

## Developing mathematical modelling skills: The role of CAS

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**Abstract:** Mathematical modelling as one component of problem solving is an important part of the mathematics curriculum and problem solving skills are often the most quoted generic skills that should be developed as an outcome of a programme of mathematics in school, college and university. Often there is a tension between mathematics seen at all levels as 'a body of knowledge' to be delivered at all costs and mathematics seen as a set of critical thinking and questioning skills. In this era of powerful software on hand-held and computer technologies there is an opportunity to review the procedures and rules that form the 'body of knowledge' that have been the central focus of the mathematics curriculum for over one hundred years. With technology we can spend less time on the traditional skills and create time for problem solving skills. We propose that mathematics software in general and CAS in particular provides opportunities for students to focus on the formulation and interpretation phases of the mathematical modelling process. Exploring the effect of parameters in a mathematical model is an important skill in mathematics and students often have difficulties in identifying the different role of variables and parameters. This is an important part of validating a mathematical model formulated to describe a real world situation. We illustrate how learning these skills can be enhanced by presenting and analysing the solution of two optimisation problems.

**Kurzreferat:** *Entwicklung von Modellbildungsfähigkeiten: Die Rolle von CAS.* Mathematische Modellbildung als eine Komponente von Problemlösung ist ein wichtiges Element des mathematischen Curriculums, Problemlösefähigkeiten sind die am häufigsten genannten generischen Fähigkeiten, die im Rahmen der Mathematikausbildung in Schule, College und Universität entwickelt werden sollten. Häufig besteht ein Spannungsverhältnis zwischen der Sichtweise von Mathematik als ein „Wissensspeicher“, dessen Inhalt in jeder Stufe um jeden Preis weitergegeben werden soll und der Sichtweise von Mathematik als eine Sammlung von Fähigkeiten zum kritischen Denken und Hinterfragen. Im Zeitalter leistungsstarker Handheld- und Computersoftware bietet sich nun die Möglichkeit die Verfahren und Regeln, die den Inhalt des „Wissensspeicher“ ausmachen, der über hundert Jahre zentrales Element des mathematischen Curriculums war, einer Revision zu unterziehen. Mit Hilfe der Technologie ist es möglich, weniger Zeit für traditionelle Fähigkeiten aufzuwenden und Zeit für Problemlösefähigkeiten zu schaffen. Wir sind der Ansicht, dass mathematische Software im Allgemeinen und CAS im Besonderen für SchülerInnen Möglichkeiten liefert, sich auf die Darstellungs- und Interpretationsphase im Modellbildungsprozess zu konzentrieren. Die Erkundung von Auswirkungen von Parameter in einem mathematischen Modell ist eine wichtige mathematische Fähigkeit und SchülerInnen haben oft Schwierigkeiten, die verschiedenen Aspekte von Variablen und Parametern auszumachen. Dies ist ein wichtiger Aspekt der Bewertung mathematischer Modelle als Beschreibungen realer außermathematischer Sachverhalte. Anhand der Lösung zweier Optimierungsprobleme und ihrer Analyse wird gezeigt wie das Lernen dieser Fähigkeiten verbessert werden kann.

**ZDM-Classification:** C40, D30, D50, I40, M10, U70

### 1 A rationale for mathematical modelling in the mathematics curriculum

To the applied mathematician, scientist and engineer the ability to solve real problems using mathematical models is one of the most important reasons for learning mathematics and this depends on the ability to apply mathematical algorithms and rules. This link of problem solving with mathematics involving the application of traditional mathematics to known models is one view of mathematical modelling. However the words 'mathematical modelling' cover a broad range of theoretical and practical orientations. At one extreme is empirical modelling involving the collection and analysis of experimental data. At the other extreme mathematical modelling is a cyclic iterative process for solving problems consisting of three main stages. First a problem in the real world is formulated as a mathematical problem. Second, the mathematical problem is solved. Third, the solution is translated back into the original context and the results are interpreted to help solve the original problem. It is this process view of modelling that forms the theoretical basis for this paper.

Problem solving skills are often the most quoted generic skills that should be developed as part of a mathematics curriculum. The NCTM Principles and Standards for School Mathematics (2000) identifies problem solving as important for all school students:

“Instructional programs from prekindergarten through grade 12 should enable all students to –

- build new mathematical knowledge through problem solving;
- solve problems that arise in mathematics and in other contexts;
- apply and adapt a variety of appropriate strategies to solve problems;
- monitor and reflect on the process of mathematical problem solving.

By learning problem solving in mathematics, students should acquire ways of thinking, habits of persistence and curiosity, and confidence in unfamiliar situations that will serve them well outside the mathematics classroom. *In everyday life and in the workplace, being a good problem solver can lead to great advantages.* (emphasis added) (NCTM, 2000).

Problem Solving in Mathematics is often thought of in three ways:

- pure mathematics investigation activities,
- mathematical modelling,
- mathematics problems and real problem solving.

Tasks that have been classified as “Mathematics Problems” and “Real Problem Solving” are those examples and exercises that are traditionally found in mathematics textbooks and other classroom resources. They usually provide students with opportunities to practice the procedures, rules and skills of the traditional mathematics curricula. “Mathematical Modelling” and “Mathematical Investigation” tasks often require students to develop their own models and explore their own conjectures in order to meet some criteria. They often provide good opportunities for students to develop problem solving and investigation skills that are useful in all areas of mathematics.

Nyman and Berry (2002) discuss the rationale for teaching mathematical modelling in the context of developing transferable skills in undergraduate mathematics programmes. The relationship between the curriculum, teaching and learning is a complicated one. The curriculum should be seen as more than a collection of facts, rules, algorithms and a collection of activities to deliver them and assess them. It must be a coherent view of mathematics that develops connections between the multiple representations of each topic. The modes of thinking that mathematics develops are often a more useful attribute than knowledge of any particular mathematical fact, algorithm or rule. A graduate who is able to work to recognize patterns, generalize, improve, verify and extend models has the skills to work at a more general systems level.

Another reason for providing school students with the opportunity to develop the skills of modelling realistic situations is demonstration by the work of Picker (2000). In her study of students' images of mathematicians and the work they do, she found that for students of lower secondary school age the work of mathematicians is essentially invisible. Students believe that mathematicians do applications similar to those they have seen in their own mathematics classes including arithmetic computation, area, perimeter and measurement. They also believe that a mathematicians' work involves accounting, doing taxes and bills, and banking; work which they contend includes doing hard sums or hard problems, yet the students could supply no specifics about what such problems entail. Here are four typical comments from students about what they think mathematicians do:

"I don't know why someone would hire a mathematician"

"I'm not sure what a mathematician does."

"I can only use a mathematician in the 8<sup>th</sup> grade just to put him in my bookbag and take him to the exam with me"

"He will do my homework and go to school in my place"

Many teachers will adopt the content focussed approach in which mathematics in school is seen as a body of knowledge to be 'taught' and assessed often by external agencies. It is possible for such an approach to encourage conceptual understanding as well as mastery of basic rules and algorithms. Judgments about those facts, algorithms and procedures that are essential will change as technology takes over the traditional algorithms. For example, most of the rules and procedures of arithmetic, algebra and calculus that have formed the heart of mathematics for many hundreds of years can now be performed with hand-held technology. Thus more time and effort can be spent on developing good conceptual understanding instead of just developing pencil and paper skills. However if curriculum designers, examiners and teachers consider the mastery of basic rules and algorithms as the important feature of mathematics teaching and learning then students will be locked into a rule-based approach instead of a concept based approach. This notion of *teacher privileging* has been investigated by Kendal and Stacey (1999) and is explained in their article in this issue of ZDM.

It is recognized fairly widely among mathematics edu-

cators in many countries that mathematics teaching and learning has six important ingredients, exposition by the teacher, discussion, appropriate practical work, consolidation and practice of basic skills and routines, (real) problem solving, investigational work. For the traditional teacher for whom mathematics is a set of skills delivered through worked examples and exercises, the introduction of problem solving, practical work, discussion and investigations may seem an awesome challenge. The six ingredients of good teaching of mathematics provide a framework for developing good mathematics lessons. There will be times when exposition by the teacher (i.e. the teacher introducing the topic and 'teaching') followed by individual skill development tasks are important and necessary. If these approaches are mixed with good discussion between 'student and student' and 'student and teacher' then this will encourage active learners involved in the learning process. We learn by doing not by watching. You could not become an accomplished violinist just by watching an orchestra playing! Mathematics is the same - students become good at mathematics by doing it and by using it. The enjoyment and appreciation of the usefulness of mathematics and the development of young mathematical thinkers comes with including all six ingredients.

Increasingly over the past few years there has been a growing movement in mathematics teaching towards the need to apply mathematics to solve 'real' problems and to develop the ability of students to 'think mathematically' through investigational activities. But the pressure of assessment and the expectations of teachers, school administrators and other education professionals about the content and character of school mathematics have led to a general lack of integration of problem solving and investigations in mathematics. Furthermore developing mathematical modelling skills is generally agreed to be more difficult to teach and to assess than developing algorithmic skills.

In teaching mathematical modelling skills, attention is often paid to defining and examining the processes and stages through which the successful modeller passes. However when assessing the skills that have been developed by the learner, it is usual to require the submission of a report (oral or written) of the whole process. We then grade model formulation, model solution and interpretation by looking at the overall outcome. Little attention is paid to describe attainment levels of students at the sub-stages of model development. From the evidence that the model formulation stage is found to be difficult for students then we would argue that effective teaching strategies need to be adopted to enhance the learning of this stage. Haines and Crouch (2001) report on the development of a research tool that has the potential to identify a student's development in mathematical modelling, in particular the use of multiple-choice format test questions to investigate the improvement in the student's skills at the sub-stages of the modelling process.

It is within this framework of teaching mathematical modelling skills, in particular the formulation stage of the process, that this paper is set. We will discuss such questions as: What role does the task play in developing

modelling skills? How do the available tools support the modelling process? What are the traditional skills that students still require when they use technology to find the mathematical solution?

## 2 Models, modelling and technology

### 2.1 *The tension between teaching models and modelling*

'Mathematical Modelling' is an umbrella term that has been used extensively since the 1970's to describe the process of solving problems set in the real world. Blum (1990) suggests that

"by real world I mean the rest of the world outside mathematics i.e. school or university subjects or disciplines different from mathematics, or everyday life and the world around us".

Modelling the real world is not new and it has been one of the reasons for teaching, learning and doing mathematics for many thousands of years! However the term 'mathematical modelling' is used rather widely and arbitrarily to encompass the use of models and the use of mathematics in any form of problem solving. A large number of research studies include mathematical modelling as a key word; for example a search of the ERIC data base on research articles about mathematical modelling listed 19 354 articles in the last ten years! If one searches for 'Modelling NOT Models' one still finds 2 868 references. However a closer study of these papers shows that there are less than one percent that explore the teaching and learning of mathematical modelling skills or the working styles of students as they solve problems. Most of the published research has been about teaching specific models.

Similarly, many courses and textbooks with mathematical modelling in the title do not actually address the process of modelling but focus of teaching examples of mathematical models and the mathematical techniques needed to use these models. This leads to a conflict of learning interests. I would argue that the teaching of modelling skills should be emphasized in a modelling course and that students should only use the mathematics skills that are familiar to them. Standard models and familiar mathematics skills can be used to illustrate the modelling process. Houston (1997) suggests that mathematical modelling is a way of life for everyone. He discusses the role of the three M's – methods, models and modelling, and the R's of mathematics – reasoning, reading, 'riting and 'rithmetic in promoting this way of life in post 16 school mathematics. The importance of distinguishing between models, modelling and methods is once again emphasized.

Mathematical modelling is often classified in two ways: (1) empirical modelling, in which models are fitted to data that is either given in the problem or collected by the student; (2) theoretical modelling, in which a student develops a model from a theoretical perspective focussing on the important features to be included in the model. Of course data is needed in the second perspective of modelling in order to evaluate the goodness of the model. In this paper we focus on developing theoretical mathematical models through the modelling process

summarized by the modelling cycle in the following diagram:

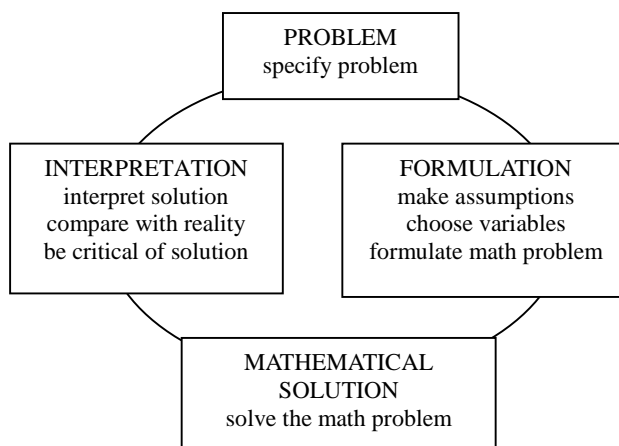


Figure 1: Mathematical Modelling cycle

The role and nature of the task are important ingredients that can be used to elicit the development of modelling skills as opposed to using standard models. A modelling approach to the teaching and learning of mathematics focuses on the mathematization of realistic situations that are meaningful to the learner (Doerr & English 2001). They point out that:

"In solving typical school word problems, students generally engage in a one or two step process of mapping problem information onto arithmetical quantities and operations. In most cases, the problem information has already been carefully mathematized for the student. The student's goal is to unmask the mathematics by mapping the problem information in such a way that an answer can be produced using familiar quantities and basic operations. In modelling tasks, the student's goal is to make sense of the situation so that s/he can mathematize it in ways that are meaningful to her/him. This will involve a cyclic process of selecting relevant quantities, creating meaningful representations, and defining operations that may lead to new quantities."

In their research, Doerr and English investigated the students' ability at ranking in order the multiplicity of features that are needed to model four realistic situations. These are good examples that develop the formulation skills of the modelling process. In contrast Jones (1997) provides an example of a problem that does not develop theoretical modelling skills. The students are given the population of the USA from 1790 and 1960 and asked to determine the present population by (i) fitting the data to a Malthus model by the method of least squares and (ii) fitting any three of the data points to a logistic model. This is an example of a closed problem for which the model is given as a differential equation and students are encouraged to use the computer software Maple to solve the problem. This example is typical of the many examples found in textbooks with titles such as *Teaching Differential Equations through Modelling*. There is often very little modelling going on; it is not about teaching mathematical modelling but about teaching mathematics in context.

Table 1 suggests a classification of mathematical tasks and proposes that each category of mathematical task has

open and closed problems. Closed tasks are predetermined in terms of problem setting, procedures and algorithms, and results and outcomes. Openness includes such characteristics as ill-defined problems, choice of a variety of procedures and methods, and non-exact outcomes (Sahlberg & Berry 2002). It is open real world tasks that we believe provide rich opportunities to learn mathematical modelling skills. For a more detailed discussion of this typology and the notion of open and closed tasks and examples of them see Sahlberg and Berry.

Table 1: Typology of tasks in school mathematics

<i>Scope of mathematical task and Category</i>	<i>Nature of task</i>
Drilling basic skills I	Closed in terms of method and outcomes
Applying a formula or algorithm II	Typically closed in terms of outcomes and also the methodology
Measuring and collecting data I – IIA	Some openness in terms of methodology but rather closed in terms of outcomes.
Real Problem Solving IIB – III	Real problems are those encountered in everyday life. They may or may not involve mathematical models. Openness of these tasks may vary from rather closed to open.
Mathematical Modelling III	Modelling tasks are typically real problems that require mathematical principles and formulas in order to be solved. Tasks are open in terms of procedures and outcomes.
Mathematical Investigations IV	Basic investigations are often closed in terms of outcomes but open to various methods. Extended investigations are typically open tasks.
Designing projects and studies in mathematics V	Projects and studies are the most open mathematical tasks. The openness includes the setting of questions and selection of methods.

## 2.2 Technology and mathematical modelling

Technology can be used in many ways in the mathematical modelling process. Computer simulations and spreadsheets can be used to develop mathematical models so that students get the feel of the importance of parameters in a model. Data logging equipment provides an opportunity to collect data to validate models. Computer algebra systems can be used to do the mathematical problem solving part of the modelling process.

Zbiek (1998) investigated the strategies used by 13 pre-service secondary school mathematics teachers to develop and validate functions as mathematical models of real world situations. For each of four modelling activities the participants in the research had constant access to technology including curve-fitting software, graphing calculators and CAS. She found that the working styles of

the participants fell into four categories: the ‘fitted-function selector’ strategy using the curve fitting software and an associated goodness of fit value as a ‘black box’ to fit data to polynomial and exponential functions; the ‘potential-function generator’ strategy in which the students use the curve fitter to develop the best fitting function and then validate the model by investigating how the characteristics of the function match the features of the real world situation; the ‘scatter plot/graphing tool’ strategy begins with scatter graph plots and then explores functions other than those that the technology tool can fit; in the fourth strategy the technology is not used in the formulation of the model. In this last case the development of the model depends on the students’ knowledge and understanding of functions.

The working styles of the participants in Zbiek’s research study are typical of the inexperienced mathematical modeller in that for the first three strategies they adopt an empirical approach based on experimental data collection. Those adopting strategy 4 are developing a theoretical model. Only in one of the three empirical strategies do the students attempt to consider the features of the real world situation in validating the model.

Doerr and Tripp (1999) in their research on understanding how students develop mathematical models found that the use of technology, in this case a graphic calculator, served to precipitate shifts in students’ thinking by identifying mismatches between students predicted models and the graphs of the data. In this research we see technology being used to help students to challenge their misconceptions and formulate an improved model of several physical situations.

Research on the use of CAS in mathematics education is still relatively new and has tended to focus on the student-knowledge-technology triad. It is commonly believed that CAS will have a major impact on the teaching, learning and assessment of mathematics at all levels (Berry et. al. 1997). By freeing students from some of the syntactic manipulations, CAS will allow them to concentrate on semantic and conceptual aspects of algebraic and calculus reasoning. In the framework of modelling, in particular, solving algebraic word problems Zehavi and Mann (1999) propose that the use of CAS helps to create networks of relationships between conceptual and procedural knowledge that facilitates mathematical understanding. It is important for students to realize that CAS itself cannot translate a story into an equation; that is the important role of the mathematician that we will explore through two problems in the next section of this paper. What CAS can do is provide the power to allow students to explore the consequences of their choice of mathematical model.

Data logging equipment encourages an empirical modelling approach to problem solving that focuses on the shallow skills of data-graph-prediction. A curve fitting exercise allows the modeller to suggest the law obeyed by the data, and the thoughtful modeller may then suggest the origins of the parameters. The weakness of this paradigm is that a real phenomenon has to exist and to be observed in order for data to be collected before it is fitted. Its strength lies in the motivation of having observed the phenomenon and seen what it really does,

allowing the mathematics to be seen as relevant and meaningful. Empirical models are often employed for prediction rather than understanding since they do not probe the underlying relationships of the phenomenon observed. Our experience of working with students, for many years in modelling courses for undergraduates, is that student working styles do not follow a systematic approach. They often fail to look back or revise their models and tend to adopt a search for data followed by data fitting to linear equations as their models. The participants in the studies of Zbiek (1998) and Doerr and Tripp (1999) also demonstrate this haphazard way of working.

Even in open-ended problems where a theoretical approach is more appropriate many students adopt the empirical paradigm. In developing courses to teach students' mathematical modelling skills we need further empirical research to investigate students' working styles. Maull and Berry (2001) present the results of a case study of four groups of undergraduate students who were observed tackling a simple modelling task as part of an undergraduate module in mathematical modelling. This paper shows that when data-logging equipment is readily available at the start of a problem, students adopt the empirical paradigm often producing poor models.

What do we learn from investigating students working styles when modelling? The most important observation from the research is that the students are working in a very haphazard way. They rush in to collect data. The natural instinct, because of the modelling experiences that they have learnt in school, is to work on an empirical model. One recommendation is that to facilitate students' development as good mathematical modellers, classroom instruction should promote the need to stand back initially from the actual problem at hand and to spend time at the start reflecting on the physical situation. Students should be encouraged to write down in words what is happening physically and should agree how to approach the various pieces of the jigsaw puzzle that will likely be the outcome. Another recommendation is to facilitate students' development as good theoretical mathematical modellers we would recommend that data logging equipment should not be available at the outset and that students should be encouraged to discuss and explore the physical situation. The appropriate use technology must be carefully taught to students so that it is not used poorly.

### 2.3 Summary

In summary the published literature shows a tension and confusion between the teaching of models and modelling. During the past fifteen years it has become fashionable to teach mathematics in context and the rationale is reported by many authors (see for instance, Blum (1990), Bassanezi (1994), Skovsmose (1994) among many others). It is still generally true that teachers and students see mathematics as a body of knowledge consisting of rules and algorithms and the focus of the curriculum and assessment is on learning these rules. The role of mathematics as a modelling tool and the process of developing mathematical models pass most students by. Choosing a range of tasks that includes all of the categories in the

typology provides the opportunity for pupils to develop the appropriate mathematical, investigational and modelling skills.

The available technological tools help us in modelling. Curve-fitting software, graphic calculators and data logging equipment provide tools to support the development of empirical models and we have discussed some of the benefits and drawbacks of such technology. We rarely draw graphs by hand relying on appropriate technology and our understanding of functions to interpret the graphs in the context of the real world situation. As we develop theoretical models we can rely on CAS to do the manipulations of the algebraic expressions and calculus that may occur. If CAS is associated with software for doing algebra and calculus symbolically then there is little scope for using CAS in developing models empirically. The role of CAS is in the formula manipulation or mathematical solution phase of the modelling process shown in Figure 1. Of course when we have a formula from a graph then we might use CAS to explore the model further. However we cannot abandon the traditional skills of arithmetic, algebra and calculus. In the next section we solve two problems to demonstrate modelling in action and then in Section 4 we discuss the 'traditional' skills that are still required in the mathematics curriculum that includes CAS.

## 3 Mathematical modeling in action: two illustrative examples

### 3.1 Designing a drinks can

*The Problem:* We wish to make a drinks can to hold 330ml of drink. What are the dimensions of the can that uses the minimum amount of material?

*Formulating a mathematical model:* The first step in problems of this type is to draw a picture of the can introducing variables for the radius and height of the can.

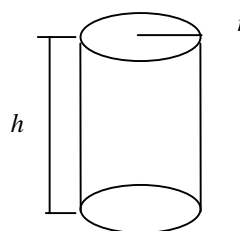


Figure 2: Modelling the drinks can

In drawing this picture of the can we are assuming that the drinks can is a circular cylinder with a top and bottom and that no materials are wasted in making the can.

If the volume of the can is denoted by  $V$  and the surface area of material used in making the can is  $S$  then:

$$V = \pi r^2 h$$

$$S = 2\pi r h + 2\pi r^2$$

The volume of the can is 330 ml so that  $V = 330$ ; write  $h$  in terms of  $r$  and substitute for  $h$  in the formula for  $S$  to give

$$S = \frac{660}{r} + 2\pi r^2$$

*The Mathematical problem and its solution:* Find the value of  $r$  for which the function  $S(r)$  is a minimum. The TI-89 screen for this activity gives

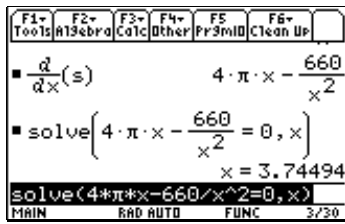


Figure 3: Finding the solution with the TI-89

Either using the second derivative test or from a graph of the function  $S(r)$  we conclude that the minimum surface area of material to make a drinks can containing 330 ml will be for a cylinder of radius 3.74 cm.

*Interpreting the solution:* At this stage we question the results or predictions of the model. This last skill is more than just saying “have I got the answer correct?”. The mathematical answer may be correct but the interpretation in the context of the real world may not be practical. The mathematical solution suggests that the drinks can should have a diameter of 7.49cm and a height of 7.49cm. The amount of material used is 264.36 cm<sup>2</sup>.

What does this mean for a physical can? Making a can with these dimensions and comparing it with a popular soft drinks can shows that the mathematical can is not very comfortable to hold or to drink from. What are the consequences for the surface area of the material to make a more realistic can?

Figure 4 shows how the surface area varies with the radius. Y1 is the surface area and Y2 is the percentage difference between the surface area and the optimal mathematical solution.

x	y1	y2
2.65	293.18	10.903
2.7	290.25	9.7944
2.75	287.52	8.7608
2.8	284.97	7.7992
2.85	282.61	6.9063

Figure 4: Table showing how the surface area varies

We see that a reduction of the radius to 2.75 cm leads to an increase in approximately 9% in material used. This would provide a design of the drinks can that is more useful to use than the ‘mathematical can’ and is not too expensive on materials.

### 3.2 Traffic flow through road repairs

*The problem:* Traffic queues often form on motorways when three lanes of traffic are forced to reduce to two lanes or to one lane because of road-works or accidents. This is a major frustration to motorists and often extends

journey times by hours.

Consider the problem of reducing two lanes of traffic to one. What speed of traffic achieves a maximum flow through the motorway section?

*Formulating the mathematical model:* The objective is to maximise the number of cars per hour passing through the motorway section; this is the flow rate that we designate as  $F$ . Suppose that vehicles enter the section  $t$  seconds apart. Then the number of cars entering per second is  $1/t$  and the flow rate is

$$F = \frac{3600}{t}$$

Suppose that the distance between the fronts of two vehicles is  $d$  metres. At a speed of  $v$  ms<sup>-1</sup> the quantities  $t$ ,  $v$  and  $d$  are connected by the equation  $d = vt$ .

Hence the expression for the flow rate becomes

$$F = \frac{3600v}{d}$$

The distance  $d$  is made up of two parts, the length  $L$  of the vehicle and the separation between the vehicles  $s$ , so that  $d = s + L$  and the expression for the flow rate becomes

$$F = \frac{3600v}{s + L}$$

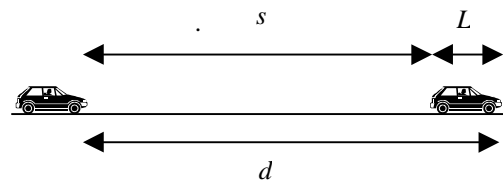


Figure 5: Modelling the car separation distance

*Modelling the separation distance:* The British Highway Code recommends overall safe stopping distances between vehicles for different speeds. Using the data from the Highway Code the thinking distance in feet is the same as the speed in miles per hour ( $v$ ). With the aid of algebra and/or graphs it can be shown that the braking distance is  $\frac{v^2}{20}$ . Hence,  $s = v + \frac{v^2}{20}$  and if we assume that

all the vehicles are average sized family cars then  $L = 13$  ft (note that we are using feet to retain the elegance of the stopping distance formula).

*The mathematical problem and its solution:* The mathematical problem is to find a maximum of the function

$$F = \frac{3600v}{13 + v + \frac{v^2}{20}}$$

The TI-89 screen for this activity gives

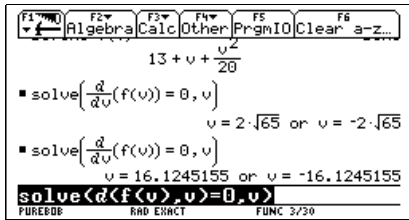


Figure 6: Finding the solution with a TI-89

The conclusion is that the vehicles should travel at about 17 mph (or 27 km/hr on European roads!).

The next step in the modelling process is to criticize and improve the model. An optimal speed of 17 mph is rather slow for most drivers; suppose that the cars do not travel at such a slow speed, how does this affect the flow rate? Figure 7 shows the percentage decrease in flow rate as the speed of the cars increases.

x	y1	y2
22.	1337.8	-2.916
24.	1313.1	-4.713
26.	1285.7	-6.698
28.	1256.9	-8.792
30.	1227.3	-10.94

Figure 7: How the flow rate varies with speed

We see that a considerable increase in speed from the optimal value of 17 mph (27 km/hr) to a more realistically acceptable speed of 30 mph (48 km/hr) reduces the flow rate by approximately 10% assuming that the cars travel at the Highway Code recommended safe stopping distances.

It is unlikely that vehicles will travel at the 'recommended safe stopping distances'. So it is the formula for stopping distances that will need to be revised. Another approach is to introduce a more general algebraic formula for stopping distances containing parameters  $a$  and  $b$

$$s = av + bv^2.$$

In this expression  $a$  is the fraction of the thinking distance, if  $a = 0$  then we are ignoring the thinking distance and if  $a = 1$  then we have the British Highway Code model. Similarly  $b$  is the fraction of the braking distance, if  $b = 0$  then we are ignoring the braking distance and if  $b = \frac{1}{20}$  then we have the British Highway Code model. It is common for students to argue that the 'thinking distance' can be reduced or even ignored. However as we shall see this will lose some essential information.

Now the mathematical problem is to find a maximum of the function

$$F = \frac{3600v}{L + av + bv^2}$$

The TI-89 screen for this activity is

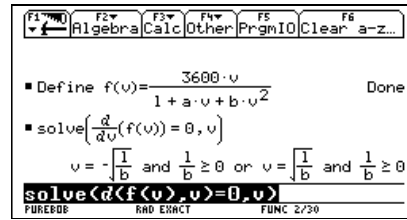


Figure 8: Finding a maximum of  $F(v)$

The algebraic approach leads to the solution  $v = \sqrt{\frac{L}{b}}$

which is a surprise for many students that the value of  $v$  for maximum flow rate is independent of the thinking distance. What does this actually mean? The value for  $v$  for which the flow rate is a maximum does not depend on the value of  $a$  however the corresponding maximum value of the flow rate does depend on the value of  $a$ .

Substitute for  $v = \sqrt{\frac{L}{b}}$  into the expression for  $F$  and simplifying then the maximum flow rate is given by

$$F_{\max} = \frac{3600}{a + 2\sqrt{Lb}}$$

For each value of  $b$  the maximum flow rate occurs for  $a = 0$  i.e. ignoring the thinking distance. (The students' intuition is correct concerning the actual maximum flow rate.) A realistic approach is for the traffic engineer to recommend a realistic speed and accept that the number of cars traveling through the road-works will not be optimal. The drivers themselves may 'feel' that they are getting through the system more quickly but in practice because the traffic flow rate is reduced there will be more traffic hold-ups and the traditional traffic queues will be longer than is mathematically necessary!

#### 4 Reflecting on the process and the required skills:

The solution to each problem suggests a linear algorithmic process in which the first step is to decide the features, then define the variables, then find relationships and so on. Of course the experienced modeller knows that this is far from the truth. However we will use these steps in the each solution to explore the skills that students need in order to be able to solve these problems. In real life modelling situations we often need to revisit the list of features to be included in the model to review the approach. Perhaps our attempt has omitted an important feature so that the model is too simple or perhaps there are too many features to sensibly include. Another situation that can occur is to attempt to develop a model where the mathematics is beyond the modeller's level of competence or knowledge. Modelling is a messy activity but in solving a real world problem we will have visited the steps in Figure 1 as least once.

Table 2 summarizes the mathematical skills needed to solve each problem.

Table 2: Mathematical skills needed in each problem

<i>skill</i>	<i>evidence in each problem</i>
defining variables	<i>Can design</i> : identifying the features (radius, height, area and volume) and the associated variables are fairly straightforward for students in this problem; the features are suggested in the problem statement. <i>Traffic flow</i> : the features are not obvious to students and they have some difficulty in this problem; considerable help is needed to formulate the model for flow rate
finding relationships	<i>Can</i> : formulas for $V$ and $S$ are known <i>Traffic</i> : the flow rate is not familiar to most students and they need to formulate the model from an understanding of the physical situation; What are the independent and dependent variables in each case?
manipulating symbols	An expert modeler takes it for granted that we need to identify one independent variable and write each expression in terms of it <i>Can</i> : 2 dependent variables $S$ & $V$ and two variables that can be taken as independent $r$ , $h$ <i>Traffic</i> : formula for $F$ needs the stopping distance formula for $s$ ; independent variables is $v$
defining the mathematical problem	<i>Can</i> : Find a minimum of the function $S = \frac{660}{r} + 2\pi r^2$ <i>Traffic</i> : Find a maximum of the function $F = \frac{3600v}{13 + v + \frac{v^2}{20}}$ These mathematical problems are the usual starting point for calculus activities but in our tasks we have done a lot of work to get this far.
solving the mathematical problem	In each mathematical problem, CAS is used to find the optimal solution using calculus.
interpretation	In each case we interpret the solution in the context of the real problem i.e. "does it make sense?" and then explore the effects of small changes in the variables.
revisions	For the traffic flow problem we introduced the parameters $a$ and $b$ and explored the effects of these on the solution. The difference between parameters and variables is an important part of the mathematical knowledge that students need to learn.

CAS is used to solve the mathematical problem that in each case involves the familiar task of finding the minimum or maximum value of a function of one variable. In a curriculum which does not encourage the use of CAS much time is needed to develop the skills of doing this optimization by hand and our experience of using the traffic flow problem with first year mathematics undergraduates with two years of school calculus is that very few of them will succeed in differentiating algebraic expressions such as the flow function  $F(v)$  accurately! It is in this part of the problem solving process that the traditional school mathematics curriculum tends to spend much of its time and student effort. The two solutions show that in the CAS rich curriculum we do not need to be able to do such mathematics by hand. However it is necessary for students to know what to do i.e. to know the mathematical steps in the process of using calculus to

optimize the functions. The time and space in the syllabus created by omitting such mathematical manipulative skills can be used to develop the formulation and interpretative skills required to solve real world problems.

## 5 Concluding Remarks

The classic approach to mathematical modelling is generally based on a set of assumptions that are then translated into mathematical expressions. To the novice mathematician, the manipulation of the mathematical expressions and their level of competence with mathematics may easily obscure the whole modelling process. This article has focused on the role that CAS can play in the process. Like other mathematical software, CAS should be available as a tool to use appropriately in mathematics and in particular in problem solving. I would argue that the significant role of CAS in mathematical modelling is in the solution and interpretation phases. It is here that students need to explore for themselves the nature and implications of the models they have formulated. CAS allows an active investigation of the properties of the function models often at a confidence level beyond a student's mental and hand skills e.g. a student may more confidently explore the algebraic form of the flow rate formula using CAS than by hand.

The discussion of the skills needed to formulate and interpret the models in the two problems in this article show that we do need an algebra rich curriculum in which pupils are able to know the difference between variables and parameters, are able to use and manipulate symbols confidently, are able to find relationships between symbols, and know what to do when using CAS to solve the mathematical problem.

Unfortunately most students do not spontaneously show the ability to formulate mathematical models to describe real world problems. Students need to study examples of good models and the reasons that they are described as good. The modelling heuristics need to be demonstrated in order to illustrate the process at work. Experts and novices differ significantly in the way that they approach a real problem. Experts tend to simplify a situation and try a 'quick dirty solution' and then after reflecting on what they have learned then generalize by criticizing the simplifications. Novices jump in at the deep end and follow that path even if it goes nowhere (Maull & Berry, 2000). Graduates of school mathematics should have had the opportunity to move beyond the novice mathematical modelling stage and CAS gives us that opportunity. We need to ensure that the school curriculum contains tasks from categories IIB (real problem solving e.g. the can design problem) and III (mathematical modelling e.g. traffic flow) i.e. tasks for which the mathematical expressions are familiar or readily available from books to tasks for which the mathematical expressions needs formulating.

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