

Basic Skills Versus Conceptual Understanding in Mathematics Education: The Case of Fraction Division

A Reply to Hung-Hsi Wu

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Abstract: The distinction between conceptual understanding and basic skills is as old as mathematics education research itself. It still remains a central issue for many disputes. In this paper, building upon professor Hung-Hsi Wu's rejection of the distinction, I explore three possible accounts of it: (a) conceptual understanding first, (b) explaining the distinction away and emphasizing "procedural-understanding" instead, and finally (c) treating understanding and procedural skill as two separate, irreducible, complementary components. In contrast to Wu who favors the second account, I argue that as far as mathematics teaching is concerned the third view is the preferable one.

Kurzreferat: Die Unterscheidung zwischen begrifflichem Verstehen und algorithmischem Können¹ ist so alt wie die Didaktik-Forschung selbst. Sie bleibt bis heute eines der zentralen Themen vieler Debatten. In diesem Aufsatz untersuche ich, ausgehend von der von Professor Hung-Hsi Wu formulierten Ablehnung dieser Unterscheidung, drei verschiedene Herangehensweisen: (a) Begriffliches-Verstehen-zuerst, (b) Beseitigung der Unterscheidung als ein Scheinproblem verbunden mit Algorithmisches-Verstehen-zuerst und schliesslich (c) eine Behandlung von Verstehen und von algorithmischem-Können als zwei getrennte, irreduzible, komplementäre Komponenten. Im Gegensatz zu Wu, der (b) bevorzugt, versuche ich zu verdeutlichen, warum in Bezug auf den Mathematikunterricht (c) die geeignetere Auffassung ist.

ZDM-Classification:

The issue of the relationship between "basic skills" and "conceptual understanding" is one of the most frequently raised and hardest to deal with in mathematics education. Few would really argue that either is irrelevant. And yet, generations of mathematicians and educational researchers found, and continue to find it, difficult to explain how these two important components can be balanced. Anyone familiar with didactic literature would easily produce scores of examples of debates which ultimately come down to taking sides on the issue "basic skills vs. thinking skills".

One of the important sources of the difficulty in finding

a balance between these components with respect to mathematics education is linked to the fact that we still lack a convincing universally accepted account of this distinction in the first place.

An interesting illustration of the difficulties raised by our distinction can be found in a paper by Hung-Hsi Wu in which the distinction is denounced as a "bogus dichotomy in mathematics education" (p.1, Wu 1999).² This relatively recent paper is paradigmatic for a characteristic manner of trying to deal with our distinction by explaining it away. Wu's bold approach to the matter makes his paper an extremely valuable and challenging contribution that deserves a closer examination.

Wu argues that the distinction "basic skills – conceptual understanding" is a theoretical fiction. Even though I agree with Wu that there cannot be good mathematics teaching without considerable emphasis on standard basic skills (that is, on formal standard symbolic technique), at the same time, I find it difficult to share his view according to which emphasis on what could be called "procedural understanding" (I will explain this below) rather than on "conceptual understanding" provides a sufficient foundation for good teaching.

In order to make this point clearer, I will briefly discuss one of the examples considered by Wu: fraction division.

Wu begins by noting that "Nowadays 'invert and multiply' has become almost synonymous with rote learning." (p.2, Wu 1999). According to Wu, those advocating "conceptual understanding" deplore the fact that in cases such as this students are often expected to swallow down the rule without having been given the opportunity to develop the corresponding conceptual understanding on which the rule is based.

The advocates of "conceptual understanding" before "symbolic skill" believe that we should begin by making it possible for the students to get a semantic appreciation of certain situations in which the procedure "invert and multiply" comes so to speak natural, before emphasizing symbolic technique. This, we are told, can be achieved, for instance, by introducing the division of fractions through a series of meaningful examples such as $1/4:1/2$. Students should be urged to reformulate the exercise in words (for instance: "how many halves are $1/4$?"). This is supposed to help students understand the meaning of the task. Once understanding is in place, the answer to the problem can be reached by interpreting the given question with the aid of some visual pictorial representation of the task. A simple diagram would show that the answer is "twice a quarter", i.e., $1/2$.

Those who advocate "conceptual understanding first" regard the presentation of a series of examples such as the previous one as a sufficient foundation for a meaningful introduction of fraction division.

Wu objects to this approach, emphasizing that it is not universal:

¹ Heinrich Bauersfeld verwendet folgende Terminologie: „prozedural/konzeptionell“ und „algorithmisch-mechanisch/begriffsbezogen-integrativ“ für die Unterscheidung „basic-skill/conceptual-understanding“ (vgl. Bauersfeld in Weiser/Wohrling 2001, S. 14 f.).

² Wu's contribution emerged in the context of the ongoing debate opposing the "New-New-Math" and the "Mathematically Correct $2+2=4$ " (<http://www.mathematicallycorrect.com/>). The distinction "understanding"/"basic skills" is at the heart of this debate.

“The worm in the apple is the need to deal with division problems when fractions are not as simple. For example, what do the above brand of logical and visual thinking skills have to say about $2/97:31/17$ ” (p.2, Wu 1999).

If this is indeed the case, then we should look for an alternative didactic approach to the matter.

What does Wu’s reconstruction of the “conceptual understanding before symbolic skills” approach to fraction division state so far? It emphasizes that semantic interpretations of the kind used in the case of dividing “nice fractions” fail to provide a universal meaningful rationale for reaching a general rule in dealing with this operation.

There is indeed an inconvenient even in the case of “nice” fractions for, even there, according to the “conceptual understanding”-first view, the only way of connecting, say, $1/6:1/2$ and $1/6 \times 2/1$ is via the mediation of some sort of pictorial artifact that has to be imagined on a case to case basis. Thus, this sort of “conceptual understanding” proves not only insufficiently general in the sense that it holds for “nice” fractions alone, but it proves, in a sense, cumbersome even in those cases in which it can be used.

Besides that, even if we take the “semantic” path just described, and even if we deal with “nice” fractions, the only conclusion this leads to is that $1/4 : 1/2$ is $2/4$ or $1/2$. However, these results do not immediately suggest the formal rule “invert and multiply”. What is the alternative?

According to Wu, the alternative approach should look like this. Begin by reassessing the meaning of ordinary division in the case of natural numbers. It should not be too hard to make students appreciate that $m:n=k$ and $m=n \times k$ both stand for something like “a whole made of m objects consists of k groups of n objects each”. Let us now assume that we want to find out what a/b divided by c/d might mean. We treat this like an equation and write

$\frac{a}{b} : \frac{c}{d} = \frac{x}{y}$. By analogy with the case of the integers, we

get $\frac{x}{y}$ must be such that $\frac{a}{b} = \frac{x}{y} \times \frac{c}{d}$. To find $\frac{x}{y}$ we only

need to multiply both sides by $\frac{d}{c}$ which leads to the desired conclusion.

Wu closes his presentation of this case by arguing that:

“Thus the method ‘invert and multiply’ is a result of a *deeper understanding* of fractions than that embodied in the naive logical and visual thinking skills above. We see clearly the concordance of skills and understanding in this instance” (emphasis in the original; p.3, Wu 1999).

While Wu’s critique of the “intuitive” approach described at the beginning seems justified,³ his plea in favor of the

second approach is overstated.

First of all, it depends on a formal analogy between multiplication in the case of natural numbers and multiplication of fractions. Here, however, we encounter a strange combination: in the case of the natural numbers, semantic considerations work without restriction, yet this gives no warrant whatsoever for an extension of these considerations to fractions. Basically, what Wu is relying on is an algebraic approach which is said to give “deeper understanding” and “concordance of skill and understanding”.

There is little doubt that the approach suggested by Wu has its merits. And yet, anybody who has ever done any work on mathematics teaching in grades 5 to 8 knows that none of the previous approaches keeps the promise. The point that I want to make here is really simple: students find both constructions (and a number of alternative ones!) difficult to grasp. What are the implications of this situation?

One of the most frequent reactions to this sort of difficulty is to say that we need to dig “deeper” in order to come up with some sort of approach which may turn out to be less artificial. How can we accomplish this? Various ways have been proposed. Some authors delegate the task of fostering understanding to the alleged natural constraints of everyday contexts assumed to be familiar to the students. On this view, all that is needed, is to give students the opportunity to work in contexts that are rich and semantically meaningful (to them) and to encourage them to communicate with their peers. Characteristic here are the positions advocated by neo-Vygotskian approaches to mathematics education such as those pursued in “ethnomathematics” and “situated cognition” (see, for instance, Vera/Simon 1993; Powell/Frankenstein 1997; Seeger/Voigt/Waschescio 1999).

Such approaches have produced a wealth of interesting ideas which are important both for the teaching practice and for theoretical debates. It seems to me, however, that they do not manage to overcome the sort of objection raised by Wu. It seems doubtful that mathematics can be taught by emphasizing this sort of spontaneous capacity of the students to develop their knowledge as long as the formal symbolic technique is not recognized as a major and independent ingredient of the process. In this respect Wu’s critique is valid.

At the same time, it seems to me that Wu’s critique does not go far enough. His commitment to the idea of “concordance between skill and understanding” as a prerequisite for good teaching leading to “a deeper understanding” of the fractions and of the operations performed on them is, I think, reminiscent of an attitude that might be called “understanding first”. Thus, in a sense, Wu’s alternative appears to have something in common with the position defended by those advocating “conceptual understanding” that Wu wants to demolish. Thus, even if I tend to agree with Wu that the only way to teach mathematics is the “honest way by teaching both

Tsvetkovich (Davydov, Tsvetkovich 1991). But this is not the path that I wish to pursue here. The intention of my argument is not to defend conceptual understanding against Wu’s attack but to explain that there is more to gain from treating “understanding” and “symbolic skills” separately.

³ It could be argued that it is possible to design better “conceptual understanding”-first approaches, such that would make it possible for the students to grasp the semantics of the four arithmetical operations on fractions in a suitable way and at the same time make them discover the formal symbolic rules as a consequence of that understanding. One of the most insightful approaches of this kind was proposed by Davydov and

skills and understanding” (p.7, Wu 1999), it seems to me that the account of understanding given here is to much reminiscent of the old illusion that it is possible to uncover some unique, natural, “best approach” to teaching fractions or whatever, in which natural semantics and formal tools harmonize in a natural way.

Contrary to this view, I am inclined to think that “understanding” is much less and much more than that: it is the constant effort of applying the formal tricks that we know and of trying to integrate them with other tricks and a variety of applications and to explore the extent to which our techniques work. I think that teachers can go on teaching “invert and multiply” as technical skills provided they combine this with substantial sessions in which various ways of entrenching this formal trick in semantically meaningful situations are explored. We do not need verbalization of our procedures either as a justification or as an absolute prerequisite for understanding. We can rely on it as a way of exploring how and if it is possible to make sense of our formal procedures in terms of various other problems and situations and explore the limits of that. It is also important to explore analogies to formal mathematics of the kind proposed by Wu. One should also consider other approaches. This leaves enough room for creativity, for developing all sorts of understanding and semantic links.

Why not teach students “invert and multiply” and then ask them to corroborate this rule with other rules? Take

$$\frac{a}{b} : \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$$

and compare it with $u:v$ when u and v are

$$\text{natural numbers. Is there a conflict here? Why no! The reason is that } u:v = \frac{u}{1} : \frac{v}{1} = \frac{u}{1} \times \frac{1}{v} = \frac{u}{v}$$

which is compatible with the analogy between fractions and division. What

$$\text{about } \frac{a}{b} : \frac{c}{d} = \frac{a:c}{b:d} ? \text{ It is easy to show that this rule is}$$

also compatible with “invert and multiply”. “Invert and multiply” has the additional merit of establishing a relationship between fraction division and fraction multiplication. Next, we can take such rules and try to interpret them in connection with hands-on problems. What situations can be imagined? Which fractions lead to “natural” interpretations and which not?

The thesis I am advocating here is that the expectation of a “concordance between skills and understanding” mentioned by Wu is a part of the problem rather than of the solution to the problem of teaching/learning mathematics. The alternative to this is to recognize symbolic mathematics as an independent realm of mathematical experience: an interface linking ideas implicit in communities of practice, ideas in the mind, and symbolic re-constructions. From a didactic point of view it is better to regard symbolic mathematics as something like a practical arena in its own right where it is possible for us to experiment with signs, trying to coordinate the experiences we make. It seems to me that what needs to be done is to drop the assumption that mathematical thinking is or should be straight-forward or linear. Skills *as well as* understanding require teaching and we should not expect that any of the two can justify or lay the absolute foundation for the other. It is rather as

Felix Klein once said that in the case of mathematics we always find ourselves in the middle, stretching our roots downwards for better foundations and richer semantics, and stretching our branches upwards towards the sun for increased generality. As far as teaching is concerned, the issue of whether there is a universal link between the various skills and between skills and understanding is theoretically problematic and brings no advantages in respect to teaching.

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