# Learning mathematics in a computer algebra environment: obstacles are opportunities

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**Abstract**: Using computer algebra is not as easy as it may seem. Students often encounter obstacles while working in a computer algebra environment. In this paper, global and local obstacles are distinguished, and obstacles from both categories are identified. The theory of instrumentation provides a framework for interpreting an obstacle as an unbalance of the conceptual and technical aspects of an instrumentation scheme. It is argued that making the obstacles explicit and trying to overcome them leads to conceptual development. Therefore, obstacles are opportunities for learning.

**Kurzreferat**: *Mathematiklernen in einer Computeralgebra-Umgebung: Hindernisse sind Möglichkeiten*. Die Verwendung von Computeralgebra ist nicht so einfach wie sie erscheinen mag. SchülerInnen begegnen beim Arbeiten in einer Computeralgebra-Umgebung häufig Hindernissen. In diesem Beitrag wird zwischen globalen und lokalen Hindernissen unterschieden und Hindernisse beider Kategorien werden indentifiziert. Die Theorie der Instrumentierung bietet einen Rahmen, ein Hindernis als eine Unausgewogenheit zwischen konzeptuellen und technischen Aspekten eines Instrumentierungsschemas zu deuten. Es werden Argumente dafür angeführt, dass eine Explizierung von Hindernissen und der Versuch sie zu überwinden zu einer konzeptuellen Entwicklung führt. Hindernisse bieten daher Möglichkeiten zu lernen.

**ZDM-Classification**: C30, C70, D30, D70, H20, I20, U70

## 1 Introduction

When I first encountered computer algebra, in the late 1980s, I was immediately fascinated by the power and the speed of the system, in my case Derive. Part of the fascination was the feeling that doing mathematics with such a system on the one hand seemed so simple, while on the other hand expertise is needed to use it efficiently. To make the nature of this expertise tangible is an intriguing but difficult task.

For example, it is easy to expand an expression such as $(x+y)^3+1$ (see line #1 in Figure 1). Going the other way however, getting Derive to transform $x^3+3x^2y+3xy^2+y^3+1$ into $(x+y)^3+1$, is not so easy. The factor command that often works for 'undoing expansion', leads to a different and more complex result in this case (see line #3 in Figure 1). If one doesn't know that the expression $x^3+3x^2y+3xy^2+y^3+1$ is the expansion of $(x+y)^3+1$, one has to 'see' that $x+y$ might be a factor. The symmetry of $x$ and $y$ in the expression suggests this. Then one can give $x+y$ a name, for instance $z$, then substitute $y = z–x$ (line #6) or $x= z–y$ and finally substitute $z = x+y$ (line #8). Easy for a mathematician who has the expertise to notice the symmetry of $x$ and $y$ in the expression, but for a student, who lacks mathematical expertise and experience with the computer algebra environment?

This example leads to the didactical issue that this paper deals with: what obstacles do students encounter when they work in a computer algebra environment and how can the teacher deal with these obstacles?

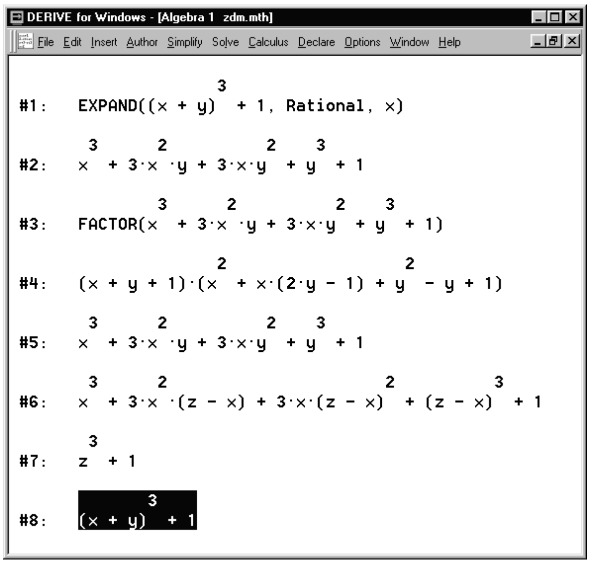


Figure 1: Finding back the factor $x + y$ with Derive

Why such a 'negative' perspective to concentrate on obstacles? In the early years of using computer algebra systems (CAS) in mathematics teaching and learning there was much optimism among educators concerning the possible benefits of computer algebra tools for the learner, but questions concerning the pitfalls and obstacles were hardly addressed (see, for example, the topics addressed in Heugl & Kutzler 1993). A few years later, Artigue (1997) argued that an analysis of constraints is relevant for the understanding of the potentials of a technological tool:

"Le travail analytique d'identification des contraintes est essentiel pour comprendre les fonctionnements possibles du savoir permis par un logiciel donné, pour analyser les d´écalages nécessairement existants avec les fonctionnements scolaires usuels de ce savoir et identifier les conflits et les problèmes de légitimité institutionnelle qui peuvent en résulter." (Artigue 1997, p. 139)

In this ZDM-issue, Guin and Trouche (2002) elaborate on this by distinguishing different types of constraints. My point of view is that the relation between the potentials of computer algebra environments and the obstacles they may generate is important, because students' difficulties and errors can be opportunities for real learning, and also may provide insightful data for the teacher or the researcher.

This paper describes obstacles that students encounter while working in a computer algebra environment. A distinction between global and local obstacles is made. It is argued that these obstacles offer opportunities for the learning of mathematics. Classroom experiences during three subsequent teaching experiments lead to a growing list of such obstacles.

Section 2 contains a starting definition of an obstacle, and an inventory of such obstacles observed in the first teaching experiment. Two categories are distinguished, the global obstacles and the local ones. In section 3 the theory of instrumentation is used to refine the category of the local, algebraic obstacles. This leads to a revised definition and to new obstacles that were identified during the second teaching experiment. Section 4 concerns the relation between mental conception, computer algebra technique and paper-and-pencil technique. This issue leads to a new global obstacle, observed during the third teaching experiment. Section 5 provides an example of student behavior, also observed during the third teaching experiment. It shows the complexity of the often-intertwined global and local obstacles. Section 6 summarizes the findings and discusses some consequences for teaching.

## 2 Global and local obstacles

My first probing and more systematic experience with student obstacles in a computer algebra environment took place during a teaching experiment in grade 11, the German SII. Obstacles are defined as barriers provided by the CAS that prevent the student from carrying out the utilization scheme that s/he has in mind (Drijvers 2000). The observations lead to the following, non-exhaustive list of obstacles that students encounter while doing mathematics in a computer algebra environment:

(1) *The difference between the algebraic representations provided by the CAS and those students expect and conceive as 'simple'*. This concerns difficulties in recognizing that, for example, -($x$ – 12), given by the CAS, is equivalent to 12 – $x$, that the student had in mind, or that $\sqrt{\frac{s}{4}}$ equals $\frac{1}{2}\sqrt{s}$. Recognizing equivalent expressions is a central issue in algebra, and still is when working in a computer algebra environment.

(2) *The difference between numerical and algebraic calculations and the implicit way the CAS deals with this difference*. For many students $\sqrt{2}$ is not a real answer: they consider 1.41 as the ultimate result. They do not really understand the difference in status of the two answers: $\sqrt{2}$ 'still has some algebra in it', whereas 1.41 is purely numerical. The CAS is not always clear about this difference in status.

(3) *The flexible conception of variables and parameters that using a CAS requires*. In a computer algebra environment 'all letters are equal', to paraphrase Orwell. However, in a specific problem context the variables have different meanings and roles, such as the role of unknown, parameter or changing quantity. The meaning and the role of the letter are 'in the eye of the beholder'. Working efficiently with a CAS requires that one deals flexibly with the roles of the variables involved and with the context-bound meanings they may have outside the software and the abstract way of dealing with them within the software.

(4) *The tendency to accept only numerical solutions and not algebraic solution*s. Students often are not satisfied with answers such as $x = ½\, s - ½\, v$. In the end they want to know what value $x$ stands for. This is called the 'expected answer obstacle'.

(5) *The limitations of the CAS, and the difficulty in providing algebraic strategies to help the CAS to overcome these limitations*. Sometimes, as in the example in the introduction, there is no direct command to perform a task, or the CAS is unable to carry it out without any help from the user. In such cases, cooperation between users' expertise and CAS capacities is needed to find a result.

(6) *The inability to decide when and how computer algebra can be useful*. Experienced users know what the CAS can be used for, and how to let it work for them in a certain problem situation. Novice users however don't have this sense of what can be reasonably expected from the tool.

(7) *The black box character of the CAS*. Usually the CAS provides no insight in the way it obtains its results. This means that students are often unable to verify the procedure. To them, the CAS has a black box character. Students may feel uncomfortable with this, as they are 'at the mercy of' a hardly controllable engine.

Some of these obstacles can be related to the theory of Realistic Mathematics Education (RME). In the RME philosophy, a bottom up approach with opportunities for reinvention and progressive formalization is essential. From this perspective, the top down and black box character of the CAS tool, often combined with a certain amount of idiosyncrasy, can be expected to produce obstacles in the learning.

Looking back at the list of identified obstacles, two remarks can be made. First, the obstacles have different natures. Obstacles 5, 6 and 7 share a global character: they deal with getting the machine to work for you in general, and with the relation between the problem solving plan and the implementation in the computer algebra environment. The first four obstacles, on the other hand, have a more local character: they deal with a particular mathematical topic – in this case algebra – and with the way it is treated by the CAS. The word 'local' does not suggest that it is not important: the issue of equivalent expressions in the first obstacle, for example, is essential for algebraic understanding, and in that sense global within the domain of algebra. However, it does not involve the overall strategy of using computer algebra for solving a mathematical problem. In the following, global obstacles and 'micro-didactical' local ones are distinguished.

The second remark concerns the question of whether the obstacles are indeed provided by the CAS. Rather, it seems that they can be considered as already existing cognitive obstacles, that are simply becoming more manifest and more important by working in the computer algebra environment.

The observations of the local obstacles motivated me to concentrate my research on the learning of algebra, in particular the concept of parameter, in a computer algebra environment. Therefore, most of the examples in the rest of this paper concern algebra.

## 3 Local obstacles and the theory of instrumentation

As described in the previous section, some of the local obstacles can be related to the theory of RME. A second theoretical framework that is fruitful for understanding the difficulties of effective use of technology is the perspective of instrumentation (Rabardel 1995; Artigue 1997; Lagrange 1999ab, 2000; Trouche 2000). For an explanation on the theory of instrumentation I refer to Guin and Trouche (2002) in this issue. For the purpose of this paper, the main points of the theory are the notion of instrumental genesis and the concept of the instrumentation scheme.

In short, instrumental genesis is the process that the user has to go through while learning to work with a (technological) tool. In order to be able to use a tool in a productive way, the learner must develop instrumentation schemes. These schemes together with the physical tool, the 'artifact', form the instrument. Often, this process requires time and effort.

Within an instrumentation scheme two components are intertwined: a technical component and a mental component. The technical part concerns the sequence of actions that one performs on a machine in order to obtain a certain goal. In the case of mathematical technological tools the mental part consists of the mathematical objects involved, and of a mental image of the problem solving process and the machine actions. Such mental mathematical conceptions are part of the instrumentation scheme, and can even develop further during the development of the scheme. Technical skills and algorithms on the one hand, and conceptual insights on the other are inextricably bound up with each other in the instrumentation scheme. In Guin and Trouche (2002) the concept of instrumentation scheme is elaborated on in detail.

Thus the instrumental genesis is the process of building up schemes that consist of both techniques and conceptions that give meaning to the techniques. In this sense the theory of instrumentation is in line with current views on the role of symbols and symbolizing in mathematics education (Gravemeijer et al. 2000). The point of departure here is that there is a dialectic relation between symbolizing and the construction of meaning. The elaboration of the relation between the theory of instrumentation and theories of symbolizing, however, is beyond the scope of this paper.

The value of the instrumentation theory is that it provides a specific way to look at the interaction between student and technological tool, and in particular it shows how seemingly technical obstacles can be related to conceptual difficulties. Paying attention to technical obstacles therefore often will involve conceptual aspects, and therefore may provoke conceptual development.

Let us make the theory concrete. The example from the introduction concerns rewriting $x^3 + x^2y + xy^2 + y^3 +1$ as $(x+y)^3 + 1$. What is needed to be able to perform the successful sequence of actions in lines #6, #7 and #8 of Figure 1? First, the student should have the skill to look at the expression that will be transformed, and realize that it is symmetric in $x$ and $y$. This is a conceptual skill. Second, the student should conclude that in that case it makes sense to try $x+y$ as a factor, a mental activity again. Then the question is how to perform this in the computer algebra environment. This requires not only the mastering of the syntax of substitution, but also the insight that one cannot substitute $x+y$ but that the substitution of $y = z – x$ is a good alternative. This leads to line #6 and simplification yields #7. Then once more some mental activity is needed: if we want an expression in $x$ and $y$, the $z$ should be replaced by $x+y$, and this time the substitution can be performed directly. This combination of mental conceptions and technical actions can develop into an instrumentation scheme if it is performed several times in similar situations, so that it becomes part of the 'repertoire' of the student.

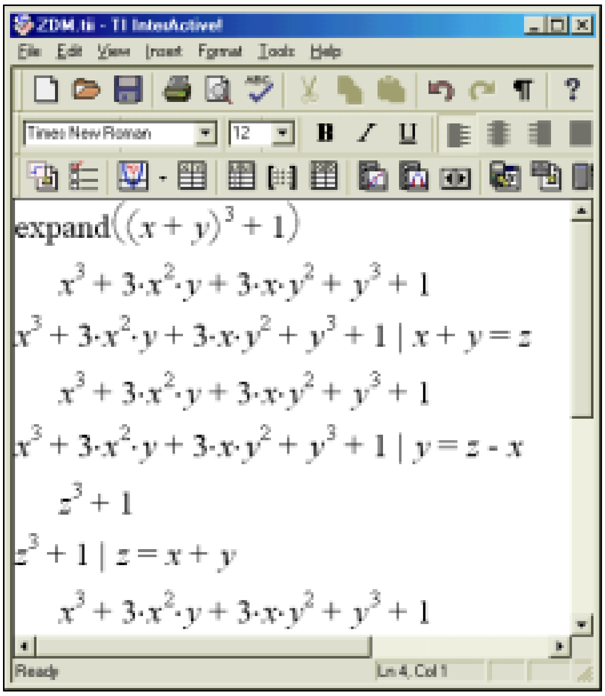


Figure 2: Substitution in TI-Interactive

In similar situations I observed students trying to substitute $x + y = z$. In TI-Interactive (see Figure 2), a relatively new software package that integrates word processing and CAS, this substitution has no effect and this may puzzle the user. It shows a lack of knowledge of the syntax of substitution, but also a limited conception of the substitution procedure. Substituting $y=z–x$ is a step in the right direction and results in $z^3+1$. A second substitution of $z = x+y$ does not give the expected $(x+y)^3+1$ because of the automatic simplification of the software. This illustrates that the relation between mathematical conception and technical skills is tool-dependent: the automatic simplification of TI-Interactive makes a difference compared to Derive. If Maple were used, the availability of the algsubs-command would affect the instrumentation.

In the second teaching experiment, that took place in grade 9, the theory of instrumentation is applied to the substitution of expressions and the solution of (parametric) equations (Drijvers & Van Herwaarden 2000, 2001). This leads to three new local, algebraic obstacles to add to our growing list:

(8) *The limited conception of algebraic substitution.* Students often think that substitution is limited to 'filling in numerical values'. That conception has to be extended to algebraic substitution of expressions.

(9) *The limited conception of algebraic solution.* Students often think of solving as 'calculating a numerical value'. This conception has to be extended to algebraic solution, including expressing one variable in terms of others .

(10) *The conception of an expression as a process.* Students often see an expression as a compact means to describe a calculation process. This conception needs to be extended to the expression as an object, as a 'thing' that can be moved, for example substituted into an equation.

The last obstacle is the most relevant of the three. In algebra, the duality process – object is very important. Several authors (Sfard 1991; Tall & Thomas 1991; Gray & Tall 1994; Tall at al. 2000) argue that mathematical objects have both a procedural and a structural aspect. To flexibly deal with this dual nature is an important algebraic skill. Wenger (1987) and Arcavi (1994) both point out that the ability to see (part of) a formula as an entity that can be moved, transformed or substituted is a strong indicator of 'symbol sense'. This issue is close to obstacle (1) in section 2 on equivalent formulas.

In the light of the instrumentation theory the description of an obstacle can be rephrased: An obstacle occurs when the technical and the conceptual part in an instrumentation scheme are not balanced, either because the technique is not accompanied by appropriate meaning and conception, or because the technical skills for the performance of a conceptual idea are lacking. This description is more symmetric than the one given in section 2, and does not 'blame' the CAS alone for all problems. Furthermore, it also suggests that conceptual development is part of the process of instrumentation in general and of the process of overcoming obstacles in particular. In that sense, obstacles are opportunities for learning.

## 4 Global obstacles and the triangle screen-paper-mind

In the previous section the relation between techniques in a computer algebra environment and the conceptions in the mind of the student was considered as crucial for appropriate instrumentation. However, students who start to work in a computer algebra environment usually already have a lot of experience with techniques in another medium: 'by-hand' techniques using paper-and-pencil. This is their main frame of reference, so it seems appropriate to consider the relation between the new 'screen techniques' and the old 'paper-and-pencil techniques' (Drijvers 2002).

It has already been indicated that, contrary to what one might think, techniques do not disappear or lose relevance in a computer algebra environment. As Lagrange points out:

> "We found a common assumption: CAS lightens the technical work in doing mathematics, and then students will focus on application or understanding. [...] Our survey of the French classrooms showed neither a clear lightening in the technical aspects of the work nor a definite enhancement of pupils' conceptual reflection. […] Technical difficulties in the use of CAS replaced the usual difficulties that pupils encountered in paper/pencil calculations. Easier calculation did not automatically enhance students' reflection and understanding." (Lagrange 1999b, p. 144)

Ideally speaking, the mental conception, the paper-and-pencil technique and the computer algebra technique should be the three connected edges of an integrated triangle that support each other. For paper-and-pencil and computer algebra techniques, two conditions are required: the computer algebra technique should be congruent with the paper-and-pencil technique, and the computer algebra technique should be transparent to the student.

By congruence is meant that a technique performed in both environments can be recognized as such, and is perceived as a different implementation of the same technique, instead of two different, unrelated techniques. For example, the syntax and the notation in the computer algebra environment should be 'natural' from the paper-and-pencil perspective. This does not mean that the differences between the paper-and-pencil medium and the computer algebra environment is ignored; for example, it is important that students are aware of the flexibility of paper-and-pencil and the power of computer algebra.

Transparency means that the student is able to 'look through' the way the computer algebra environment finds and presents its results, on the basis of his paper-and-pencil experience. This is difficult, as the computer algebra algorithms usually are more sophisticated than the paper-and-pencil techniques. For example, substitution in a CAS is often transparent, whereas solving or simplification often are not. However, simple features of the computer algebra tools such as the presentation of results in a comprehensive form and providing sensible information of eventual errors may enhance the transparency.

If the conditions of congruence and transparency are fulfilled, students may relate the different kinds of techniques, for example by transferring paper-and-pencil problem solving approaches to the computer algebra environment or by using computer algebra notations in their paper-and-pencil work. Two observations from the third teaching experiment illustrate this interference between computer algebra technique and paper-and-pencil technique. First, Martin is solving the system of equations:

$$x \cdot y = 540$$

$$\sqrt{x^2 + y^2} = 39$$

In the computer algebra environment, Martin prefers to isolate $y$ in each of the two equations and then set them equal to each other, while it is shorter to isolate $y$ in only one equation and then substitute the result in the other equation. My conjecture is that this behavior in the computer algebra environment stems from the similar problem solving approach that Martin is familiar with from the paper-and-pencil environment: his text book often uses functions and graphs as contexts for equations, and usually $y$ is already isolated in the formulas. By the way, many CASs can solve the system $x \cdot y = 540$,

$x^2 + y^2 = 39^2$ directly.

Second, Dustin is solving a system of equations with paper-and-pencil. This was after a TI-89 experiment, but the computer algebra calculator was no longer available. As Figure 3 shows, Dustin uses the vertical bar to indicate the substitution, which clearly is inspired from the TI-89 notation. The CAS notation is so natural to him, that transfer to paper-and-pencil takes place.

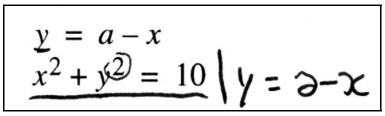


Figure 3: Transfer of notation

In general, my classroom observations suggest that such transfer did occur for the technique of substitution, but only to a limited extent for the solution of equations.

If transparency and congruence are conditions for connecting computer algebra techniques and paper-and-pencil techniques, what are its antagonists? I think the pseudo-transparency of computer algebra, the double reference phenomenon and the black box character, already mentioned as obstacle 7.

Pseudo-transparency (Artigue 1997) means that the technique in the computer algebra environment is close to the paper-and-pencil technique, but not exactly the same, with sometimes quite subtle differences. For example, if one enters ($x$+5)/3, in many CAS's one needs to use brackets, but on the screen the formula appears without brackets. In fact, the horizontal fraction-bar in $\frac{x+5}{3}$ can be considered as a special notation for brackets in the numerator, but the students often are not aware of that. In a similar way the brackets used to enter the square root of an expression disappear immediately after entering. As a consequence of this pseudo-transparency, students working in a computer algebra environment sometimes are not 'discovering mathematics', as we expect, but may be 'discovering the software' with all its peculiarities. That is what is meant by double reference: referring to specific representations of the computer algebra tool instead of referring to mathematical concepts.

The black box character of most computer algebra tools (obstacle 7) may prevent students from seeing the congruence between the machine use and the paper-and-pencil techniques. This seems to be an obstacle for students who may feel uncomfortable when they are not able to perform some techniques by hand and have to 'trust technology' without having means for verification. I attribute the fact that students seem to have difficulties with transfer with respect to solving equations to the black box character of the solving routine in the computer algebra environment.

This section can be summarized by stating that the integration of mental conceptions, paper-and-pencil techniques and computer algebra techniques is an important factor in the learning of mathematics using computer algebra. The observation of the difficulties in achieving this adds a new, global obstacle to our list:

(11) *The difficult transfer between CAS technique and paper-and-pencil technique because of the lack of congruence between the techniques in the two media.* Factors in this are the black box character and the non-transparency of the computer algebra tool.

## 5 Alternating global and local obstacles: an example

So far a distinction has been made between global and local obstacles. Several obstacles have been identified and the description of an obstacle has been related to the instrumentation theory. This section contains a long classroom observation of a grade 10 student, Maria, that shows how the global and local obstacles interfere. Before the observation, Maria and her colleagues used the TI-89 symbolic calculators for three weeks. They worked through assignments where the value of a parameter was changed with a slider bar. This way they studied the effect on the parabola, and thus perceived the parameter as a 'letter that determines the position of the parabola'. Maria works on the assignment in Figure 4.

Below you see a sheaf of graphs of the family $y = x^2 + b \cdot x + 1$. We pay attention to the vertices of the parabola.

- **a.** Mark and connect the vertices. What kind of curve do you seem to get?
- **b.** Express the coordinates of the vertex of a 'family member' in $b$. Hint: the minimum lies between the zeroes, if there are zeroes.
- **c.** Find the equation of the curve through all the vertices.

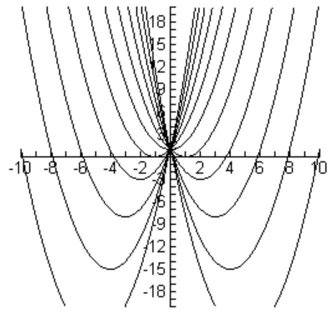


Figure 4: Sheaf of graphs

Maria thinks that question **a** concerns reproducing the picture with the TI-89. She enters `Y1=x^2+a*x+1` in the function list of the TI-89. Then she sees that the parameter is called $b$, tries to correct this but doesn't manage, clears the complete function and re-enters: `Y1=x^2+b*x+^1|b=`. Then she wonders what the values of $b$ are. She chooses values –5, -4, -3, … 4, 5. She sets the viewing window dimensions, first using the wrong minus key, corrects and gets a nice picture (see Figure 5). She managed to overcome the small, local obstacles and she is proud and happy.

Maria: It works! This is a revolution in my math calculator thing!

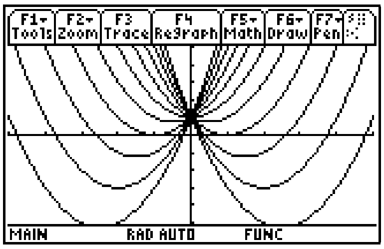


Figure 5: Maria's sheaf of graphs on the TI-89

Question **b** suggests to find the zeroes first. Maria enters solve(x^2+b*x+1,x), which gives an error message as '=0' is lacking. Some other computer algebra tools are not so strict on the difference between expressions and equations, and would have accepted this input. This is an instrumentation problem with solve, obstacle 9. Maria doesn't like the resulting error message:

Maria: It hates me, that calculator!

Then she tries solve(x^2+b*x+1,x|. She seems to consider substitution (obstacle 8). Then she changes *x* into *b*, solve(x^2+b*x+1,b), but that doesn't work either. An example of role confusion, obstacle 3. Then she reverts to her graph, to show it to the rest of the class by means of the projection display. She doesn't overcome the obstacles concerning solving and substitution and the lesson is over.

The next lesson, Maria enters the formula again. Apparently, it was cleared between the lessons. This time, however, she forgets the multiplication sign between *b* and *x*: Y1=x^2+bx+1. No graph, not even after adding the parameter values. When the observer suggests that the multiplication sign is missing, she corrects and gets the same picture as the lesson before.

Maria: Oh that has to be 'times'. Yes that's always me, I never do it completely right.

Maria once more wants to find the zeroes. She enters solve(x^2+b*x+1=y,b), so she now knows that there needs to be an equation instead of an expression, but she has *y* instead of 0 and solves with respect to *b* instead of x. The machine replies: b=(-x^2-y+1)/x. She seems to focus on *b*, probably because *b* is mentioned in the assignment, and she does not distinguish between the roles of the different letters (obst. 4).

Then she enters solve(b=(-x^2-y+1)/x)|b=5). She probably wants a numerical value instead of the general *b* (obst. 3)? However this doesn't work because of the errors in the solve command (obs. 9). The next trial includes a letter with respect to which the equation is solved: solve(b=(-x^2-y+1)/x),x|b=5) but the brackets are not well placed. She clears the line and substitutes 5 by hand: solve(5=(-x^2-y+1)/x,x). This gives an expression for *x* in terms of *y* that doesn't seem to help her.

Maria: I just don't understand it, I am simply too stupid. But I don't have a *y* of course.

Here we see the global obstacle of the inability to implement the problem solving strategy into the technological environment (obst. 6). One can wonder if Maria has a clear solving strategy in mind.

With help from the observer Maria notices that she should substitute 0 for *y* in order to find the intersection points with the *x*-axis. This gives the *x*-coordinates of the zeroes in terms of *b* (see Figure 6). A nice result, that could have been obtained in a more direct way.

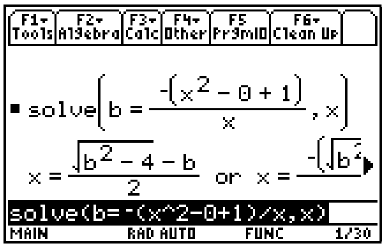


Figure 6: Zeroes in terms of parameter *b*

Maria copies this formula into her notebook, but has difficulties in reading the solution: the square root sign is too long in the second solution (see Figure 7).

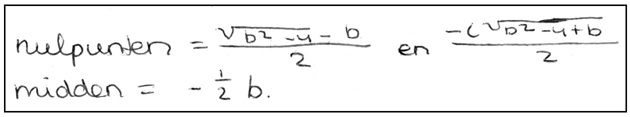

Figure 7: Copying solutions into the notebook

Now that she has the general solution for the zeroes, she seems to want concrete values for *b* before being able to proceed:

Maria: But then you have to fill in a value for *b*, don't you?

And, later:

Maria: But there is no use in knowing that the extreme value lies in the middle of the two zeroes if you cannot calculate it?

Here we see obstacle 2 on the preference for numerical results and obstacle 10 on the perception of expressions as answers. Maria seems to suffer from the so-called 'lack-of-closure' obstacle: unless you have a value for the parameters, you cannot calculate anything and you cannot go on.

However, with some help from the observer Maria accepts proceeding with expressions. With some struggle she takes the average of the two general solutions for *x* and finds that the *x*-coordinate of the vertex equals $-b/2$. She is satisfied with that:

Maria: Well I did find this out quite nicely.

In order to find the *y*-coordinate of the vertex, she substitutes $x=-b/2$ using the solve-command: solve(y=-0.5*b^2+b*-0.5*b+1,y), encountering obstacles 8 and 9. She forgets the brackets around $-b/2$, so that only *b* is squared. The result, $y=1-b^2$, is not correct. Also, using solve to carry out a substitution is not an efficient way of working. By the way, she doesn't seem to worry about the difference between ½ and 0.5 (obst. 2).

For question **c**, Maria writes down in her notebook that the vertices have coordinates $(-½b,\ 1-b^2)$ and concludes that the equation of the curve through the vertices is

$y=(1-b^2)\cdot x-\frac{1}{2}\cdot b$. However, probably with the help of her neighbor, she corrects her error and ends up with the right formula for the curve through the extreme points:

$$y = 1 - x^2$$

Looking back at Maria's problem solving behavior, many of the previously identified obstacles can be recognized. Also, the observation shows how the different kinds of obstacles may alternate and interfere.

Some of the more technical obstacles do not frustrate the problem solving process. For example, at the end Maria misses the difference between exact and decimal numbers, but that doesn't hinder her. Also, the wrong minus key while setting the window dimensions was corrected easily. On other occasions, for example when Maria forgets to put a multiplication symbol between two variables, technical details can be very time consuming, can frustrate the process and lead to losing track of the problem solving strategy while trying to overcome them. Such relatively simple syntactical problems, that may be due to the idiosyncrasy of the computer algebra tool, can have important influences indeed.

A more serious category of obstacles shows up when the technical problems are related to the conception of the mathematics involved. Take for example Maria's problems with solving equations. She tries to solve expressions instead of equations, she solves with respect to the wrong variable or forgets to indicate the variable, and she seems to confuse solving and substituting. The obstacles she encounters prevent her from following her original plans and lead to erroneous behavior. I conjecture that a more mature conception of solving, including the 'isolation' of one variable in order to express it in terms of some others, would have helped her. The conception of a formula as an object is a factor in that, too.

Maria's problem solving behavior seems to miss a clear direction. The impression is that the overall solving strategy is not clear for her, either because the geometrical context, the role of the parameter in the dynamics of the graph, and the meaning of the question are not clear to her, or because her global plan is frustrated by the local obstacles encountered.

As a new element, the observation shows that a meaningful interpretation of the computer algebra output is not evident. Maria's expression of the lack of closure, as well as her error while copying the solution formula, suggest that the meaning of these machine results for her is not clear. This leads to a final obstacle:

(12) *The difficulty in interpreting the output of the CAS.*

Maria's comments on what is happening, her feelings of victory when she succeeds and her frustration when things do not work indicate that the obstacles may create strong emotions that prevent the student from advancing further. That alone is a reason to take these obstacles seriously.

Several conclusions can be drawn from this case. First, the observations show how local obstacles lead to losing track of the global problem solving strategy. Second, many of the previously identified obstacles show up in conjunction in this episode. Third, the relation between technical and conceptual difficulties, as stressed in the theory of instrumentation, is observed. Finally, a new obstacle is identified that concerns the inability to understand the output of the computer algebra device.

## 6 Conclusion: obstacles are opportunities

This paper identifies the following obstacles that students encounter while working in a computer algebra environment.

1 The difference between the algebraic representations provided by the CAS and those students expect and conceive as 'simple'.
2 The difference between numerical and algebraic calculations and the implicit way the CAS deals with this difference.
3 The flexible conception of variables and parameters that using a CAS requires.
4 The tendency to accept only numerical solutions and no algebraic solutions.
5 The limitations of the CAS, and the difficulty in providing algebraic strategies to help the CAS to overcome these limitations.
6 The inability to decide when and how computer algebra can be useful.
7 The black box character of the CAS.
8 The limited conception of algebraic substitution.
9 The limited conception of algebraic solution.
10 The conception of an expression as a process.
11 The difficult transfer between CAS technique and paper-and-pencil.
12 The difficulty in interpreting the CAS output.

Two kinds of obstacles are distinguished: the global obstacles (5, 6, 7, 11, 12) and the local ones (1, 2, 3, 4, 8, 9, 10). For the latter category, the theory of instrumentation stresses the relation between machine technique and mathematical conception. Also, the lack of congruence between machine technique and paper-and-pencil technique can also play a role, as well as the lack of transparency of the computer algebra device.

It is worthwhile to consider the limitations of this inventory. First, it does not claim to be exhaustive; the identified obstacles just imposed themselves on different occasions. Second, the inventory of the local obstacles is subject dependent: it emerged from the perspective of the role of computer algebra in algebra education. With another focus, however, the list of specific obstacles might be different. I conjecture, however, that the dual character of the local obstacles, that share an interference of technical and conceptual aspects, exceeds the subject-dependency: if the topic was not algebra with parameters, the local obstacles might have a similar dual character. That would support the theory of instrumentation of technological tools.

There are two reasons why the observed obstacles should be taken seriously in the classroom. The first reason is indicated in the previous section: encountering obstacles can elicit feelings of irritation and frustration by the students. Although dealing with frustration is a part of doing mathematics, in some cases it can be counterproductive. Ignoring the obstacles in teaching can amplify this effect.

A second and probably more important reason to take the obstacles seriously in teaching is that they offer opportunities for learning. Obstacles, according to the description in section 3, often integrate a technical and a conceptual aspect. Therefore, working on overcoming an obstacle often also means working on the conceptual development of the mathematics involved. Many of the obstacles seem to be, at least partially, existing cognitive obstacles that are simply becoming more manifest in the computer algebra environment. Therefore, to investigate what the problem really is, to find out what the 'logic' is behind a specific syntax, to discover the meaning of the output, to invent a new strategy that is more feasible in the computer algebra environment, such activities offer opportunities for a better understanding, an improved conceptual development and a good mathematical attitude. In that sense, obstacles are opportunities for learning that can be exploited in interaction with individual students and in classroom discussions.

As a consequence for teaching, I recommend a pedagogical strategy of considering obstacles seriously, paying attention to them and taking advantage of the opportunities they offer. As Simon (1995) stated in a more general sense than the case of working in a computer algebra environment:

> "Conceptual difficulties that I have previously observed in students are not to be avoided; rather, they provide particular challenges, which if surmounted by the students, result in conceptual growth." (Simon 1995, p. 139)

Instead of trying to ignore the obstacles encountered, I suggest to make them the subject of classroom discussion in which the meaning of the techniques and the conceptions is developed. The mathematical ideas behind the obstacles should be considered explicitly, and the computer algebra environment used as an inspiring object of study instead of an 'oracle'. I believe that such an approach turns the obstacles of computer algebra use into opportunities for learning, and enriches mathematical discourse in the classroom.

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