

## How Pythagoras' theorem is taught in Czech Republic, Hong Kong and Shanghai: A case study

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**Abstract:** This paper attempts to explore certain characteristics of the mathematics classroom by investigating how teachers from three different cultures, namely, the Czech Republic, Hong Kong and Shanghai, handle Pythagoras' theorem at eighth grade.

Based on a fine-grained analysis of one lesson from each of the three places, some features in terms of the ways of handling the same topic were revealed as follows: the Hong Kong teacher and the Shanghai teacher emphasized exploring Pythagoras' theorem, the Shanghai teacher seemed to emphasize mathematical proofs, while the Czech teacher and the Hong Kong teacher tended to verify the theorem visually. It was found that the Czech teacher and the Hong Kong teacher put stress on demonstrating with some degree of student input in the process of learning. On the other hand, the Shanghai teacher demonstrated a constructive learning scenario: students were actively involved in the process of learning under the teacher's control through a series of deliberate activities. Regarding the classroom exercises, the Shanghai teacher tended to vary problems implicitly within a mathematical context, while the teachers in the other two places preferred varying problems explicitly within both mathematical and daily life contexts.

**ZDM-Classification:** D10, E50, G40

### 1 Background

Mathematics education in different countries is strongly influenced by cultural and social factors that shape goals, beliefs, expectations, and teaching method (An *et al.*, 2002), and cross-cultural comparison can help researchers and educators understand explicitly their own implicit theories about how teachers teach and how children learn (Stigler & Perry, 1988). It will also help educators to reflect upon their own practices, and recognize ways of improving their practices (Bishop, 2002). Recently, international comparative studies in mathematics have focused on classroom teaching (Leung, 1995; Lee, 1998; Stigler & Hiebert, 1999). For example, the TIMSS Video Study (Stigler & Hiebert, 1999) and the on going TIMSS-R Video Study aim at identifying different patterns of classroom instruction in different cultures by examining a representative sample of eighth-grade mathematics classrooms in each participating country. The Learners' Perspective Study, on the other hand, aims at examining learning and teaching in eighth grade mathematics classrooms in a more integrated and comprehensive fashion (Clarke, 2002a). However, these studies do not pay due attention to how a particular mathematics topic is handled in different cultures. Since a teacher must handle

or present a certain mathematics topic in a certain way so that students can understand the content, a study on how the same topic is taught in different cultures will highlight the similarities and differences among different cultures. This study intends to capture the features of mathematics classrooms in the Czech Republic and China (Hong Kong and Shanghai) by investigating the ways teachers tackle Pythagoras' theorem in the three different places.

### 2 Methodology

#### 2.1 Source of data

Firstly, eight Hong Kong videos and five Czech videos on Pythagoras' theorem (at grade 8) were selected from the TIMSS-R Video Study. Secondly, eleven Shanghai grade 8 lessons on Pythagoras' theorem were videotaped by the researchers in March 2001 following the method used in the TIMSS-R Video Study (see website <http://www.lessonlab.com> for details). Based on a thorough analysis on all the Hong Kong and Shanghai videos, the patterns of teaching the theorem in these two cities were identified (Huang, 2002). Then one representative video on the first lesson of the unit on Pythagoras' theorem was chosen from each of the three places. For the Czech Republic, unfortunately none of the five videos was the first lesson in the unit, and so one video which was the earliest lesson in the unit was chosen. Totally, three videos, their transcripts, and relevant documents such as the teacher questionnaires, lesson plans, copies of textbooks and worksheets constituted the data for this study. The backgrounds of the three lessons are shown in Table 1 (see next page).

The Table indicates that the class size in Shanghai was the largest, while the teacher in Shanghai was the least educated. The Hong Kong teacher and the Shanghai teacher majored in mathematics, but the Czech teacher majored in science.

#### 2.2 Data analysis

The English transcripts of those videos taken from the TIMSS-R Video Study were already available. The Shanghai videos were transcribed verbatim in Chinese by the teachers who delivered the lessons, and were then translated into English. Through watching the videos, and reading their transcripts and relevant documents again and again, the three lessons were compared. The data analysis focused on the following aspects:

- Structure of the lesson;
- Ways of teaching Pythagoras' theorem;
- Patterns of classroom interaction.

How the first two aspects are analyzed will be illustrated in the next section. The way of describing the classroom interaction will be discussed here in more detail.

	Czech Republic (CZ)	Hong Kong (HK)	Shanghai (SH)
Rank and score in TIMSS 1995 at grade 8 <sup>1</sup>	6 <sup>th</sup> 564(4.9)	4 <sup>th</sup> 588(6.5)	N/A
Rank and score in TIMSS 1999 at grade 8	15 <sup>th</sup> 520(2.7)	4 <sup>th</sup> 582(4.3)	N/A
Class size	20	42	60
Duration of lesson (minutes)	45	40*2 <sup>2</sup>	45
Educational level of teacher	Bachelor in Mathematics and Physics	Bachelor in Mathematics	Diploma <sup>3</sup> in Mathematics

**Table 1:** The backgrounds of the three lessons

According to Sfard (2001), “learning is nothing else than a special kind of social interaction aimed at modification of other social interactions”, and communication should be viewed not as a mere aid to thinking, but as almost tantamount to the thinking itself. In this study, the features of classroom interaction were investigated from two aspects: Firstly, the feature of questioning will be measured quantitatively. Secondly, the patterns of interaction will be investigated qualitatively.

The teachers’ questions were classified into three categories: 1. Requesting a simple yes or no as a response, asking students to confirm an answer, or questions of management [Yes /No]; 2. Asking students to name an answer without asking for any explanation of how the answer was found, or asking for the solution to an explicitly stated calculation [Name]; 3. Asking for an explanation of the answers given or procedures carried out, or asking a question that requires students to find facts or relationships from complex situations [Explanation]; 4. Asking students for generalizing or evaluating an answer [Generalization]. Only the first three types of questions were found in the three lessons, and examples for the three types of questions are illustrated below:

Question 1: Have you heard of Pythagoras’ theorem before? [Yes/No]

Question 2: What are the relationships between a, b, and c. [Name]

Question 3: Why is the quadrangle at the center a square? [Explanation]

<sup>1</sup> The data on the results of TIMSS 1995 and TIMSS 1999 are from those reports (Mullis *et al.*, 1997; 2000). The information related to the teachers is from the teacher questionnaires collected by the TIMSS-R Video Study.

<sup>2</sup> It is a common practice that there are two consecutive lessons without a break between them. The lesson selected here consisted of two sections. The first part focused on introducing Pythagoras theorem which took around 60 minutes, while the second part was devoted to classroom exercise, which lasted around 20 minutes. This study only focused on the first part.

<sup>3</sup> This is a document issued by an educational institution in mainland China, such as a college or a university, testifying that the recipient has successfully completed a particular course of study which usually requires two or three full-time years after finishing Grade 12 in Secondary schools.

### 3 Results

#### 3.1 The structure of the lesson

It was found that all the lessons in this study, by and large, include an opening, introducing the new topic (except the CZ one), justifying, practising, and ending, as shown in Table 2 (see next page).

The Table shows that the Czech teacher spent relatively high proportion of time on organizing and revision, while the Hong Kong teacher and Shanghai teacher spent a substantial amount of time on justification of the theorem. All three teachers emphasized classroom exercises. The following discussions will be focused on introducing the theorem, justifying the theorem and applying the theorem.

#### 3.2 Ways of teaching Pythagoras’ theorem

##### 3.2.1 Introduction of the theorem

**Hong Kong lesson.** After reviewing how to identify the hypotenuse of right-angled triangles in different positions, the teacher pointed out this lesson was about Pythagoras’ theorem. Through demonstrating the area relationship between two diagrams as shown in Figure 1(1) and Figure 1(2), Pythagoras’ theorem was found and proved.

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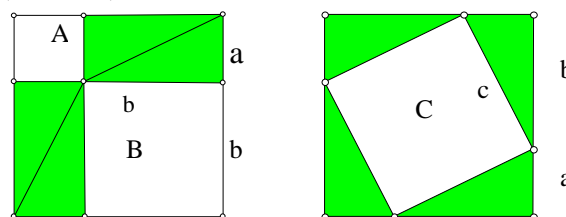


Figure 1(1)

Figure 1(2)

Mainly through teacher’s demonstration, and occasionally through questioning students, the teacher explained that the area of Figure 1(1) is:  $A+B+4 \times ab/2$ , while the area of Figure 1(2) is:  $C+4 \times ab/2$ . By taking away the four congruent right-angled triangles from both large squares which are of the same area, the students found the relationship:  $A+B=C$  and Pythagoras’ theorem.

Place	Segment	Description	Duration (Minutes)
<b>Czech Republic (CZ)</b>	1	Evaluating the previous written test and collecting exercise book	6.4
	2	Reviewing the previous lesson	3.1
	3	Introducing new proofs of the theorem; Explaining the theorem	12.2
	4	Examples and exercises; demonstration using physical objects	19.9
	5	Summary and assignment	2.1
	Total		44
<b>Hong Kong (HK)</b>	1	Distributing worksheets	1.7
	2	Reviewing relevant knowledge	4.1
	3	Introducing the proofs of Pythagoras' theorem	30.6
	4	Examples and exercises	18.1
	5	Summary	2.5
	Total		57
<b>Shanghai (SH)</b>	1	Establishing a situation for learning the new topic	1.5
	2	Doing experiment and making a conjecture	7.2
	3	Proving and explaining Pythagoras' theorem	19.2
	4	Examples and exercises	16.4
	5	Summary and assignment	1.1
	Total		45

**Table 2:** Overview of the three lessons

**Shanghai lesson.** The Shanghai teacher started the lesson by asking the questions “what is the relationship between the three sides of a triangle?” and “what is the relationship between the three sides of a right-angled triangle?”. The teacher then motivated students to explore the unknown feature, which was actually the topic for this lesson. After that, the class was divided into groups of four (two students in the front row turned about, so the four students around a desk formed a group); and each group was assigned to do the following tasks: (1) Drawing a right-angled triangle ABC,  $\angle C=90^\circ$ ; (2) Measuring the lengths of three sides a, b and c; (3) Calculating the values of  $a^2$ ,  $b^2$  and  $c^2$ ; and (4) Guessing the relationship among  $a^2$ ,  $b^2$  and  $c^2$ . Based on drawing, measuring, calculating and communicating, the students were asked to make a conjecture on the relationship between the three sides of a right-angled triangle, i.e.  $a^2 + b^2 = c^2$

**Discussion**

The aforementioned description indicates that the Hong Kong teacher and Shanghai teacher tended to explore the theorem through deliberate activities.

**3.2.2 Justification of the theorem**

All the three teachers tried to justify the theorem by different methods. The following describes how the teachers proceeded with the justification.

**Czech lesson.** Pythagoras' theorem and one “proof” of the theorem had already been introduced in the previous lesson. In this lesson, the Czech teacher further brought in new proofs. Firstly, the teacher introduced a “proof” through demonstrating a model similar to the diagrams in Figure 1. The students were asked to identify each part of the figures and explain the relationship between the two diagrams (as shown in Figure 1(1) and Figure 1(2)).

Finally, the students were convinced that “the sum of the areas of the squares constructed from the adjacent sides (of a right-angled triangle) is equal to the area of the square constructed from the hypotenuse”, and this was called the Pythagoras' theorem by the teacher. Then the students were asked to read a paragraph in the textbook entitled “venture paradise” in which three “proofs” were introduced based on three diagrams (similar to Figure 2), and students were asked to complete the proofs after the lesson.

**Hong Kong lesson.** The teacher organized a cutting-and-fitting activity to verify the theorem. The students were asked to play a game by following the instructions on the wall chart as follows:

- (1) Two squares (one whole square (A) and another (B) made of 4 pieces I, II, III, IV)) are attached to the two shorter sides of the triangle as shown in Figure 2

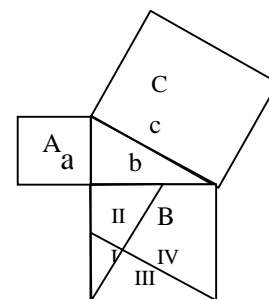


Figure 2

- (2) Fit the 4 pieces of puzzles cut from the square B, together with the square A to make a square (C) on the longest side.

Based on this game, the students found the relationship among the areas of the three squares:  $A+B=C$ , namely, Pythagoras' theorem  $a^2 + b^2 = c^2$ .

**Shanghai lesson.** In the lesson, two proofs were explored by the following steps: specializing to an isosceles right-angled triangle, and then generalizing to a

right-angled triangle.

**A. Specializing: isosceles right-angled triangle**

A group activity (in groups of four students) was organized: fitting a square by using four congruent isosceles right-angled triangles. Students were asked to present their diagrams by projector, as shown in Figure 3(1) and Figure 3(2), and explain the reasonableness of the diagrams. Then the students were asked to calculate the area of the square (let the hypotenuse be  $c$ , and the adjacent side be  $a$ ) in different ways, and the equation:  $a^2 + a^2 = c^2$  was arrived at. Thus, the theorem was proved for an isosceles right-angled triangle.

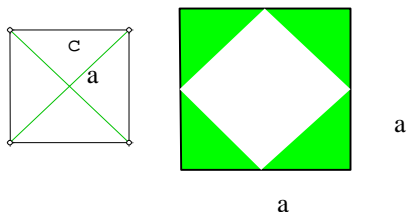


Figure 3(1) Figure 3(2)

Through exploring the special case, not only was the theorem proved in a special situation, but also more importantly, the ways of proving the theorem in the general case came up progressively.

**B. Generalizing: general right-angled triangle**

To prove the theorem in the general case, the teacher organized another group activity: fitting figures that include a square by using four congruent general right-angled triangles. Based on the activity, the students were asked to present their diagrams, as shown in Figures 4(1) and 4(2), and provide the relevant explanations. As soon as the above diagrams were presented on the screen, the students were encouraged to calculate the areas in different ways (the adjacent sides are  $a$  and  $b$ , and the hypotenuse is  $c$ ).

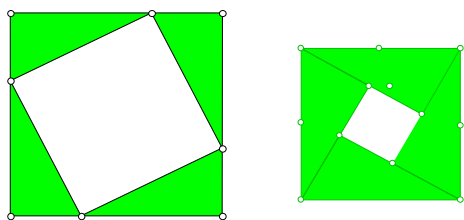


Figure 4(1) Figure 4(1)

Regarding Figure 4(1), one method is:

$$S_{\text{large square}} = (a+b)^2$$

The other is

$$S_{\text{small squares}} + 4S_{\text{triangles}} = c^2 + 4 \times \frac{1}{2} ab ;$$

Since the two methods are used to calculate the area of

the same square, then

$$S_{\text{large square}} = S_{\text{small squares}} + 4S_{\text{triangles}}$$

Namely,  $(a + b)^2 = c^2 + 4 \times \frac{1}{2} ab$

Simplifying, we get  $a^2 + b^2 = c^2$

Similarly, with the help of Figure 4(2), another proof was introduced.

**Discussion**

With respect to the justification of the theorem, diverse pictures evolved. The Shanghai lesson seemed to emphasize multiple mathematical proofs. Regarding the justification in the Czech lesson and the Hong Kong lesson, the two teachers tended to verify the theorem through physical and visual activities. In distinguishing them from each other according to whether the justification is bounded up with mathematical symbols, the Hong Kong lesson is closer to the Shanghai one, since no mathematical symbols were used in the Czech lesson at this stage. It seems that the Shanghai teacher stressed deductive mathematical reasoning, while the Hong Kong teacher and the Czech teacher preferred visual reasoning and verification.

**3.2.3 Application of the theorem**

Giving problems for students to practice is very important for students to learn the theorem. The kinds of problems teachers use also reflect teachers' interpretations of the objectives of teaching the theorem

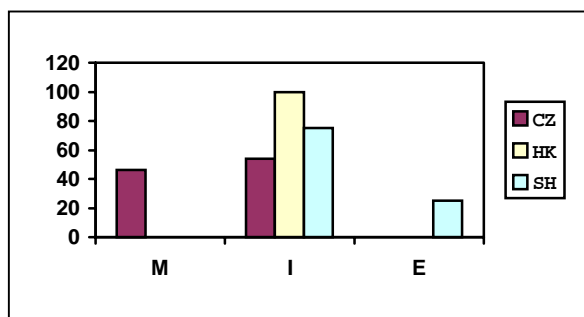
**Distribution of problems**

According to Gu (1994), problems can be conceived as comprising three basic elements. The initial status A is the given conditions of the problem. The process of solving the problem B is the transition of approaching the conclusion based on the existing knowledge, experience and given conditions. The final stage C is the conclusion. A problem is considered as a prototype if it consists of an obvious set of conditions, a conclusion and a solving process familiar to the learners. The prototype can be transformed into a closed variation or an open variation by removing or obscuring one or two of the three components respectively. Furthermore, Gu merges Blooms' six teaching objectives: knowledge, calculation, interpretation, application, analysis and synthesis, into three categories. They are memorization (including knowledge and calculation), interpretation (including interpretation and application) and exploration (analysis and synthesis). It is found that the three types of teaching objectives can be achieved by providing students with three kinds of varieties of problems respectively. Regarding Pythagoras' theorem, these categories can be illustrated in Table 3.

Category of problem	Specification to Pythagoras' theorem
Memorization [M]	Calculating the power and square root; Naming hypotenuse in different right-angled triangles in terms of their positions and their labels; Writing up the formulae corresponding to different right-angled triangles in terms of their positions and their labels;
Interpretation [I]	Calculating the length of the third side of a right-angled triangle when two sides are given explicitly; Judging whether an angle is a right angle when the lengths of the three sides are given explicitly. Calculating the length of the third side of a right-angled triangle when the lengths of two sides are given explicitly in a daily life context;
Exploration [E]	Calculating the length of the third side of a right-angled triangle when the lengths of two sides are given implicitly; Applying the theorem by constructing right-angled triangles; Ill-structured problems (trial and error, discussion with different situations)

**Table 3:** Categories of problems

According to this classification, the problems used in these three lessons can be depicted in Figure 5.



**Figure 5.** Distribution of problems in the lessons

It is found that in the Czech lesson, 46% of the problems are at memorization level while the others are at the interpretation level. However, in the Hong Kong lesson, all the problems are at the interpretation level. In the Shanghai lesson, three-fourth of the problems are at the interpretation level and all others belong to the exploration level. Overall, the Shanghai teacher provided more challenging problems than others, and the Czech teacher offered the easiest problems to her students. Examples of problems will be given in the following section.

**Problems and explanation**

In the Czech lesson, the following types of problems were used:

- Calculating the hypotenuse when the two sides adjacent to the right angle are given in different right-angled triangles in terms of their positions and size
- Naming the hypotenuse in different right-angled triangles in terms of their positions and their labels;
- Writing down the formulas corresponding to different right-angled triangles in terms of their positions and their labels;
- Determining if triplets such as [9,40,41], [9,12,13] etc. are Pythagorean triplets.

Apart from the types of problems given above, the teacher organized the following activity for students :

Three students were asked to stretch three line segments of lengths 3 meters, 4 meters, and 5 meters, so as to create a right-angled triangle in the classroom (each student stood at a vertex). Students were to judge a perpendicular relationship between the two adjacent sides (CZ, Interpretation).

After this activity, students were asked to give a piece of advice to their fathers at home if they do not know how to mark out a right angle for example when constructing a fence.

In the Hong Kong lesson, the basic type of problems is to find the third side when two sides are given in different right-angled triangles in terms of their position and size. In addition, the teacher presented a daily life problem:

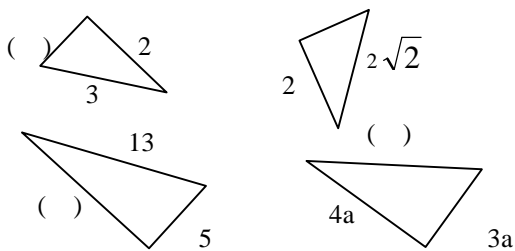
An army of soldiers wants to attack a castle, which is separated from them by a river and a wall. The river is 15 m wide and the wall is 20 m high.

- How can the soldiers reach the top of the wall (if they cannot fly over the wall by any means)<sup>4</sup>?
- What is the shortest distance to get to the top of the wall?

In the Shanghai lesson, the following basic problems were provided:

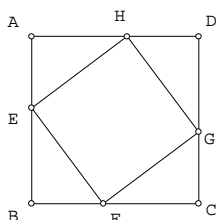
**Exercise 1:** In the following right-angled triangles, given the lengths of two sides, fill the length of the third side into the relevant brackets:

<sup>4</sup> In ancient China, a common way for soldiers to reach the top of a wall in the enemy's city is to construct a ladder for soldiers to climb over the river to arrive at the top of the wall.



In addition, one problem was made up from a figure fitted by students themselves as follows:

**Exercise 2:** In the following figure, if the area of square EFGH is  $20\text{ cm}^2$ , and the ratio of the lengths of the two adjacent sides is 1:2, can you calculate the perimeter?



From the problems used in these lessons, it was found that the Czech teacher and the Hong Kong teacher emphasized more the daily life application and the connection between mathematics and society. In particular, the Czech teacher stressed the history of discovering Pythagoras’ theorem, the beauty of Pythagoras’ triplets and the real context application of Pythagoras’ theorem such as judging a perpendicular relationship when constructing a fence. On the other hand, the teacher in Shanghai tended to provide a variety of problems with different mathematical contexts, which require several steps and different concepts and skills to solve.

**Variation in the exercises**

In addition to noting the types of problems provided in the lessons, it is also interesting to identify the ways the problems varied. If the changes from the prototype of a problem (in which the learnt knowledge can be applied directly) to its variations are identified *visually and concretely* (such as variations in number, positions of figure etc.), but the relevant concepts and strategies can still be applied explicitly, then this kind of variation is regarded as explicit (see Exercise 1 in the Shanghai lesson). On the other hand, if the changes from the prototype to its variations *have to be discerned by analysis abstractly and logically* (such as variation in parameter, subtle change or omission of certain conditions, change of contexts, or reckoning on certain strategies etc.) so that the conditions or strategies for applying the relevant knowledge are implicit, then this kind of variation is characterized as implicit (see Exercise 2 in the Shanghai lesson). According to this explicit-implicit distinction, it was found that both types of variation often appeared in the Shanghai lesson, but

basically only explicit variation appeared in the Czech lesson and the Hong Kong lesson. By using implicit variation in the exercise, not only will the problems be more difficult, but they will also be more open-ended as well.

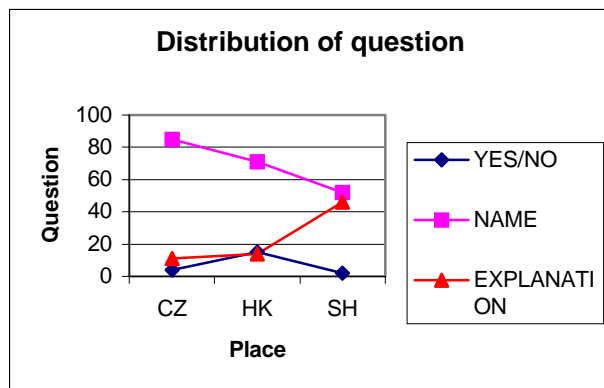
**Discussion**

It seems that the teachers in all three places paid attention to practice at the “interpretation level”. Yet, in the Czech lesson, there were one quarter of the exercise at the “memorization” category but there was no such kind of exercise in the Hong Kong lesson and the Shanghai lesson. Moreover, the Hong Kong teacher provided a problem with a real life context for students to practice, and the Shanghai teacher assigned several problems with complicated mathematical contexts for students to solve. It seems that the Hong Kong teacher and the Shanghai teacher gave more challenging problems for their students to tackle.

**3.3. Classroom interaction**

**3.3.1 Questioning**

As mentioned above, there was no question of the fourth category in all the lessons. The total numbers of questions are 102, 48 and 52 in the Czech lesson, Hong Kong lesson and Shanghai lesson respectively. The distribution of the questions was shown in Figure 5.



**Figure 5.** Distribution of questions

The above table shows that the second category of questions was the most commonly used in the lessons. In addition, it was found that the Czech teacher and the Hong Kong teacher adopted a similar pattern of questioning, with more than 70% of the questions requesting a simple yes or no response, and less than 15% of the questions in the other two types. It is interesting to take note of the Shanghai teacher’s pattern: the Shanghai teacher only asked less than 5% of the questions to request a simple yes or no response and asked around half of the questions to elicit students’ explanations.

**3.3.2. Pattern of classroom interaction**

The following section tries to characterize the features of classroom interaction in each country or city by quoting some excerpts.

**Czech Republic.** After discussing the homework, the

teacher introduced a proof. The exchanges were demonstrated as follows:

1. T I will show you one more. I have it prepared here. Yeah. I hope it will not fall down from it. The magnet needles are already weak.
2. T So, the basic thing is the board here [there are three congruent right-angled triangle models in the board]. We will start from it. Here I have a right-angled triangle. This is ...?
3. Ss Right angle.
4. T One leg. The second leg.
5. Ss The second leg. The hypotenuse.
6. T The hypotenuse. Well, and now- all these three triangles are congruent. The aid is already old, so, don't look at those worn-out vertexes. They are congruent, because they ...
7. Ss Coincide.
8. T They coincide if we place them this way [overlapping two models together]. Yeah. And now I place them this way. I hope I will do it correctly. So. They stick, yeah [putting a diagrams as shown in Figure 1(1)] . What do you see here?
9. Ss A square.
10. T A square above ... one leg. And this?
11. Ss A square above the second one.
12. T It is a square above the second leg. You can see it well, yeah? So, the blue part is ...
13. Ss A square.
14. T The sum of squares above both legs. Is it so?
15. Ss Yes.
16. T That is this side. Well, and now I set it another way. I set it in such a way that I will make the square above the hypotenuse. I can keep this here [putting up a diagram as shown in figure 1(2)]
17. T The aid is old, but I think it is very graphical. Here we have the hypotenuse and the blue part which is there ...
18. Ss A square.
19. T The square above the hypotenuse. And can you deduce from this that those two squares together and this square- that they have the same size?
20. Ss Yes.
21. T Well, yeah or no?
22. Ss Yes.
23. T Well, it fills again the whole board and there are the three triangles which were there before, too [demonstrating the previous diagram as shown in Figure 1(1)].
24. T So the two squares which you saw

before ... hence the sum of the areas of the squares above both legs is the same as the square constructed above the hypotenuse.

25. T That is another way how to prove Pythagoras' theorem. And there are many other ways. And these ways are even interesting in such a way that they can be used as puzzles.

(CZ,transcript, p.5-6)

#### Teacher demonstration with some students input.

During the process of demonstrating the proof, the teacher often asked students simple but critical questions to check if they understood what the teacher tried to demonstrate. The teacher clearly demonstrated and explained the components of the models (1~7) and the diagrams put up (8~18), and the relationship between the two diagrams (19~24) with the help of frequent questioning (2, 4, 6, 8, 10, 12,14, 16, 17, 19, 21).

**Hong Kong.** The process of proving the theorem can be divided into the following: Calculating the area of the diagram by using Figure 1(1); calculating the area of the diagram by using Figure 1(2); explaining the equivalence of the two methods; and (3) deducing Pythagoras' theorem. The following excerpt manifests the second way of calculating the area of the squares.

1. T Okay, can anyone tell me, actually this figure - this figure A plus B - the area of this figure and the area of this square C, what is the relationship?
2. Ss A plus B ...
3. Ss A ...
4. T What did you say?
5. Ss A plus B equal C.
6. T Equal. A student said the areas of these two figures are equal. Why?
7. T Why did you tell me that they are equal?
8. Ss Because... the four...
9. T Yes. A student has told me. He noticed what I did before was that—I took away three purple right-angled triangles, right?
10. T That means—I knew then that the areas of these three right-angled triangles are equal.
11. T If I take away these four right-angled triangles at the same time—we have said that they are the same, right?
12. T These two—their length and width are both a plus b. Their areas are equal.
13. T If I take away four right-angled triangles, I'm taking away the same things. So the resulting area—should the area of A plus B equal to C?
14. Ss Yes.
15. T Yes. We get—this is the result.
16. T Okay, this square A—this area, plus the area of square B are equal to C, the area

- of this square. Okay?  
17. Ss Yes.

### Teacher demonstration with little student input.

Based on the teacher's visual demonstration, the students were able to find out the relationship among area A, B, C (5), but they could not offer a clear explanation (8). Then the teacher demonstrated and explained the area relationship (9-17) in accordance with the principle that "taking away the same thing from the same area results in the same remainder".

**Shanghai.** The conjecture made by the students was proved by the following steps: examining a special case and then exploring the proof in the general situation. The discourse of introducing the first proof was illustrated as follows.

1. T:  $a^2+a^2=c^2$ . Then is there such a relationship in general right-angled triangles? Let's go on putting up the figures. *Please change them into four congruent general right-angled triangles. How many ways of putting it up do you have?* Let's have a try. [The students put up a square in groups of four. The teacher gives the students advice when making an inspection. The students who have finished putting up a square on the projector]
2. T: Well, let's stop to have a look. The group of Chen Tingting has put up a square like this just now. Well, Chen Tingting stand up to answer. *Why is it a square that you put up?*
3. S: They are four congruent triangles. The adjacent sides of two groups are equal. The sum of them is equal, too
4. T: The sum of them is equal
5. S: The square can be put up.
6. T: The square can be put up.
7. T: Sit down, please. There was another method to put up a square just now. Let's have a look.
8. T: (Because of the light color on the projector) Deepen the color a little. Well, the student who put up the square just now stands up to answer. *You put up a square. Why? Please, Shen Lingjuan?*
9. S: Because they are congruent triangles. Their hypotenuses are equal.
10. T: hypotenuses are equal.
11. S: So what I put up is a square.
12. T: Well, sit down. The students who put up this square put your hands up (Figure 6.7(4)). (The students raising their hands. It is found that the majority of the students put up their hands). Put them down! Is there anyone who put (the square) like Shen Lingjuan (did)? Yeah. So did your group. Well, put down your hands. What can we get through putting

up the picture? First of all let's look at the square that Chen Tingting put up just now. *If the short adjacent side is a, the long adjacent side is b, and the hypotenuse is c, what is the area of the big square?* (The teacher repeating) Well, Zhang Ling, please.

13. S: It should be  $(a+b)^2$
14. T: What does  $(a+b)^2$  indicate in the square?
15. S:  $(b+a)^2$  is  $(b+a)^2$ .
16. T:  $(b+a)^2$  is  $(b+a)^2$
17. T: Any other way? *Any other way?* Well, Zhou Yu, please.
18. S:  $4 \times \frac{1}{2}ab + c^2$ .
19. T: *What area is it?*
20. S: It's the area of the four small triangles added to  $c^2$ .
21. T: What area is this [pointing to the central small figure]?
22. S: It's the area of the small square.
23. T: it's the area of the small square.
24. T: *What relationship can you get?*
25. S:  $(a+b)^2 = 2ab + c^2$ .
26. T: Please put it in order.
27. S:  $a^2 + 2ab + b^2 = 2ab + c^2$ ;  $a^2 + b^2 = c^2$
28. T: Sit down, please. We can get what  $a^2 + b^2$  is equal to through putting up (the picture) just now.
29. Ss:  $c^2$ .

(SH09, transcript, p.9)

### Teacher control with much student input.

In the above excerpt, the teacher tended to encourage students to present their diagrams on the screen, and give their reasons (2, 8). Moreover, the students were asked to tell the whole procedure of their calculation (12, 17, 19, 24, 26).

### Discussion

The scenarios described here seemed to be contradictory to the stereotype that the classroom in East Asia is knowledge transmitting and teacher-centered. While the Czech teacher and the Hong Kong teacher preferred to demonstrate visually the process of proving with some elicitation of student's involvement, the Shanghai teacher seemed to put much emphasis on how to help students make their own conjectures and proofs, and encourage students to express their ideas by organizing deliberate activities. This observation may be due to the specificity of the particular content and the idiosyncrasy of the particular teacher, but at least it should point to a rethink about a clear-cut distinction between student-centeredness and teacher-centeredness.

### 4 Discussion and Conclusion

Based on this study, the following observations can be made:

#### 4.1 Mathematical proof versus visual verification

The Shanghai teacher seemed to put much stress on the

process of discovery and the proof of Pythagoras' theorem. Not only were students asked to make conjectures based on drawing, measuring and calculating, but also multiple mathematical proofs were explored by engaging in certain activities. Moreover, the teacher was adept at providing scaffoldings for students to develop the proofs by themselves. It seems that the Shanghai teacher tended to justify the theorem mathematically and logically.

On the other hand, the Czech teacher and the Hong Kong teacher tended to introduce multiple "proofs" by solving puzzles or demonstrating diagrams visually. It seems that the Czech teacher and the Hong Kong teacher preferred to justify the theorem visually.

Recently, the role of proof and the ways of treating proof have been re-vitalized in the new mathematics curriculum internationally. For example, *Principles and Standards for School Mathematics* (NCTM, 2000) suggests that the mathematics education of pre-kindergarten through grade 12 students should enable all students "to recognize reasoning and proof as fundamental aspects of mathematics, make and investigate mathematical conjectures, develop and evaluate mathematical arguments and proofs, and select and use various types of reasoning and methods of proof" (p. 56). It seems that the teachers in this study all meet this reformed orientation to a certain degree. However, the question is what kind of proofs should be introduced? The mathematical proofs like those in the Shanghai lesson, or the visual proofs like those in the Czech lesson and the Hong Kong lesson? The answer depends on what kind of mathematics are students expected to learn.

#### 4.2 Knowledge constructing or transmitting

It is quite surprising to find that the Shanghai teacher paid much attention to the process of re-discovering the theorem: from making conjecture to justifying the conjecture through a series of well-designed activities. The teacher in the Czech Republic and the teacher in Hong Kong put emphasis on introducing and verifying the theorem by using visual aids. Moreover, it was found that students in the Shanghai lesson were quite involved in the process of learning such as putting up and presenting diagrams, and explaining their understanding (although the teacher guided these activities heavily). This is quite contrary to other observations that East Asian students are passive learners (Paine, 1990; Morris, et al., 1996). The teacher in the Czech Republic and the teacher in Hong Kong tended to demonstrate and explain the process of knowledge construction and try to elicit student's responses to a certain extent. This finding seems to challenge the stereotypes presented in much of the literature.

#### 4.3 Explicit variation versus implicit variation

All teachers in the study emphasized classroom exercise. It is important to take note of the ways of varying the problems in classroom teaching, which may affect the quality of doing exercise. This study seems to suggest that the teacher in Shanghai was adept at providing both explicit and implicit variations of problems for students

to explore, while the other two teachers preferred using only explicit variation of problems for students to practice. Moreover, implicit variation of problems may provide students with more challenging and open-ended problems to tackle, which may be helpful for developing students' ability of problem solving.

#### 4.4 Concluding remarks

As Clarke (2002b) argued, an international comparative study "should be undertaken in anticipation of insights into the novel, interesting and adaptable practices employed in other school systems of whatever cultural persuasion, and of insights into the strange, invisible, and unquestioned routines and rituals of our own school system and our own mathematics"(p.15). This study aims at describing mathematics classroom scenes vividly in different cultures for readers to reflect upon their own practices. Moreover, this study employed a methodology that proves to be useful in capturing "the novel, interesting and adaptable practices employed" in the three places concerned. In particular, the methodology supports the notion that keeping the content invariant when comparing mathematics classrooms in different cultures will make a study more profitable (Alexandersson, 2002).

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