addition, Cai (2000a) also selected a set of 12 open-ended tasks to assess non-routine problem-solving skills (Silver & Lane, 1992). Half of these non-routine problems were process-constrained tasks. A process-constrained task can be solved by applying a "standard algorithm," but students do need to figure out what algorithm to apply and actually carry out a procedure or a set of procedures to solve the problem. The Pizza Ratio Problem is an example of a process-constrained non-routine problem (See Figure 1). To solve this problem, students need to set up a ratio for the number of pizzas and the number of people and then compare the fraction representations. The other half of the non-routine problems were process-open tasks. A process-open task cannot be solved by following a "standard algorithm." A task that is process-open may not require the execution of a procedure or a set of procedures; instead, it may require an exploration of the problem situation and then the solution to the problem. The Block Pattern Problem (See Figure 1) is an example of a process-open problem. This pattern problem is process-open since there is no "algorithm" students can follow to solve it. Instead, students need to use inductive reasoning to determine the rule or regularity.

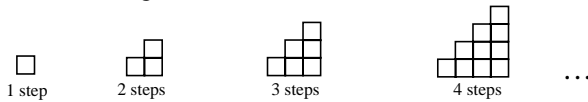
Figure 1:
Examples of Process-constrained and Process-open Non-routine Problems

THE PIZZA RATIO PROBLEM (process-constrained):
Here are some children and pizzas. 7 girls share 2 pizzas equally and 3 boys share 1 pizza equally.



- Does each girl get the same amount as each boy? Explain or show how you found your answer.
- If each girl does not get the same amount as each boy, who gets more? Explain or show how you found your answer.

THE BLOCK PATTERN PROBLEM (process-open):
Look at the figures below.



- A. How many blocks are needed to build a staircase of 5 steps? Explain how you found your answer.
- B. How many blocks are needed to build a staircase of 20 steps? Explain how you found your answer.

Table 1 shows the mean scores of the U.S. and Chinese students on each component of the routine problem-solving tasks as well as mean scores on both process-constrained and process-open non-routine problems. Chinese students outperformed U.S. students on execution, translation, and planning components, but there is no statistically significant difference between two samples on the integration component. For the non-routine problem solving, the Chinese sample had a higher mean score than did the U.S. sample on the process-constrained problems, but the U.S. sample had higher mean score than did the Chinese sample on the process-open tasks. Interestingly, the difference pattern on the two types of non-routine problems for the U.S. sample is different from that of the Chinese sample. For the U.S. sample, the mean score on the process-open tasks is higher than the mean score on the process-constrained tasks. For the Chinese sample, however, the mean score on the process-constrained tasks is higher than the mean score on the process-open tasks.

The results shown in Table 1 suggest the need for examining international differences beyond simple ranking. Most importantly, it reveals that the nature of the tasks is a crucially important factor to consider in international comparisons. Indeed, even comparing the performance of students from different nations on correctness of individual tasks is not too revealing without some exploration of the thinking and methods that led students to their correct answers.

Table 1:
Mean Scores of Chinese and U.S. Students on Each Type of Problems

	US (n=232)	China (n=310)	Possible Score
Execution component* Of Routine Problem Solving (n=13)	6.27 (2.97) ^a	11.36 (1.29)	13.00
Translation Component* of Routine Problem Solving (n=6)	3.30 (1.65)	4.68 (1.14)	6.00
Integration Component of Routine Problem Solving (n=6)	3.74 (1.33)	4.08 (1.21)	6.00

Planning Component* of Routine Problem Solving (n=6)	3.21 (1.08)	5.12 (1.10)	6.00
Process-Constrained* Nonroutine Problem Solving (n=6)	12.85 (7.16)	17.97 (5.19)	24.00
Process-Open* Nonroutine Problem Solving (n=6)	14.70 (6.07)	13.59 (4.81)	24.00

^aNumbers in parentheses are the standard deviations.

* Chinese students have a significantly higher mean score than the U.S. students on this type of tasks

#U.S. students have a significantly higher mean score than the Chinese students on this type of tasks.

2.2 Assessments beyond correctness

It is useful to know the difference between U.S. and Chinese students' solving various problems in terms of mean scores, but such simple comparisons provide little guidance for understanding and improving students' mathematics learning. If cross-national studies are to provide information for seeking ways in which future learning can be supported and enhanced, it is essential for cross-national comparisons to include evidence of students' thinking and reasoning beyond reporting mean scores and percent correct. Several cross-national studies have used open-ended tasks to examine U.S. and Asian students' thinking and reasoning beyond correctness of answers (e.g., Becker, Sawada, & Shimizu, 1999; Cai, 1995, 1998, 2000a; Silver, Leung, & Cai, 1995). The use of open-ended tasks makes it possible to identify students' solution strategies and representations through analyzing their written work to problems. In other words, examining the strategies that students apply and the success of the applications can provide information regarding students' mathematical thinking and reasoning. Their solution representations reflect the ways students process a problem and communicate their mathematical ideas and thinking processes. Previous studies suggested that such an analysis is an appropriate and informative research method to capture students' mathematical thinking and reasoning involved in problem solving. Let's take a look at two examples.



The first example is related to the Pizza Ratio Problem (see Figure 1), in which U.S. and Chinese sixth-grade students were asked to determine if each girl or each boy gets more pizza when seven girls share two pizzas and three boys share one pizza equally (Cai, 2000a). A greater percentage of Chinese students (59%) than U.S. students (42%) provided complete and convincing arguments to justify that each girl gets a different amount than each boy and that each boy gets more than each girl. Table 2 describes these arguments and the percentage of the students who provided each argument. Of those providing complete and convincing arguments, the majority of the Chinese students provided a numerical argument (Argument 1). None or only a few of the

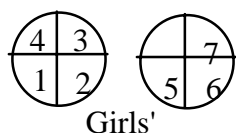
Chinese students used any of the other types of arguments. However, the U.S. students tended to use drawing-oriented types of justifications.

The second example is related to the Block Pattern Problem (see Figure 1). To solve the Block Pattern Problem, students are required to find the number of blocks needed to build staircases of 5 and 20 steps and to explain how they found their answers. About 90% of the U.S. and Chinese students correctly found the number of blocks needed to build a staircase of 5 steps. The percentages of the U.S. and Chinese students (28% and 22%, respectively) who correctly found the number of blocks needed for 20 steps were close. Comparing the two samples of students' solution strategies in these two parts, there were differences in the percentage distributions both for the number of blocks needed to build the staircase of 5 steps and the number of blocks needed to build the staircase of 20 steps. In particular, 61% of the U.S. students actually drew a staircase of 5 steps to find the number of blocks in it, but only 13% of the Chinese students drew the staircase to find the answer. Similarly, a larger percentage of the U.S. students (25%) than the Chinese students (5%) actually drew a staircase of 20 steps to find the answer.

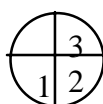
The findings from these examples showed that U.S. students are much more likely than Chinese students to use drawing strategies and representations; Chinese students, on the other hand, are much more likely than U.S. students to use numerical or symbolic strategies and representations. Is it possible that these Chinese students might have used visualization mentally, but they expressed their solutions in non-visual forms (e.g., algebra equations)? On the other hand, is it possible that U.S. students have just used these drawing strategies because teachers told them so and they did not necessarily think visually? In the study (Cai, 2000a), strategies and representations were identified based on an analysis of visible record of students' written work. Although one would expect the close connections between students' external and internal representations, future studies are needed to fully answer these questions. Nevertheless, these two examples show the contributions of assessments beyond correctness to capture students' thinking in international comparisons. Clearly, the analysis of students' solution strategies and representations will not only help us understand if there are international differences, but also help us understand the nature of the international differences.

Table 2
Various Convincing Arguments for the Pizza Ratio Problem

Argument	Description	Percent of Students	
		U.S. (n=42)	China (n=59)
<i>Convincing Argument 1:</i>	Each boy will get 1/3 of a pizza, and each girl will get 2/7 of a pizza. If you compared 1/3 with 2/7, you would know that 1/3 is bigger than 2/7 by transforming them into common fractions decimals.	21	90
<i>Convincing Argument 2:</i>	If there were six girls, each girl and each boy would have the same. But you have 7 girls, so each girl gets less than each boy.	14	2
<i>Convincing Argument 3:</i>	Three girls share one pizza, and another three girls share another pizza. Each of these six girls will get the same amount of the pizza as each boy. But one of the girls has no pizza. So, each boy will get more.	7	2
			
<i>Convincing Argument 4:</i>	Three girls share one pizza and remaining four share one pizza. Each piece that each of the remaining four girls get are smaller than those boys get. So boys get more.	29	2
			
<i>Convincing Argument 5:</i>	Seven Girls get two pizzas, and 3 boys get one pizza. The girls have twice as many pizzas as boys. But the number of girls is more than twice that of the boys. So the boys get more.	10	3
<i>Convincing Argument 6:</i>	Each pizza was cut into 4 pieces. Each girl gets 1 piece, and 1 piece left over. Each boy gets piece, and 1 piece left over. The 1 piece left over must be shared by the 7 girls, but the 1 piece left over will be shared by the three boys. So the boys get more.	7	0



Girls'



Boys'

Convincing Argument 7: $7/2 = 3.5$ and $3/1 = 3$. Therefore 3.5 girls will share one pizza, and 3 boys will share one pizza. Thus, each boy gets more.

Convincing Argument 8: Each pizza is cut into 21 pieces. Each girl will get 6 pieces, and each boy will get 7 pieces.

10 2

2 0

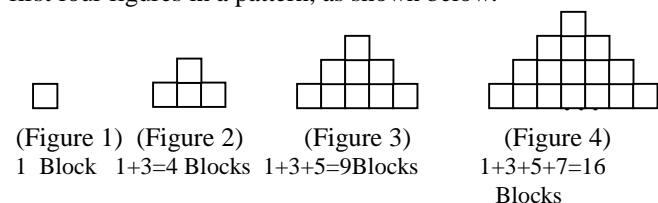
2.3 Assessments beyond problem solving

Mathematical problem posing is one of the key components of mathematical exploration. In scientific inquiry, formulating a problem is often more important than finding solutions to the problem. It is important to

examine U.S. and Chinese students' mathematical thinking beyond problem solving because the examination of problem solving and problem posing focuses on the investigation of students' thinking from different perspectives (Cai, 1998, Cai & Hwang, 2002; English, 1997; NCTM, 2000; Silver, 1994). For example, examining U.S. and Chinese students' problem solving can help us understand the solution strategies and representations students used to solve given problems. In contrast, the examination of U.S. and Chinese students' mathematical problem posing can help us understand how U.S. and Chinese students present the problems they generate. Since U.S. students are much more likely than Chinese students to use drawing-based strategies and representations in problem solving, is it possible that U.S. students would be more likely to use pictorial representations to present the problems they generate than Chinese students? Assessments beyond problem solving allow us to determine what similarities and differences existed between students in the two nations in the areas of problem posing. In addition, such assessments also allow us to examine the generality of the relatedness of students' performance in problem solving and problem posing from a cross-national perspective.

Figure 2:
One Pair of the Problem-Solving and Problem-Posing Tasks

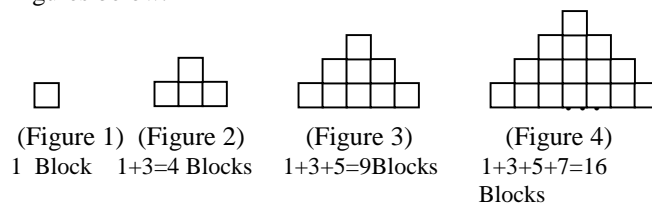
THE PROBLEM-POSING TASK: Mr. Miller drew the first four figures in a pattern, as shown below.



For his students' homework, he wanted them to make up three problems based on the above pattern: an easy

problem, a moderately difficult problem, and a difficult problem. These problems can be solved using the information in the pattern. Help Mr. Miller make up the three problems and write these problems in the space below.

THE PROBLEM-SOLVING TASK: Look at the Figures below.



- A. Draw the 5th Figure.
- B. How many blocks are there in the 6th figure? Explain how you found your answer.
- C. Find the sum of $1+3+5+7+9+11+13+15+17$. Show how you found your answer.
- D. How many blocks are there in the 50th figure? Explain how you found your answer.

A recent study was completed to investigate a sample of 155 Chinese sixth grade students and a comparable sample of 98 U.S. sixth-grade students' thinking in three pairs of problem-solving and problem-posing tasks (Cai & Hwang, 2002). Each pair of the problem-solving and problem-posing tasks has the same mathematical situation, but the problem-posing task requires students to generate one easy, one moderately difficult, and one difficult problem, based on the situation. The problem-solving task, however, requires students to answer a number of questions based on the situation (see Figure 2). Students were asked to do problem-posing tasks first; then on the second day, they were asked to work on the problem-solving tasks.

2.3.1 Summary of problem-solving results

Table 3 shows the percentages of U.S. and Chinese students who obtained correct answers to each of the four questions in one of the problem-solving tasks. About 70% of the U.S. and Chinese students drew the fifth figure correctly. A larger percentage of the Chinese students obtained correct answers in Questions B, C, and D. Similar to what was found in other studies (e.g., Cai, 2000a), a considerable number of U.S. students used drawings to answer some of the questions. For example, nearly 35% of the U.S. students actually drew the sixth figure and counted the number of blocks to answer Question B, but only 3% of Chinese students did

so. On the other hand, 30% of the Chinese students listed the expression of $1 + 3 + 5 + 7 + 9 + 11$ or $(1 + 11) \times 6 \div 2$ to find answer for Question B, but only 2% of the U.S. students did so. Of those who answered Question C correctly, over 80% of the U.S. students just added numbers sequentially, but only about 15% of the Chinese students did so. On the other hand, nearly 85% of the Chinese students either made tens and then added or paired numbers (e.g., $(1 + 17) \times 9 \div 2$) to answer Question C. Interestingly, two U.S. students still tried to draw ninth figure to answer Question C. For Question

D, over half of the Chinese students listed $1 + 3 + 5 + \dots + 99$ to answer it, but only about 10% of the U.S. students did so. Of U.S. students who used appropriate strategies, the vast majority of them used guess-and-check strategies. Four U.S. students still attempted a drawing in order to answer Question D. The results from this study provide additional data that Chinese students are likely to use generalized strategies in problem solving, while U.S. students are likely to use drawing and guess-and-check strategies.

Table 3:
Percentages of U.S. and Chinese Students Who Obtained Correct Answers

Questions	Percent of Students	
	U.S. (n=98)	China (n=155)
A. Draw the 5 th Figure	70	74
B. Blocks in 6 th Figure*	49	63
C. $1+3+5+7+9+11+13+15+17=?*$	68	92
D. Blocks in 50 th Figure*	16	39

*The Chinese students performed significantly better than the U.S. students.

2.3.2 Summary of problem-posing results

Table 4 shows the percentage distributions of the problems generated by U.S. and Chinese 6th grade students based on one of the problem situations. The problems students posed were coded in three levels. The coding scheme for analyzing student responses to the problem-posing task was based on prior research in solving pattern problems in general and solving pattern problem in specifically in mathematics. In solving pattern problems, one needs to induce a rule based on given elements of a pattern, then to extend the pattern using the rule (Simon, 1979). First, the problems were classified into mathematical problems or irrelevant problems/No responses. Second, each mathematical problem was classified into an extension or a non-extension problem. An extension problem refers to a problem questioning the pattern beyond the four given figures. A non-extension problem refers to a problem questioning the given figures in the pattern. Third, a non-extension problem was coded as questioning the number of blocks in a figure, number of blocks in two or more figures, comparing number of blocks in one or more figures, or asking to draw a figure in the pattern. Besides the categories described for classifying a non-extension problem, an extension problem can also be a rule-based problem. Here are two examples of rule-based problems: "What is the pattern in these figures?" and "How do blocks in each figure increase?" Table 4 also includes these coding categories.

There are several obvious similarities between U.S. and Chinese students. As expected, both U.S. and Chinese students tended to generate more extension

problems in their third problem than in their first or second problem. The most frequently generated problems are those asking to find the number of blocks in a figure. About 40% of the U.S. students generated at least two extension problems, which is similar to the percentage of Chinese students who generated at least two extension problems (36%). The percentage distributions of extension and non-extension problems are very similar between U.S. and Chinese students for their first, second, or third generated problems.

However, the percentage distributions for the kinds of extension or non-extension problems are different between U.S. and Chinese students. Chinese students are likely to generate problems comparing number of blocks in 2 or more figures (e.g., How many more blocks are needed to construct the 10th figure than the 5th figure?). For example, nearly one-third of Chinese students' third problem involving comparisons of blocks in two or more figures, which is significantly higher than that for the U.S. students (6%).

Another notable difference is related to U.S. students' asking to draw a figure in the pattern. In fact, about 10% of U.S. students' first, second, or third extension problems are those asking to draw a figure in the pattern. No Chinese students generated such problems in their first and second problems. Only in the Chinese students' third generated problems, 6% of them posed such problems. A few U.S. students also generated non-extension problems asking to draw a figure. Since the students did not always generate all three problems asking to draw a figure in the pattern, we can also look at the percent of students who generated at

least one problem asking to draw a figure. This analysis shows that 19% of the U.S. students generated at least one problem asking to draw a figure; this is in contrast

to 6% of the Chinese students who did so. This finding suggests that U.S. students are also more likely to pose problems involving drawing than are Chinese students.

Table 4:
Percentage Distributions of Various Posed Problems

		98 U.S. Students			155 CH Students		
		P1	P2	P3	P1	P2	P3
<i>Extension</i>	<i>Math</i>						
	<i>Problem</i>						
	# of Blocks in One Figure	10	8	14	12	9	15
	# of Blocks in More than One	3	4	2	1	3	3
	Comparing # of Blocks	0	2	3	4	7	10
<i>Non-Extension</i>	<i>Math</i>						
	<i>Problem</i>						
	Draw a Figure	9	8	12	0	0	6
	Rule-Based	8	15	18	6	16	10
	Total % of	26	37	49	23	35	44
		<i>Extension Problems</i>					
<i>Non-Extension</i>	<i>Math</i>						
	<i>Problem</i>						
	# of Blocks in One Figure	27	21	15	20	16	10
	# of Blocks in More than One	15	13	20	10	4	2
	Comparing # of Blocks	8	5	3	18	25	21
<i>No/Irrelevant Responses</i>	<i>Math</i>						
	<i>Problem</i>						
	Draw a Figure	3	1	1	0	0	0
	Rule-Based	53	40	39	48	45	33
	Total % of	53	40	39	48	45	33
		<i>Non-extension Problems</i>					
		17	23	12	29	20	23

2.3.3 Links between problem posing and problem solving

Table 5 shows how students who generated at least 2 extension problems performed in the problem-solving task compared to those who did not generate at least 2 extension problems. For both samples, although those students who generated at least 2 extension problems had higher success rates in Questions A, B, and C of the problem-solving task than did those who did not, the difference is not statistically significant. For both samples, only a statistically significant difference existed in Question D, in which the students who generated at least 2 extension problems had a higher success rate than did those who did not generate at least two extension problems. Answering Question D requires skills and knowledge to generalize the pattern. This result shows that students who can generate extension problems may understand the regularities of the pattern better and be able to generalize the pattern.

Recall that 19% of the U.S. students generated at least one problem asking to draw a figure in the pattern. Would these U.S. students more likely to use drawing and counting strategies to answer Question B of the problem-solving task? They were not, based on the data from this pair of problem-solving and problem-posing tasks. In fact, the U.S. students who did not generate any problem asking to draw a figure in the pattern were as likely as those who generated at least one problem asking to draw a figure to use drawing strategies in answering Question B. The above results confirmed the argument that the quality of the problems students pose might serve as an index of how well they can solve problems (Kilpatrick, 1987). However, these results do not show a direct link between the kinds of problems students pose and the kinds of strategies they employed to solve a problem with the same mathematical structure.

Table 5:
Percentages of Students Who Obtained Correct Answers for the Problem-solving Task with or without at Least Two Extension Problems to the Problem-Posing Task

Questions	% of U.S. Students		% of Chinese Students	
	With Two Extension (n=38)	No Two Extension (n=60)	With Two Extension (n=56)	No Two Extension (n=99)
A: Draw the 5 th Figure	72	69	77	73
B: Blocks in 6 th Figure	55	45	68	61
C: 1+3+5+7+9+11+13+15+17	69	67	95	91
D: Blocks in 50 th Figure*	27	10	57	29

3. Examining the impact of two factors on U.S. and Chinese students' mathematical thinking

The available evidence clearly shows that U.S. and Chinese students performed differently on different tasks. Even on equivalent tasks, U.S. and Chinese students may use different representations to solve or present problems. Why are there these observed differences? In this section, two studies were reviewed to understand the differences between U.S. and Chinese students' mathematical thinking. The first study examined the impact of learning opportunity of algebra on students' thinking. The second study examined the impact of teachers' beliefs and expectation on students' thinking.

3.1 Early algebra learning

Because of the central role classroom instruction plays in understanding the dynamic processes and the organization of students' thinking and learning (Rogoff & Chavajay, 1995; Schoenfeld, 1992), it is possible that the difference in the use of solution strategies and representations is due to the variations of classroom instruction that U.S. and Chinese students receive. In the previous studies (e.g., Cai, 2000a), the U.S. sixth grade students were not formally taught algebraic concepts, but the Chinese sixth graders had about 20 lessons on the topic. Is it possible that the early learning algebra may have had an impact on Chinese students' using more generalized, symbolic-based strategies? If learning algebra does have an impact on students' using more generalized, symbolic-based strategies, it should be the case that U.S. eighth or ninth grade students who have been formally introduced to algebraic concepts may use generalized, symbolic-based representations similar to those used by Chinese students. It is also likely that Chinese fourth or fifth grade students, who have not been formally introduced to algebraic concepts, may use similar concrete, drawing strategies, just as U.S. students did.

A recent study examined the extent to which U.S. and Chinese students' mathematical thinking is related to their learning of algebra (Cai, 2000b). In the study, the U.S. sample consisted of 115 sixth grade, 109 seventh grade, and 110 eighth grade students and the Chinese sample consists of 196 fourth grade, 213 fifth grade, and 200 sixth grade students. The unmatched grade selection for the U.S. and Chinese samples allowed us to examine the solution strategies and representations of U.S. and Chinese students who had and who had not formally learned algebraic concepts. The Chinese fourth and fifth grade students had not formally learned algebraic concepts, but the sixth grade Chinese students had. The U.S. sixth and seventh grade students had not

formally learned algebraic concepts, but the eighth graders had formally learned these concepts, including the concept of variables, equation and equation solving, and solving word problems by setting up equations.

Table 6 shows the results from the Pizza Ratio Problem that has two versions for the study (Cai, 2000b). In the first version (Version 7), 7 girls share 2 pizzas and 3 boys share one pizza. In the second version (Version 8), instead of 7 girls, 8 girls share 2 pizzas and 3 boys share one pizza. Since it is easier for a student to "visualize" cutting 2 pizzas so that they can be shared by 8 girls, version 8 might invite students to use visual drawings in their solutions. Because it is very difficult to cut 2 pizzas so that they can be equally shared by 7 girls, version 7 might invite students to use numerical symbols in their solutions. The results of the study showed that U.S. eighth grade students formally introduced to algebraic concepts were more likely to use numerical representations than were U.S. sixth and seventh grade students, but they were still much less likely than Chinese sixth grade students to use numerical representations. Only a very few Chinese students in each grade level used drawing strategies. Thus, the Chinese sixth-grade students' being less likely to concrete, drawing strategies than U.S. students cannot be explained by the Chinese students' opportunity to learn algebra. It should be indicated that this study (Cai, 2000b) is just a part of the effort of examining instructional impact on U.S. and Chinese students' mathematical thinking. Continuous effort is needed to investigate the instructional impact on U.S. and Chinese students' mathematical thinking.

Although the early learning of algebra cannot fully explain why the Chinese students preferred to use symbolic-based representations and why the U.S. students preferred to use pictorial-based concrete representations, the Chinese students' early successful introduction of algebraic concepts is still worth investigating. In particular, we need to document and analyze how the Chinese curriculum and instructional practice support students' development of algebraic thinking by helping them progress smoothly from arithmetic to algebraic thinking and thereby helping them appreciate the usefulness of algebraic approaches in problem solving. There is no doubt that the examination of the practice in one nation can provide a broader point of view on how a topic should be treated. Such an international perspective can increase researchers' and teachers' experiences in other countries when they try to face challenges in developing students' algebraic thinking.

Table 6:
Percentages of U.S. and Chinese Students Getting Correctly Justified Answer and Using Each Representation for the Pizza Ratio Task

	CH4th (n7=101) (n8=95)	CH5th (n7=106) (n8=107)	CH6th (n7=98) (n8=102)	US6th (n7=55) (n8=60)	US7th (n7=58) (n8=51)	US8th (n7=51) (n8=59)
CORRECTLY JUSTIFIED ANSWER						
Version 7	15	35	56	25	29	43
Version 8	21	57	93	42	53	59
VISUAL DRAWINGS						
Version 7	5	5	2	59	50	36
Version 8	4	3	4	65	53	49
NUMERICAL SYMBOLS						
Version 7	44	61	71	28	32	45
Version 8	35	51	68	27	36	47
WRITTEN WORDS						
Version 7	51	34	27	13	18	19
Version 8	61	46	28	8	11	4

3. 2 Teachers' beliefs and expectations

Cultural beliefs do not dictate what teachers do, but teachers do draw upon their cultural beliefs as a normative framework of values and goals to guide their teaching (Bruner, 1996). Although Chinese sixth grade students were more likely than the U.S. sixth grade students to construct mathematical expressions in their solutions, still a considerable number of them did not construct mathematical expressions or algebraic equations in their solutions. Why did these Chinese students not use concrete, drawing approaches to solve the problems as U.S. students did? Concrete, drawing strategy may provide an entry-level facility to solve the problems. In fact, for those Chinese students who did not use mathematical expressions, a drawing strategy might be beneficial in solving the problems. However, many Chinese students chose not to use such drawing strategies. For example, about 80% of the Chinese 4th graders were unable to provide correctly justified answer for Version 8 of the Pizza Ratio Problem, but only 4% of them used drawing approaches. Is it possible that teachers in China do not support the drawing strategies? If so, why do Chinese teachers discourage their students to use concrete, drawing strategies?

A study is being undertaken to address these research questions through both extensive interviews of 11 U.S. and 9 Chinese experienced teachers and analysis of their lesson plans. The analysis of lesson plans and interview transcripts contributed information about U.S. and Chinese teachers' conceptions of various representations from three aspects: (1) generating pedagogical representations for classroom instruction, (2) knowing students' representations and strategies in problem solving, and (3) evaluating students' representations and

solution strategies. Primary results from the study did show that U.S. and Chinese teachers tend to hold different expectations for their students and value solution strategies differently (Cai, 2002). For example, let's look at how U.S. and Chinese teachers evaluate the following three solution strategies to solve the problem related to arithmetic average: *Angela is selling hats for her Mathematics Club. She sold 9 hats in the first week, 3 hats in the second, and 6 hats in the third. How many hats does she have to sell in the fourth week so that the average number of hats sold in four weeks is 7?*

Algebraic Approach: Let x be the number of hats sold in Week 4. $9+3+6+x=4 \times 7$. $x=10$ by solving the equation for x . So Angela should sell 10 hats in Week 4.

Drawing Approach: The average (7) was viewed as a leveling basis to "line up" the numbers of hats sold in the week 1, 2, and 3. Since 9 hats were sold in week 1, it has two extra hats, which were moved to week 2. Since 3 hats were sold in week 2, 4 additional hats are needed in order to line up the average. Week 2 got 2 hats from Week 1 so only 2 more was needed. Since 6 hats were sold in week 3, it needs 1 additional hat to line up the average. In order to line up the average number of hats sold over four weeks, 10 hats should be sold in Week 4.

Arithmetic Approach: $9 + 3 + 6 = 18$. $4 \times 7 = 28$. $28 - 18 = 10$. Therefore, Angela should sell 10 hats in Week 4 so that the average number of hats sold in four weeks is 7.

Using a general five-point scoring rubric (ratings of 0-4), U.S. teachers scored all three responses highly: the

mean is 3.91 for the response with algebraic approach and 3.82 for responses with arithmetic and drawing approaches. However, Chinese teachers did not score the response with drawing approach as high as the responses with algebraic and arithmetic approaches: the mean is 3.00 for the response with drawing approach and 3.78 for the responses with arithmetic and algebraic approaches. Almost all the U.S. and Chinese teachers liked the responses with the algebraic approach the best when compared to other responses for the problem. However, the U.S. teachers seem to have different expectations than the Chinese teachers. All the U.S. teachers except one believed that sixth grade students in the United States in general are not expected to solve problems using algebraic approaches. For example, one of the teachers explicitly said, "I wish my sixth graders can do this. But in our school, only seventh grade students are taught algebraic concepts. Sixth-graders are only learning pre-algebra and are not expected to solve them using this kind approach involving x 's." On the other hand, all Chinese teachers expect their sixth graders to solve problems using algebraic approaches.

The Chinese teachers took the nature of solution strategies into account in their scoring of students' responses, so they gave lower scores to the responses with a drawing approach because they thought the approach is not efficient. While most U.S. teachers realized that drawing is not a sophisticated strategy and is very time consuming, they all agreed that it is a viable and acceptable approach to apply. Moreover, almost all U.S. teachers stated that the response using the drawing approach clearly shows how students think about the concept of average and how they solved the problem, therefore, the response should receive the highest score.

4. Interpreting the differences in U.S. and Chinese students' mathematical thinking

Cross-national studies in mathematics provide unique opportunities to explore how we can improve students' learning. It is important to explore why there are these observed differences in U.S. and Chinese students' mathematical thinking. It is also important to explore the lessons we can learn from the cross-national studies that examined U.S. and Chinese students' thinking to improve students' learning of mathematics. This section particularly discussed two possible lessons we may learn from these findings about the differences between U.S. and Chinese students' mathematical thinking.

4.1 Having computational and routine problem-solving skills does not imply having non-routine problem-solving skills

Using different types of tasks, we are able to see different performance patterns between U.S. and Chinese students. The Chinese students performed better than the U.S. students on tasks requiring computation and applications of formulas, but the U.S. students performed better than the Chinese students on tasks requiring sense-making of problem situations. These results suggest that having computation and routine problem-solving skills does not mean having non-routine problem-solving skills. The

findings of the performance differences on different tasks may suggest the ineffectiveness of Chinese classroom instruction in developing their students' non-routine problem solving skills, although it does show the effectiveness of developing Chinese students' computation and routine problem-solving skills. It is possible that Chinese teachers overemphasized abstract and numerical representations, so their students showed less flexibility in using innovative ways to solve problems in classroom.

On the other hand, the performance differences on various tasks may also indicate the ineffectiveness of U.S. classroom instruction in developing students' algorithmic thinking and computation skills. Both national and international assessments showed that U.S. students not only lack computation skills with fractions, but also lack computation skills with whole numbers (Silver & Kenney, 2001; U.S. Department of Education, 1996). For example, only 68% of the U.S. 6th graders are able to correctly find sum of $1 + 3 + 5 + 7 + 11 + 13 + 15 + 17$ (see Table 3). Obviously, both basic skills and high-order thinking skills in mathematics are important, but having basic mathematical skills does not imply having higher-order thinking skills or vice versa. Would it be possible to design the kind of instruction that promotes both routine problem solving and non-routine, creative problem solving? Obviously, extensive research is needed to characterize the kind of instruction. However, a combination of concrete representation-based instruction and abstract, substantive content-based instruction might be a good place to start that search.

The findings that U.S. and Chinese students performed differently on different tasks also suggest the complexity of interpreting the cross-national performance differences in mathematics. Using national representative samples, the TIMSS Video Study (Stigler & Hiebert, 1999) showed that compared to German and Japanese teachers' teaching, U.S. teachers focused more on mathematical procedures and techniques and practice of routine problem solving, and less on mathematical exploration and conceptual understanding. If that is the case, why did U.S. students perform so poorly on computation and routine problem-solving tasks requiring procedural knowledge, in contrast to Chinese students (see Table 1) and students from other countries (U.S. Department of Education, 1996)? One may argue that since U.S. teachers focused more on procedures, U.S. students may not understand the principles behind the procedures (Hiebert, 1986), thus, they performed poorly on computation and routine problem solving. However, if U.S. teachers focused less on mathematical exploration and conceptual understanding, according to Stigler and Hiebert (1999), and Chinese teachers have a better understanding of elementary mathematics than their U.S. counterparts, according to Ma (1999), why did the U.S. students perform better than the Chinese students on the process-open non-routine problems? Furthermore, why did U.S. students in Cai's study (2000a) perform better on the process-open problems than on the process-constrained problems and vice versa for the Chinese

students? To answer these questions, more research is needed.

4.2 Concrete experiences do not automatically lead to generalization and conceptual understanding

Through assessing U.S. and Chinese students' mathematical performance beyond correctness and computation, we are able to see remarkable differences between U.S. and Chinese students' mathematical thinking and reasoning, as exhibited in their uses of different solution strategies and representations. The U.S. students frequently used pictorial representations, while the Chinese students were more likely to use symbolic (arithmetic or algebraic) representations in their solution processes. When both U.S. and Chinese students used symbolic representations, the U.S. students primarily used arithmetic representations while the Chinese students were more likely to use algebraic representations. This finding indicates the ineffectiveness of using concrete representation to develop students' algorithmic thinking and substantive content knowledge. The U.S. teachers are much more frequently than do teachers in other countries to use manipulatives and concrete representations (Stigler & Hiebert, 1999). Concrete representations or manipulatives can facilitate students' conceptual understanding if they are appropriately used. However, research showed that manipulatives or concrete representations do not guarantee students' conceptual understanding (e.g., Baroody, 1990). While concrete representations help students' conceptual understanding of abstract nature of mathematics, they do not automatically lead to generalization and conceptual understanding.

The relative advantages of the U.S. students in items with a concrete nature over items with an abstract nature in comparison to students in other countries can also be seen in one of the TIMSS performance assessment tasks, Folding and Cutting Task. The Folding and Cutting Task consists of four items. The first three items involve actually folding and cutting physical shapes and the last item involves predictions based on the first three activities. Harmon et al. (1997) reported that the overall difference between U.S. and Chinese students in Hong Kong is not due to the differences on the first three items, but instead it is due to the difference on the last item involving a prediction. In fact, both the U. S. and Chinese students had similar average percentage score on each of the first three items, but the average percentage score for the U.S. students (45%) was much lower than that for the Chinese students (70%) on the last item.

The concrete, drawing strategies used by some U.S. students did provide them with entry-level facilities in solving problems, but such concrete strategies have limitations. They are context- or task-specific strategies in problem solving. They will limit students' thinking and further learning unless we can help students shift to more generalized approaches (Dreyfus & Eisenberg, 1996; Janvier, 1987; Krutetskii, 1976). Concrete experiences can neither automatically lead to generalization nor to conceptual understanding. If a central aim of classroom instruction is to foster the transfer of acquired knowledge beyond the initial learning context to new circumstances,

concrete experiences are only effective if they support for generalization and conceptual understanding (Presmeg, 1997). Mathematics instruction which uses concrete representations to facilitate students' conceptual understanding and reasoning does not imply that teachers should not teach algorithms and basic skills and students should never practice these (Hiebert, Wearne, & Taber, 1991; Thompson, 1992, 1994). Students should be given the opportunity to construct their own representation of mathematical concepts, rules, and relationships. At the same time, we should expect students to have an understanding that goes beyond "concreteness." For example, teachers may start with concrete and manipulatives to encourage students to use their own strategies for solving problems and making sense of mathematics. After that, teachers should gradually push steps further to encourage students to develop more efficient and generalizable solution strategies (Cai, 2001).

5. Conclusion

The main purpose of educational research is to improve the learning opportunities for all students. The cross-national studies in the teaching and learning of mathematics provide unique opportunities for us to understand students' thinking and reasoning, and to suggest ways future learning can be supported and enhanced. Using an emerging understanding from a series of cross-national studies involving U.S. and Chinese, this paper addressed issues related to assessing, describing, and interpreting international performance differences in mathematics. At this point, we are still in an early stage to understand the possible factors that may contribute to the differences in U.S. and Chinese students' mathematical thinking. This paper also proposed a number of research questions to continue the endeavor of understanding and improving students' learning of mathematics. To actually make the idea of improving students' learning a reality, the systematic investigation of research questions raised in paper, as well as issues related to assessing and understanding students' thinking, is needed. Cross-national studies in mathematics are powerful ways to make the idea a reality.

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Author

Cai, Jinfan, Prof., Dr., 523 Ewing Hall, Department of Mathematical Sciences, University of Delaware, Newark, DE 19716, USA.

E-mail: jcai@math.udel.edu