Learning to prove: The idea of heuristic examples

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Abstract: Proof is an important topic in the area of mathematics curriculum and an essential aspect of mathematical competence. However, recent studies have revealed wide gaps in student's understanding of proof. Furthermore, effective teaching to prove, for example, by Schoenfeld's approach, is a real challenge for teachers. A very powerful and empirically well founded method of learning mathematics, which is also relatively easy to implement in the classroom, is learning through worked-out examples. It is, however, primarily suited for algorithmic content areas. We propose the concept of using heuristic worked-out examples, which do not provide an algorithmic problem solution but offer instead heuristic steps that lead towards finding a proof. We rely on Boero's model of proving in designing the single sub-steps of a heuristic example. We illustrate our instructional idea by presenting an heuristic example for proving that the interior angles in any triangle add up to 180°.

Kurzreferat: Es ist ein wichtiges Ziel des Mathematikunterrichts in der Sekundarstufe, das die Schülerinnen und Schülern ein Verständnis für mathematisches Argumentieren und Beweisen entwickeln. Doch verschiedene neuere Studien belegen, dass das Erreichen dieses Ziels mit erheblichen Schwierigkeiten für Schüler und Lehrer verbunden ist. Nun ist empirisch gut belegt, dass das Lernen mithilfe ausgearbeiteter Lösungsbeispiele in der Mathematik zu guten Ergebnissen führen kann und darüber hinaus auch leicht in den konkreten Unterricht integriert werden kann. Diese Methode ist allerdings im Wesentlichen für algorithmische Inhalte geeignet. Als eine didaktisch sinnvolle Erweiterung wird im Folgenden das Konzept heuristischer ausgearbeiteter Lösungsbeispiele betrachtet. Dabei steht nicht ein Lösungsalgorithmus im Vordergrund, sondern die Aufeinanderfolge geeigneter heuristischer Schritte. Wir verwenden das Modell des Beweisens von Boero zum Aufbaus des Konzepts. Am Beispiel des Satzes von der Winkelsumme im Dreieck werden die grundlegenden Ideen konkretisiert.

ZDM-Classifikation: C30, C40, C70, D50, E50, G40

1 Argumentation and proof in the mathematics classroom

1.1 Mathematical competencies and the understanding of proof

Proof and logical argumentation are highly important topics in the area of mathematical sciences. Despite the fact that mathematics may even be regarded as a *proving science* the role of proof in the school curriculum has not always reflected this importance. In the 1970s and 1980s, there was a intensive discussion as to whether proofs should even be part of the mathematics curriculum in secondary schools. Mathematics educators argued that proving in the classroom had developed into a topic area where formal aspects were emphasized but mathematical understanding was disregarded (Hanna, 1983). This view is still shared by many researchers, but its consequences have been modified. Proof is regarded an important topic of the mathematics curriculum (NCTM, 2000) and an essential aspect for mathematical competence, but *proof* is not necessarily used as a synonym for *formal proof*. Researchers such as Hanna and Jahnke (1993), Hersh (1993), Moore (1994), Hoyles, (1997), Harel and Sowder (1998) have pointed out that in both mathematical research and school instruction, proving spans a broad range of formal and informal arguments. The understanding or generating of proofs is a significant component of mathematical competence. Moreover, mathematical argumentation has been identified as the essential element of higher order mathematical competence in the TIMS study. Klieme (2000) identified four items in the 12th grade (respectively 13th grade) TIMSS test that could be assigned to the highest level of students' competence It was also found that three of these four items ask for mathematical argumentation based on the interpretation of complex diagrams.

Some empirical surveys of North American high school students (Senk, 1985; Usiskin, 1987) and pre-service teachers (Martin & Harel, 1989) have revealed wide gaps in the respondents' understanding of proof. Healy and Hoyles (1998) showed in a systematic investigation that 10th grade students had deficits in their understanding of proofs, their ability to construct proofs, and their views on the role of proof. The students were hardly able to construct mathematical proofs and were more likely to rely on empirical verification. This empirical orientation does not necessarily support the construction of correct mathematical proofs as was stated by Hoyles and Healy (1999). Their subjects were not able to identify mathematical arguments for a proof after an intensive empirical investigation of the context. Lin (2000) argues that these students were probably not able to master the transfer between the methods and results of an empirical investigation and the methods needed for general mathematical argumentation. The gap between case-based reasoning in empirical investigations and generalized argumentation may not be bridged.

A similar study in Germany with 13th grade students gave additional evidence to the finding that most students performed poor in constructing proofs and in judging the correctness of proofs (Reiss, Klieme & Heinze, 2001). The students were asked to solve TIMSS geometry items and to determine whether specific mathematical arguments (in a geometry context) could be regarded as mathematical proofs. Some of the students participated in individual interviews and were asked to work on geometry problems and, while doing so, to comment on all their activities during the problem-solving process. The study revealed that with respect to the specific elementary geometry problem-solving context the students had the declarative knowledge necessary for solving the problems. For example, almost all students knew that the interior angles in any triangle add up to 180°. Moreover, most of them knew that the base angles in an isosceles triangle are identical. But only a few of them could apply this knowledge to a specific problem. The results showed that combining even a small number of mathematical arguments was beyond most students' abilities (Reiss & Heinze, 2000; Reiss & Thomas, 2000). It was easier for the students to judge the correctness of proofs. In particular, most of them recognized that empirical argumentation was not a valid mathematical method of proof (Reiss, Klieme & Heinze, 2001). The individual interviews gave evidence that the differences between performing proofs and judging proofs were of a systematic nature. Many students argued in these interviews that they were not able to find something similar to a starting point in a proof or to identify correct arguments with respect to the specific context of proof.

1.2 Boero's expert model of mathematical proof and its consequences for proof in the mathematics classroom

Mathematical proving is a complex activity which cannot be simply reduced to the correct use of deductive argumentation. It combines, for example, processes of concise logical argumentation as well as the more heuristic processes of producing a conjecture and looking for plausible arguments to support the conjecture.

The product of this activity – which is a mathematical proof to be published in some journal –reflects only specific parts of the actual process of proving. This process is most likely to remain non-transparent for students learning proof in the mathematics classroom. Boero (1999) gives an overview of the different steps which lead a mathematician to a correct mathematical proof. This model clarifies the activities of producing a conjecture and constructing a mathematical proof in some detail. The model consists of six phases which lead an expert's proving process, but it is not meant to be a linear model. Conjecturing, exploring, testing results, and writing a formal proof are activities, which are most likely to be performed repeatedly during the process. The first phase described by Boero (1999) is (1) the production of a conjecture. This includes the exploration of the problem situation as well as the identification of arguments to support the evidence. Boero refers to this phase as "the private side of mathematicians' work", which is usually not shared with the mathematical community. (2) The formulation of the statement according to shared textual conventions is the second phase. This phase aims at providing a precisely formulated conjecture, which will then be the basis for all further activities. It may be revised in the forthcoming processes but this revision would have consequences for most activities performed by the mathematician. The third phase combines (3) the exploration of the (precisely stated) conjecture and the identification of appropriate arguments for its validation. This is also part of the "private work" since exploration might, for example, lead to errors or at least to complicated formulations in the proof. Only the last three phases are subject to public communication. They include (4) the selection and combination of coherent arguments in a deductive chain, (5) the organization of these arguments according to mathematical standards, and perhaps (6) the proposal of a formal proof.

The expert model of proof illustrates that proving is a complex cognitive activity. It is not only characterized by logical argumentation but by an exchange between explorative, inductive, and deductive processes. The mathematician has to identify a suitable choice of the elements involved in this process and arrange them in a logically consistent scheme.

Proof is an important aspect of mathematics instruction (Schoenfeld, 1994) but it is not meant to be an end in itself. The ability to explore a problem-solving situation, exchange concise arguments, and organize these arguments in a logically consistent series is also important for mastering other subject matters. Thus, mathematics instruction should not focus on all aspects of the proving process in an equal manner. In particular, with respect to an understanding of proof and with respect to the needs of a more general education, the practices of the mathematics classroom suggest that the first four phases are important for the learning of mathematics. It is obvious that these phases are not only difficult for a learner to understand, but that students will hardly succeed without the help of a teacher.

The phase model of Boero (1999) shows that the proving process of an expert is not linear but a part of a sequence of intertwined activities, which specifically include, the heuristic processes of exploration and investigation. These heuristic processes are important for mathematical understanding, and students should be encouraged to use heuristic methods in their problem solving processes. Therefore learning environments, which provide opportunities for individual explorations on one hand and foster mathematical argumentation on the other hand, should be implemented in the mathematics classroom. It is important that a learning environment for mathematical proof and logical argumentation should initiate heuristic problem solving, exploration and investigation. Results of Healy (2000) as well as those of Hoyles and Healy (1999) suggest, that explorations and investigations of a problem does not necessarily lead students to ideas for mathematical proving. Accordingly, a learning environment for proof and argumentation should not only allow for explorations and investigations but it should also provide specific help and support with respect to the proving process.

One possibility to provide such help and support is to employ worked-out examples, which model the heuristic processes of proof finding. Before we introduce the concept of heuristic examples, we will first outline the advantages of example-based learning that has been shown in algorithmic mathematical sub-areas.

2. Learning from worked-out examples

Worked-out examples consist of a problem formulation, solution steps, and the final solution itself. During the past two decades, learning from worked-out examples in well-structured domains such as mathematics has received increasing attention from cognitive as well as from educational psychologists (e.g. Caroll, 1994; Renkl, 2001; Zhu & Simon, 1987). There are quite a number of reasons why this type of learning constitutes an important research topic. For example, VanLehn (1986) concluded from his analysis of subtraction errors made by elementary students that learning by examples is a major path for acquiring such skills in school mathematics.

Further evidence for the importance of learning from worked-out examples are the observations made by Anderson, Farrell, and Sauers (1984) and by Pirolli and Anderson (1985). They found that novices in LISP programming, at least in the beginning, tended to ignore verbal descriptions of LISP procedures and relied on examples. Accordingly, the initial creation of their own LISP procedures was characterized by the use of examples. Recker and Pirolli (1995) found that their participants, while learning LISP in a hypertext environment, preferred examples (in 77% of the cases) when they had the possibility to choose between an example and textual information.

LeFevre and Dixon (1986) investigated the preference of learners to relying on examples more directly. They presented an instruction in the form of an abstract text and worked-out examples to their participants. The information encased in these two sources contradicted each other. Thus, through inspection of the participants' solutions to the problems it could be differentiated as to the informational source preferrence. The results of LeFevre and Dixon (1986) showed that learners strongly preferred examples. This preference was maintained even when the information in the examples was reduced, the text and principles were made to be more redundant and detailed, and when an attempt was made to convince the participants of the importance of the abstract text information.

The preference of learners for worked-out examples is functional. For example, Zhu and Simon (1987) found that their carefully designed and sequenced mathematical examples were sufficient to induce skill acquisition and abstract problem representations without providing explicit instruction. In a field study, they even showed that it was possible to work through a three-year curriculum in only two years (without achievement deficits) by the employment of example-based learning. In addition, numerous studies performed by Sweller and his colleagues (e.g. Sweller & Cooper, 1985; Tarmizi & Sweller, 1988; Ward & Sweller, 1990; for an overview see Sweller, van Merriënboer, & Paas, 1998) showed that the employment of worked-out examples during the *initial* acquisition of cognitive skills is usually very effective. They compared the typical procedure that can be found, for example, in many textbooks (i.e. introduction of a rule or principle ➔ one worked-out example ➔ problems to be solved) with employing a series of example-problem pairs after an initial introduction of a rule or principle and one worked-out example. The second method was reliably superior. Thus, worked-out examples are of special use when a new topic is introduced (see also Renkl, Schworm, & vom Hofe, 2001).

The superiority of example-based learning is explained by the argument that problem solving requires such a large amount of working memory capacity when the learning contents are new to the students that it interferes with learning in the sense of schema acquisition. More specifically, it is argued, that in order to solve problems, novices (i.e.,learners) employ means-ends-analyses. This implies that the learner has to simultaneously focus on the following aspects: actual problem state, desired problem state, difference between actual and desired problem states, relevant operators, and sub-goals. Given this load, there are few resources left for the processes of understanding and for inducing abstract and generalizable problem solving schemata (cf. Sweller, 1988, 1994).

In addition to the capacity arguments, there is probably another important advantage of worked-out examples in comparison to problem solving--an advantage that is strongly related to motivation. When learners are told to solve problems, their primary goal is, of course, to solve problems and not necessarily to learn. The learners adopt a performance orientation. In contrast, when confronted with worked-out examples, there is no demand to perform. The only task confronted by the learners is to *understand*. Thus, worked-out examples foster intentional learning (Bereiter & Scardamalia, 1989), or in other words, a learning orientation (see also the distinction between performance and learning orientation by Dweck & Leggett, 1988).

Lastly, another advantage of using worked-out examples ought to be mentioned, one that is especially relevant when the implementation of learning methods on a long-term and large-scale basis is considered: the integration into the classroom instruction. Whereas many methods for learning and teaching have been proposed in recent years (e.g., problem-based learning) which require a profound restructuring of classroom instruction (and maybe even of the school culture), learning by worked-out examples can be easily integrated into common classroom instruction. Working with examples is already part of traditional classroom instruction. What has to be achieved is "only" a prolongation and an optimization of the "example phase."

In summary, it has been shown that worked-out examples are an important source of information for learners, that learning from examples is preferred by learners, that it is typically very effective in initial skill acquisition, and that it can be easily implemented in the classroom. Although worked-out examples have all these advantages, their employment as a learning methodology does not, of course, guarantee effective learning.

A very important factor that moderates the effectiveness of learning from examples is how the deeply the students process the examples. More specifically, Chi, Bassok, Lewis, Reimann, & Glaser (1989) showed that the extent to which learners profited from the study of worked-out examples (content domain: physics/mechanics) depended on how well they explained the rationale of the presented solutions to themselves. This was called the “self-explanation effect”. These results could be replicated several times in other studies (e.g. Pirolli & Recker, 1994; Renkl, 1997). It has been found that the successful and the unsuccessful learners differ with respect to the following main points: (1) The successful learners frequently assign meaning to operators by identifying the underlying domain principle (principle-based explanations). (2) They frequently assign meaning to operators by identifying the (sub-)goals achieved by those operators (identifying a problem's sub-goal structure). (3) With increasing prior knowledge, they tend to anticipate the next solution step instead of looking it up (anticipative reasoning). (4) They do not hold "illusions of understanding" due to superficial example processing (meta-cognitive monitoring).

Unfortunately, most students do not spontaneously show effective self-explanation activities. Most students process examples in a passive or superficial manner (Renkl, 1997). Hence, students usually need some guidance in studying examples.

3 The concept of “heuristic examples": A possible remedy for learning proofs and proving?

As discussed above, mathematical proof is regarded as an important topic in mathematical instruction, albeit students have great difficulties in generating proofs. We have already argued that one important factor impeding learning to prove is the belief that a mathematical proof is a straight-forward, deductively deduced, systematic and logical sequence of steps formulated in a formal mathematical language. The dysfunctionality of this belief makes it problematic to employ the types of worked-out examples that have been successfully used in other mathematical sub-areas. As mentioned above, worked-out examples consist of a problem formulation, solution steps, and the final solution itself. A "traditional" worked-out example of a proof would contain the statement to be proved and the sequence of solution steps as a kind of ideal solution process. However, such an “ideal” solution process of a mathematical proof does not reflect the real solution process, even not that of an expert. It simply displays a type of an end product of a proving process. Thus, employing the usual type of worked-out examples would reinforce the dysfunctional student's belief that proving is a straightforward deductive activity.

Therefore, the problem-solving heuristics have to be displayed in order to teach proof. For example, Schoenfeld developed a method for teaching mathematical problem solving using modeling, coaching and fading.

Experts employ various heuristic methods when approaching a given problem and they are able to manage these heuristics properly. Schoenfeld (1983) emphasizes the particular importance of managerial actions, and metacognitive behavior in general, during the problem-solving process. Accordingly, he taught students some of the heuristics used by experts and showed them how they ought to be applied in different kinds of math problems (Schoenfeld, 1983, 1985). He used cognitive modeling (Collins, Brown & Newman, 1989) to exhibit the expert’s thinking processes, that is, he thought aloud when he solved problems that were posed by his students. Then, he presented a problem to his students that required the usage of the heuristics formerly introduced and guided their way through the generation of alternative solution paths. Schoenfeld found important constituents to be the exploration in the search for a good method and the evaluation of that exploration. These key factors of proof construction need to be illustrated in some way (Schoenfeld, 1983; Collins, Brown & Newman, 1989).

Experts and novices differ significantly in the way they approach a real problem. Novices spend most of their time in (uncontrolled) exploration which Schoenfeld (1993) comments as “read, make a decision quickly, and pursue that direction come hell or high water”. Experts spend most of their time analyzing the problem constraints und thus making sense of the problem. After a training of self-regulation students will probably not act like experts, but they will be more likely to identify more or less probable problem-solving attempts. Moreover, Schoenfeld (1993) describes a concomitant increase in problem-solving success.

A problem with such an approach is that would require a very intensive teacher training and the whole instructional process would have to be restructured, which would also lead to various difficulties. A possible way to combine the advantages of worked-out examples (i.e., effectiveness and easy classroom implementation) and of Schoenfeld's approach (i.e., making heuristics explicit) is to design worked-out examples that do not provide a ready proof's solution steps, but the steps of the heuristics strategies that led to the final solution steps. A sensible model of steps and processes of effective proving has been recently formulated by Boero (1999; see above). Hence, the type of worked-out examples we have in mind instantiate the heuristic processes proposed by Boero's model of effective proving. We call this type of written-down models "heuristic examples".

4 A heuristic worked-out example for mathematical proof: Properties of the angles of triangles.

Heuristic worked-out examples are based on a complex mathematical problem which has been presented to the students. The students are not required to give its correct mathematical formulation. Nonetheless, the identification of arguments, which support evidence for the solution of the problem, is an important part of a worked-out example. Students are asked to explore the context of a problem in order to fully understand the conjecture. At the end of the exploration process a heuristic worked-out example presents a correct and detailed proof of the conjecture. Accordingly, a heuristic worked-out example will include most of the steps of a problem solving (or proving) process described by Boero (1999).

The following heuristic worked-out example is intended to provide an overview of the most important aspects of this type of example. This specific example is not only meant to prove that the interior angles in any triangle add up to 180° but also helps demonstrate to students various aspects of proving in general.

The problem: *Alex and Chris have drawn different triangles and respectively measured the sum of their three angles,. Both were surprised to discover that this sum was 180° for all of the triangles. Alex and Chris were sure that this was not an accidental result. Their conjecture was: „In every triangle, the sum of its interior angles is 180°.“*

In the following, we will look closely at their problem-solving work. You are encouraged not only to read this solution but to repeat all the problem solving steps that Alex and Chris followed.

(1) *Exploration of a problem situation*:

Equipment: a pair of scissors, a protractor, paper.

(a) Draw a triangle ABC, mark its angles α, β, and γ. Measure the size of these angles. What is the sum of α, β, and γ?
Note this size:
………………………………

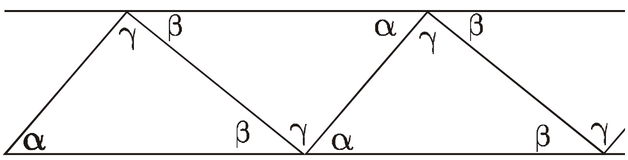
Repeat the experiment three or four times. Note the size of all the angles you get:
………………………………

(b) Draw a triangle ABC, mark its angles α, β, and γ. Take your scissors, cut it out, tear off the corners of the triangle and put them together to form a new angle. What size does this angle probably have?

Note this size:
………………………………

Repeat the experiment three or four times. Note the size of all the angles you get:
………………………………

(c) Draw a triangle ABC, mark its angles α, β, and γ. Take your scissors, cut it out. Using ABC as an outline draw, get more triangles. Cut them out and combine them so that you get a straight line at the bottom.



There is probably a straight line on top. This would provide evidence that congruent triangles may be used to completely inlay a plane. What does this mean for α, β, and γ?

Accordingly, all these experiments suggest that the angles of an arbitrary triangle add up to 180°.

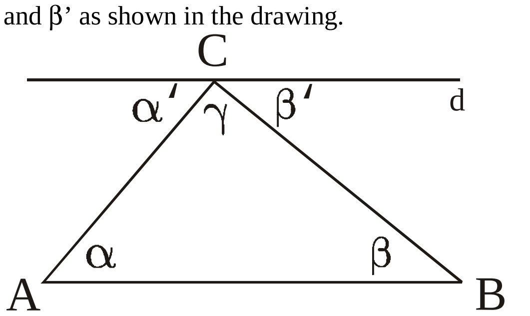
(2) *Conjecture*:
Let ABC be a triangle, and α, β, and γ its angles. Then $\alpha + \beta + \gamma = 180°$.

Mathematical conjectures need to be proved. In order to perform this mathematical proof it is necessary to
- clarify what one knows about angles and respectively about triangles,
- identify arguments which might be important for the proof and to,
- organize correct arguments in a logical sequence.

(3) *What information is available on angles?*
There are some possible prerequisites for the proof, which are well known about angles. In particular you may recall the following facts:
- A straight line is regarded to cover an angle of 180°.
- Vertical angles are congruent.
- When parallel lines are cut by a transversal then the corresponding angles are congruent.
- When parallel lines are cut by a transversal then alternate interior angles are congruent.
Comparing this information and the experimental data may lead to an idea of the proof.

(4) *The idea of a proof*:
A straight line is regarded to cover an angle of 180°. Accordingly, one may argue that the angles of an arbitrary triangle are congruent to angles which add up to a straight line.

(5) *The proof of the conjecture*:
There is a triangle ABC, and α, β, and γ are its angles. Draw the line d, which is the parallel line to AB and which touches C. Mark the angles α' and β' as shown in the drawing.



We know, that α and α' and respectively β and β' are alternate interior angles, and that AB and d are parallel lines. Accordingly, $\alpha = \alpha'$ and $\beta = \beta'$. As d is a straight line, it is obvious that $\alpha + \beta' + \gamma' = 180°$ and, therefore, you can conclude, that $\alpha + \beta + \gamma = 180°$ as well.

(6) *Looking back:*
As a result of the problem solving process we know now for sure that the interior angles in *every* triangle add up to 180°. In the language of mathematics one would say that we found a *proof* for this conjecture.

The solution: *Alex and Chris have found a proof for their conjecture. They proved: „In every triangle, the sum of its interior angles is 180°."*

This heuristic worked-out example reveals the most important aspects of this type of presentation. It provides particularly information on the use of heuristics in the problem-solving process. Firstly, the exploration of the situation gives evidence for the conjecture. Secondly, the drawing suggests a specific problem-solving context. It does not necessarily lead to correct (or useful) mathematical arguments. For example, this proof does not use the (correct) argument that vertical angles are congruent. It is important to accept that mathematical work means making an adequate choice out of possible arguments. It is usually not at all clear in the beginning which (significant) arguments will be integrated in the proof. However, if the correct arguments are combined in a correct way, we come to a conclusion for all possible triangles.

There are many ways to use worked-out examples in the mathematics classroom. However, as outlined above, most students process examples in a superficial or passive way so that they need some guidance. In addition, young students are hardly able to fully concentrate on elaborated examples for a longer period of time. With respect to the heuristic example detailed above, the students' work should be supervised and summarized at at least a few points during the problem-solving process: after the exploration of the problem, after the identification of suitable arguments, and after their combination in a proof. Furthermore, a teacher could present the problem to the students and let them explore it. Thereafter, the students would be encouraged to reflect on their actions by reading and discussing these parts of the worked-out example.

Heuristic worked-out examples cannot be used as isolated tools for teaching, but are meant to be integrated frequently in the mathematics classroom. Students need to learn how to extract important information and to concentrate on a series of problem solving steps while viewing an example. Accordingly, students will have to learn how to work independently with heuristic worked-out examples.

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