Information structuring – a new way of perceiving the content of learning

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Abstract: Dominating information by formation seems to be the only realistic solution to overcome the crisis generated by the higher and higher accumulation of information in each domain. The paper is proposing a model of building structured knowledge able to generate in the child's mind strategies for efficient processing of information.

Kurzreferat: Die Information durch Bildung zu beherrschen scheint die einzige realistische Lösung zu sein, die Krise zu überwinden, die durch Anhäufung von mehr und mehr Informationen enstanden ist. Das Referat präsentiert ein Bildungsmodell von strukturierten Kenntnissen, das im Gedächtnis des Kindes eine effiziente Bearbeitung von Information ermöglicht.

ZDM-Classification: C30, D30, D40

1. The perspective of the 21-st century

Learning mathematics was perceived as a difficult task starting with the beginning of the mass school. The performances had been for a long time considered satisfactory, because the information supposed to being learned was elementary: i.e. reading and writing numbers, and computing at the level of elementary practical demands. Nevertheless, major difficulties appeared with the spread of the public lower and upper secondary schools. As an essential feature specific to the old school, we could mention the gap between school and the communities of the experts in mathematics. We meet this in Ancient Egypt and Greece, but also in 18th century France. Starting with the 19th century concepts that had been previously taught in the "Academies" were included in the schools curricula. Roughly speaking, this fact meant giving up renouncing at the simple memorial didactics and choosing the one of proving, both as a mathematical method and as a didactical method. Consequently, learning mathematics gained a new dimension. Nevertheless, such shift did not permit a real progress because it already supposed deep mathematical knowledge and this was still learned by the same poor efficient effort of memory.

The first half of the 20th century brought about a new revolution in teaching math, the one connected with the development of the intuitive method, materialized in what was called "the active school", which permitted important progress in learning numbers and geometry.

The second half of the 20th century displayed even more sophisticated methodology given the child's psychology achievements. Different researches showed that a twelve year old is able to internalize vectorial space structures, elements of topology and so on. The uniformity by abstraction seamed to be comfortable, but it was again exaggerated and, in fact, a real danger to mathematics learning. H. Freudenthal directly raised the question in 1963. He remarked that the authors of the new curricula follow implicit didactical principles but "as a didactical object, a mathematical theory is not proved by its logical consistency but by explaining the didactics it is based upon." The sixties brought into discussion mathematics teaching in connection with the compulsory schooling and, especially in connection with the development of secondary schools. It is true, the worker with "intelligent tools", the "mind-worker" had already got numeracy skills in mathematics, but to obtain mass performances the old methods were not enough. The new methods were however difficult to crystallize. Nevertheless, the pressure was so strong, that in a few years there were a lot of symposiums on this topic. Few examples: the symposium in Aarhus-Denmark (June, 1960) focused on teaching geometry in secondary schools; the symposium in Belgrade (September 1960) – for math and physics education; the seminar in Lausanne-Switzerland, (June 1961) on teaching math analysis in secondary schools and universities; the seminar in Bologna-Italy, 1961; the international Congress of mathematicians, Stockholm, 1962. The same rhythm is kept for the years to follow as well.

Summarizing the ideas, the intentions of *modernizing the teaching of mathematics* took into account the following issues:

- Learning mathematics should be shaped closer to the level covered by the mathematics evolution as a scientific domain (the problem was also familiar to the 19th century), with the purpose to diminish the discrepancy between the more and more abstract and sophisticated feature of mathematics as a science and the necessity of including some of its recent achievements in the didactical domain.
- Learning mathematics by everybody, not only by the gifted students becomes a necessity, because the post-industrial era needs a new professional "mass".
- As a consequence, is necessary to identify the levels of learning mathematics for different categories: the ones who will study mathematics as a further profession, the ones who will become workers with the intelligent tools of the new millennium and the majority, who need to manage mathematics for the concrete task of real life.

All these demand a deep change of learning techniques because the old didactics are not of much help.

The idea of searching the psychological phenomena involved in teaching and learning new concepts is becoming increasingly familiar. This is also a consequence of the fact that the abstract mathematical concepts could be learned with the old methods only at the level of university education and sometimes, not even there. Learning mathematics and teachers' practice are still affected today by the gap between the tendency of excessive formal rigorous display of the concepts, on the one hand, and a canonical collection of algorithms, on the other hand. Following this, an elite of gifted children is able to achieve the high theoretical standards; for the other students learning mathematics is done by successive accumulation of algorithms without any perspective or clear goal. The nowadays-Dutch school brings a new practice of intuition – realistic mathematics. "Make the mathematics meaningful" helps a lot the practical process of teaching and learning mathematics but asks for very well trained teachers in mathematics and also very creative teachers. There is still here another risk: one of perceiving mathematics as a practical matter and loosing in this way its essential role of *structuring thinking*.

Actually, the intentions of modernizing math teaching failed at the level of really developing new coherent didactics, not at the level of their rationality. From here, a pressure upon didactics to modernize and upon psychology to offer new ideas regarding the learning phenomenon. In connection with these assumptions, some *psychological* issues are significant:

- The classical theories of learning are operating with "linear" models, or the human learning has as specific feature the learning of complex sets.
- Examples of non-linear learning, when they appear, are referring to narrow sequences of learning and not to internalizing an entire domain of a school subject.
- The recent theories of intelligence (intelligences) seem to be very productive for an active learning but they are not systematically applied in learning a school discipline.

2. Some hypotheses

Taking into account these contradictions between the society demands and the school teaching-learning methods and content on the one hand, and between the level of the theory (in mathematics and psychology) and the didactics of the discipline, on the other hand, our research tries to gather recent acquisitions from psychology and didactics of mathematics.

The next hypotheses are steps for further developments:

- If the human intellect is an historical category, i.e. it develops together with the evolution of culture, then we cannot speak about a stable intellect but about an evolutive one, the expert of each specific scientific domain being the last level reached by the intellect evolution in that area. In this case, the research on this intellect with psychological methods should become the source of setting up formative outcomes of learning. Most of contemporary school systems did this more or less intuitively (See the attainment targets in the *National Curriculum of England and Wales*, 1989; 1995; 2000, or the New Zeeland curriculum, to take only two examples). The U.S. *Principles and Standards for School Mathematics*, 1991/2000, tried to adapt the "expert" level to the "best practices" of experienced teachers.
- If the information in mathematics, as in other knowledge area, has a continuous accumulation, then the only solution is to value thinking resources, able to dominate the information and to permit the retention by deduction, by creation and not by memorizing. Exactly in the way in which a concept could fix a class of phenomena unknown by the individual in its totality, in the same way for example, the infinity of addition word problems is dominated by the formula $a+b=x$, where x is the unknown variable. If the typology is correctly internalized, then the logic of recognizing the category works implicitly at the psychological level.
- If we agree that information is internalized with respect to the building concepts rules, then to bring the psychological level to the logical level of the domain, we need a double didactics: one for internalizing (Vygotsky, 1934/1962; Galperin, 1972), and one of banding in structures (as the perspective from structuralism and constructivism seems to develop).

The paper is presenting some results of the didactical technology we developed and applied for mathematics learning in school during 1994-2000 in Romania. This technology had been developed on the basis of the cognitive structural learning model, we built up during the research.

We call **structural cognitive learning (SCL) of a domain** the process by which abilities specific to the expert in that domain are created in the one who learns. This kind of learning implies a bipolar cognitive construction: on the one hand, the domain which is to be taught is organically integrated in a *constructed structure* emphasizing clearly specified objectives, and, on the other hand, learning each subgroup of the structure implies an *active reconstruction* of its meaning (in the sense of the constructivist definition of learning).

The structural cognitive learning of a school subject *supposes the complex structuring of information* to be internalized. In this way, a school subject appears as a logical structure, which must be learned as a structure, with natural relationships between concepts. In this context, the effective learning needs specific categories of training: *systemic training*, *random training*, and *structured training* of the intellectual capacities. Some of these capacities are general for the human intellect and some are specific to the domain that is to be learned, in our case - mathematics.

Dominating the information by formation seems to be the only realistic solution to overcome the crisis generated by the larger and larger accumulation of information in each domain. This solution becomes operational as far as information organically inculcates formation, turning the last one into explicit objectives of learning.

3. Building structured learning units

There is an issue, often quoted and exemplified by Z.P.Dienes (1973) in his book "An Experimental Study on Learning in Mathematics", that a child can learn anything, at any age, if one finds the means to transpose what is to be learned in a mental language accessible to the one who learns.

This idea spread quickly and led to the presence of certain elements previously considered as hardly accessible or even inaccessible, in the mathematics curricula. Of course, some exaggerations appeared too; the stress was laid on "*so* it's possible" and less on "*how* it is possible". Another aspect that was lost in practice refers to the fact that those experiments took into account the learning of certain well-specified and limited entities (for example, elements of topology, or elements of algebraic calculus). However, what happens when the child has to learn a whole "domain" with all the structured and interconnected knowledge within? This is the question that concerns us, and the answer to it is obviously much more complicated than the answer to the question Dienes asked himself.

The experiments showed that the SCL could be accomplished in school if and only if what it is taught belongs to a whole, which is constructed and structured gradually.

The first phase in building the content for this type of learning is breaking up the information into basic elements for identifying those *input ce*lls, whose structured union creates the domain. These cells also called *basic learning units* are defined by three components: *basic mental operations, levels of abstracting, and information.*

In this way, a *basic learning unit* (U) can be represented as a vector having three independent components:

U (basic mental operations; levels of abstracting; information).

We shall explain in the following the meaning of each category.

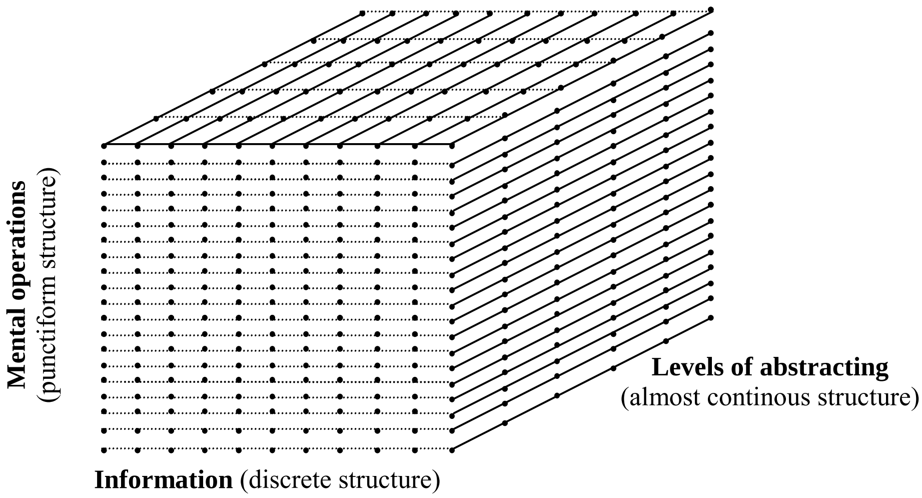

Fig. 1. The cuboid of the basic learning units

Without giving a formal definition, we are using the concept of *basic mental operation* as the simplest brain activity that could be clearly identified by differentiating from others and which is irreducible to its components in the sense that it is expressing forms of actions that are directly teachable. As the terms designating *the basic mental operations* have a broad utility scope, they will be here strictly defined from the viewpoint of the present theory, as follows:

- *Recognizing:* the action by which, in the presence of a certain entity[1], one becomes aware of the fact that he or she has come across that entity before; in other words, recognizing means identical association of an entity to itself.
- *Denominating:* the action by which a known entity is associated to a name; otherwise, denominating means associating an entity to its name.
- *Reproducing:* the action by which one creates specimens identical to an entity already known; in terms of associations, reproducing means associating the entity with a shape which is preserving its properties, be that shape rendered graphically, physically, or verbally.
- *Representing:* the action by which one creates a symbolic shape of an entity already known; in terms of associations, representing means associating the entity with a symbolic shape of it.
- *Interpreting:* the action by which an explanatory "version" is associated with an entity; in terms of associations, interpreting means associating the entity with a descriptive version of it.
- *Classifying:* the action of categorizing entities following one or several criteria; in terms of associations, classifying means associating the entity with the category it belongs to.
- *Class:* a group of entities having the same properties with respect to a certain criterion.
- *Counting:* the action by which, one associates, "element by element" a sequence of sets in ordered succession with the sequence of cardinal numbers of those sets; counting means associating a set of entities with its cardinal number.
- *Measuring:* the action by which, one associates its measure to an entity with respect to a certain criterion;

[1] The terms "entity", respectively "object" will be used in the following with the broadest possible meaning. In particular forms, they could mean, depending on the circumstances: physical object, scheme, theorem, concept, theory, etc.

measuring means associating a dimension of an entity with its measure, according to certain standard or non-standard conventions.

- *Estimating:* the action by which one appreciates or grasps the size of the measure of an entity or the result of a procedure; estimating means associating a dimension of an entity with its "predicted" measure by assessing, not by measuring.
- *Ordering:* the action by which several entities are placed in correspondence with their positions in an ordered set by an order relationship between the measures of those entities referred to a certain criterion; ordering means associating a set of entities with the positions of the measures of those entities in an ordered sequence.
- *Association:* the action by which a link or a correspondence based upon a criterion is established between different entities.

All the operations defined till now share a common feature, which is they *express different types of associations or correspondences* set up between two terms.

In addition to the correspondences previously described, the human brain is able to do many other associations, even if they are not designated in the daily language. That is the reason for preserving the operation generically called association in the list of basic mental operations. Such associations are, for instance: *inversion* (association of an entity with its reverse or opposite related to a certain criterion); *substitution* (replacement of an entity situated in a given context with another entity, keeping the context and the relationships between the entity and the context unchanged), etc.

In the following, other types of basic mental operations are described.

- *Comparing:* the action by which two entities are facing up for characterizing the similarities and the distinctions between them; comparing expresses the action by which one relates two objects.
- *Selecting-discriminating:* the action by which one separates an element with certain features out of an unstructured set; it expresses the action by which one relates an object to all other objects from a given set.
- *Checking* is a shortcut for *becoming aware of errors' outcome*. It expresses the action by which one compares the results obtained in analogous or different ways or estimates the results, with the purpose of eliminating possible mistakes; it is expressing the action by which one relates an object to an implicit or explicit pre-existent model of it. It presupposes the discrimination between the wrong and right results in a given situation. Its presence on this list is justified by the fact that the role of the control awareness of the mental actions is extremely important in order to construct and strengthen the mental structure.

These last three operations share in common the property of generating a *relationship between two terms*.

Other types of basic mental operations are presented in the following.

- *Algebraic operations:* correspondences defined on a Cartesian product with values in another Cartesian product. In school, one studies the binary operations (defined on a Cartesian product with two factors), such as addition and multiplication. In school learning, we distinguish several phases of the algebraic operations:

Pre-arithmetical operations. These refer mainly to composing and decomposing numbers. However, in this category we shall also include a list of more general operations, which, quantitatively expressed, lead to arithmetical operations, such as: magnifying, reducing, taking away, adding, combining, etc.;

Arithmetical operations (addition, multiplication, squaring, etc.);

Operations with variables (the same operations with the ones with numbers);

Operations with sets (union, intersection, difference, etc.).

- *Topological operations:* operations defined on an infinite Cartesian product. Such an operation is, for example, the calculus of the limit of a series of real numbers.
- *Logical operations:* operations applied to propositions or to predicates. These are: disjunction ("or"), conjunction ("and"), negation ("not"), logical implication ("If p, then q" ($p \rightarrow q$)), and their combinations. As these elements are enough for the objective we aim at, we do not go into further details that are related to mathematical logic.

Other significant basic mental operations are listed below.

- *Identifying patterns:* identifying the constant (the invariant) in a variant environment.
- *Grasping:* the action by which an entity is directly known, that is without proceeding discursively, by passing from one bit of information to another, but by perceiving that entity or its essence instantaneously, either through the senses, or through the intellect.
- *Generating:* the action by which one creates new entities, previously unknown, starting from entities already known (by the subject).

The basic mental operations were selected, on the one hand, using the "atomistic" criterion (the simplest forms of actions which are directly teachable) and, on the other hand, according to the idea that they are meant to be senses generators at the level of the intellect, that is they can be completely internalized.

In addition to these basic operations, the building of the mental structures also implies *transfer operations*. These are shifts from one level of abstracting (on which a specific operation acts on) to another, shifts from one operation to another, as well as other operations, which result through composing the basic ones, or through their interference with different information or chunks of information. An example of these last categories is done below in the case of *generating*.

We can speak about *random generating* and *conditioned generating.*

Random generating refers to producing or choosing entities at random, out of a given set. (For example, to train this, the students could receive tasks as follows: "Give some examples of fractions"; "Create shapes using plasticine"; "Draw a triangle, etc.").

Conditioned generating refers to engendering entities with respect to one or several conditions that have been imposed. The training consists in developing different types of conditioned generating; some of them are listed below. The brackets contain examples selected at random from the methodology developed for the SCL process.

- *Generating through representation* ("Represent in different ways a quarter, as suggestively as possible".)
- *Generating through interpretation* ("Number 7 being given, consider it as being, successively, a sum, a product, the solution of an equation, etc., and devise problems for each case.")
- *Generating through classification* refers to, on the one hand, finding as many classes as possible for a given object, and, on the other hand, finding as many representatives as possible in a given class (for instance, to enumerate as many different objects as possible which can have the same use): ("Draw geometrical figures in which a triangle should be formed by: the diagonals of a polygon; two sides and one diagonal of a polygon; the intersection of two polygons; the face of a pyramid; the base of a prism, etc."; "Find ten natural numbers which are multiples for 2 and 3 and not for 12.").
- *Generating through counting* ("Count by threes starting with 2. Vary the counting step.")
- *Generating through estimation* ("Give examples of exercises the result of which is closer to 500 than to 600." "Give examples of objects the height of which is about 2m.")
- *Generating through analogy* (Devise a problem to be solved through the same algorithm as the following: "In the trapezium ABCD (AB || BC), O is the intersection point of the diagonals and AO=OD, OB=OC. Show that the trapezium is isosceles.")
- *Generating through substitution* ("In the formula $a+7=x$, x-unknown, replace a by different values and devise problems appropriate to the formula obtained; replace a through an operation of subtraction and devise problems fitting the formula.")
- *Generating through comparison* ("Give examples of: a) natural numbers less than 7; b) fractions less than 7." or "Find the length of a square which has the area at least double than the area of a square with the length 4.").
- *Generating through selection-discrimination* ("Draw different polygons in which the diagonals are perpendicular; draw different polygons in which diagonals are not perpendicular." or "Giving a ring with the diameter of 5, describe some 3D shapes able to cross this ring.")
- *Generating through arranging and rearranging* ("We have the numbers: 12; 24; 27; 43 developing the following ordered sequences:

12	24	27	43
43	27	24	12
12	24	43	27
43	27	12	24

Which was the criterion, which has been used for the arrangement in each case? Devise other criteria and write the ordered sequences accordingly[2].")

- *Generating through increasing and decreasing* ("Consider the problem: "There were 15 trees. Other 7 were planted. How many trees are there?" Add another piece of information so that the problem should have a solution according to the formula:
$a + b - c = x$, x - unknown.")
- *Generating through combining* ("Devise a problem, which can be solved through two operations: a multiplication and a subtraction.")
- *Generating through changing (varying) the hypothesis* ("We know that if a triangle is isosceles, then the bisectors of the angles opposed to the congruent sides are congruent. What supplementary conclusions result if, in addition, the triangle is also right-angular?" or "The sum of two natural numbers is 200. What could be the numbers? Add an hypothesis so that the problem has: a) one solution; b) two solutions; c) three solutions.")
- *Generating through changing (varying) the conclusion* ("In a given theorem, replace the conclusion with a weaker one (less restrictive). What hypotheses will be sufficient to draw this conclusion?" or "A four-sided polygon has two edges on a circle. What type of polygon: a) has the other 2 edges on the same circle; b) has not the other 2 edges on the same circle?").
- *Generating through logical derivation* ("Express the reciprocal proposition, the negation, the contrary proposition, the contrary of the reciprocal, etc., for a given theorem.")

Both random generating and conditioned generating, included in various kinds of training, have as result the *development of the creative capacities* of the one who learns.

The basic mental operations act on various *levels of abstracting*. The levels of abstracting are stages in *internalizing the information*. In a working classification, we consider the next levels of abstracting: *the concrete plan* – at the level of using objects, *the unconventional symbolic plan* – at the level of using nonstandard physical representations, *the symbolic plan* – at the level of using standard physical representations, *the verbal plan* – at the level of using verbal representations, *the mental plan* – at the level of using internalized representations. According to the entity they are applied to, it is possible to consider more or less refined scales of abstracting.

The levels of abstracting as they are seen in this paper are gravitating around the idea of representation and here is the conjunction with Goldin concept of building powerful representational systems as a goal of learning (Goldin, 1996). The categories to classifying the levels of abstracting are intentionally extremely general; they are particularized, as they are associated to a certain operation or/and to specific information pertaining to a particular school curriculum.

The third component of the basic learning units is represented by *information*. In the mathematics learning process, information are normally structured into domains of mathematics and integrated in a didactical model of the discipline. We anticipate by specifying that, within the frame of the mental training, the focus will not be on information, but on the complex of actions through which a piece of information is internalized. Information in this article mean both conceptual and procedural knowledge generally prescribed in the written curriculum to be learned in school. From a curriculum developer

[2] Answer: Increasing order of the given numbers; decreasing order of the given numbers; increasing order of the sum of the figures; increasing order of the number of divisors.

perspective, this model also offers the possibility to rethink this curriculum.

In any structured domain, there are different *levels of* information *complexity*. These have been characterized from three dimensions: *the nature*, *the structure*, and *the procedure*.

If we take the *nature* of information then the criterion is the *level of generalization*; information could be than organized from the particular to the general. In the SCL process, a lot of passages from the particular to the general and vice versa are practiced.

The *structure* of information is described by the *connections among concepts*, which organize information on different stages from an unstructured (no-systemic) one to a structured (systemic) one. In training mental skills it is very important to pass through as many organizational stages as possible.

Concerning the *procedure*, the criterion is the *dynamics of connections* and from this point of view, information could be organized from algorithmic to creative procedural actions.

Summarizing, a basic learning unit can be decomposed on three independent directions, suggesting a three dimensional structure: the basic mental operations, the levels of abstracting, and the information. The three components have different structures. The set of the basic mental operations have a punctiform structure in the sense that no connection, hierarchical or of another type, can be established between them. Information have a discrete structure, in the mathematical sense of this word that is we accept the hypothesis that between two "neighboring" pieces of information there is a certain distance, that is if going deeply into the domain, other information can appear too. The levels of abstracting have an almost continuous structure, that is the "distance" between a certain level and another one can be infinitely refined, so that the passing becomes practically imperceptible.

What we previously described can be schematically represented by what we call the *cuboid of the basic learning units* (Fig.1.). Each "point" (cuboid) in this space, described by the three co-ordinates, represents a basic learning unit. Let us imagine that we add to this configuration the transfer operations, and to each bit of information the tri-dimensional structure of the information degrees of complexity. What we get will be called "the cuboid of the input capacities". Each element of this space is therefore an *input capacity*.

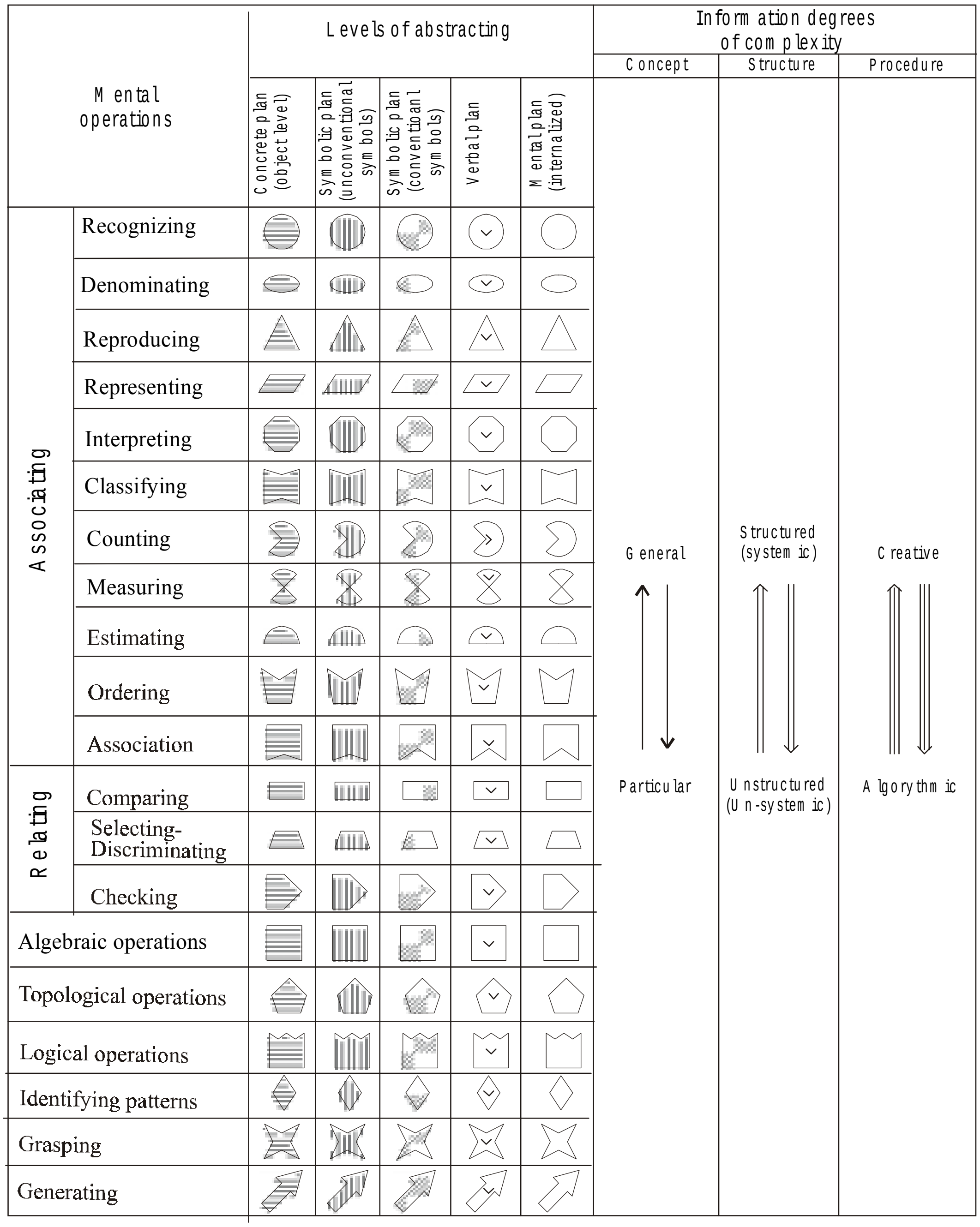

Fig. 2. The inventory of the input capacities

The diagram representing The inventory of the input capacities in Fig. 2 is a plan representation of the cuboid of the learning units. The geometrical symbols that appear in the table have been used in the workbooks developed according to this theory for pre-primary and primary grades 1 to 4, to describe the category to which a particular item contained in the books belongs to.

4. The Structural Cognitive Didactical Model of the Discipline - the Basis for the New Curriculum

In the traditional common teaching, the goal is to teach so that students assimilate information. The total amount of information considered necessary and useful to be taught with mathematics from the first school year to the last one is divided into parts fitting each school stage. Then, inside each school stage, the information specific to each school year is selected through the curricula. Further on, in the textbooks, information is split into chapters and lessons. The reverse path, from the first lesson in the first school year to the last lesson in the secondary school, is supposed to be followed by the pupil's mind with the purpose of "knowing" the mathematics assigned to the pre-university education by the experts in curriculum development. Thus, the pupil faces a mixture of rigorously detailed informational islands. The pupil must go through these. In the end, the pupil must know all he/she has learned, or at least the essential of each mathematical *procept*. Moreover, the pupil must also possess a mathematical thinking, which has a special human and social value; otherwise, the whole teaching would be pointless. Beyond the barren instruction, there is a need to develop mathematical thinking of a much greater value, therefore transcending the boundaries of the formal content. The logical rigorousness, the ability to do quantitative analyses, the quality of the professional activity is tightly related to the quality of the mathematical instruction of each person.

The need for developing the mathematical thinking has been permanently emphasized in didactics since a long time. The problem is not whether the development of the mathematical thinking represents the essential goal of mathematical instruction in school, but how to pass from the intention to the reality. Nowadays, the dominant idea is that the mathematical competence, or mathematical thinking, is a spontaneous result of instruction. The person learns and *automatically* becomes competent, according to the following rule: the one who gets the information is also able to think on the appropriate level. As we see, according to the traditional philosophy of learning, which is still very strong everywhere in the world, information represent the essential in the mathematical didactics, the competence (mathematical thinking) being just a mechanical consequence of their assimilation.

The results of this philosophy of learning are expressed by the present-day fact: the average pupil's failure to learn mathematics. This average pupil possesses, at best, just "informational islands". Researches carried out on secondary-school students reveal major flaws exactly on the level of mathematical thinking; but these flaws have their roots precisely at the beginning of the informational stairway, that is as early as the first grade. The teaching which is focused on developing capacities implies an extremely structured *knowledge organization*; that is why, in spite of our assertion about the pre-eminence of skills development over information, we had to start with building a curricular model of the informational contents of school learning.

For the description of the contents of math for primary and secondary education, we have to refer to the relation between the two meanings of mathematics: science of thinking and school subject. Starting with the science, some fields of mathematics are considered fundamental for the school path: ARITHMETIC (having as components: numbers, operations with numbers, estimations and approximations); ALGEBRA (having as components: sets, relationships, sequences, functions, equations, inequations, systems, polynomials, etc.); ANALYSIS (dealing with the study of certain properties of functions); GEOMETRY (dealing with the study of geometrical figures and solids from different perspectives: synthetical, metrical, the one of the relative positions, and the one of characterizing specific measures in terms of algebraic formulae); LOGIC; DATA ORGANIZING (collecting, recording, processing, representing and interpreting data); STATISTICS and PROBABILITIES.

It is natural to find in the school curriculum elements from each of these fields of mathematics as a science, and this is what really happens at present. However, the problem is to distribute this huge amount of information within the period of instruction so that the school graduate might subsequently use concepts and competencies that are specific to each mathematical field that he or she has studied. This is so because, for a significant segment of the school population, the end of compulsory school means getting into the professional life, as far as for another part, upper secondary education means getting into the professional life. Meanwhile, for another segment of the school population, it means going on studying, and all these aspects must be taken into consideration in outlining the didactical model of the discipline.

From this point of view, the primary school, being the most stable part of the system, is in a paradoxical situation: though it has not been preparing graduates for decades, it remained, in very many countries, in a restricted area of contents, in which learning "reading, writing and computing" still is the fundamental objective, despite the intentions stated in the curricula's rationales. The superabundance of difficult problems in textbooks is not liable to break the deadlock, but to deepen, in the child's mind, the confusions produced by the lack of perspective. It follows from the above that the change must start with the primary school, and that this change is substantial on this level, both in content (*what* it is learnt), and in methodology (*how* is learnt). This shift in the targets of learning has been done by Great Britain in the nineties and by other English languages countries, as Singapore, Australia, and New Zealand in successive steps.

Following the SCL theory, a model that describes the way the content of Mathematics as a school subject should evolve from the primary school to high school has been generated.

Some of the basic principles guiding this approach are: continuity, perspective, flexibility and focus on the mathematical concepts.

The model ensures the continuity of information for the whole pre-university school period. It creates a basis of the mathematical thinking as early as the early ages, avoiding stray learning, which, as it frequently happens nowadays, is contradicted by later mathematical acquisitions (We refer mainly to the gaps which appear between the school stages, as well as between certain mathematical fields, during the schooling period).

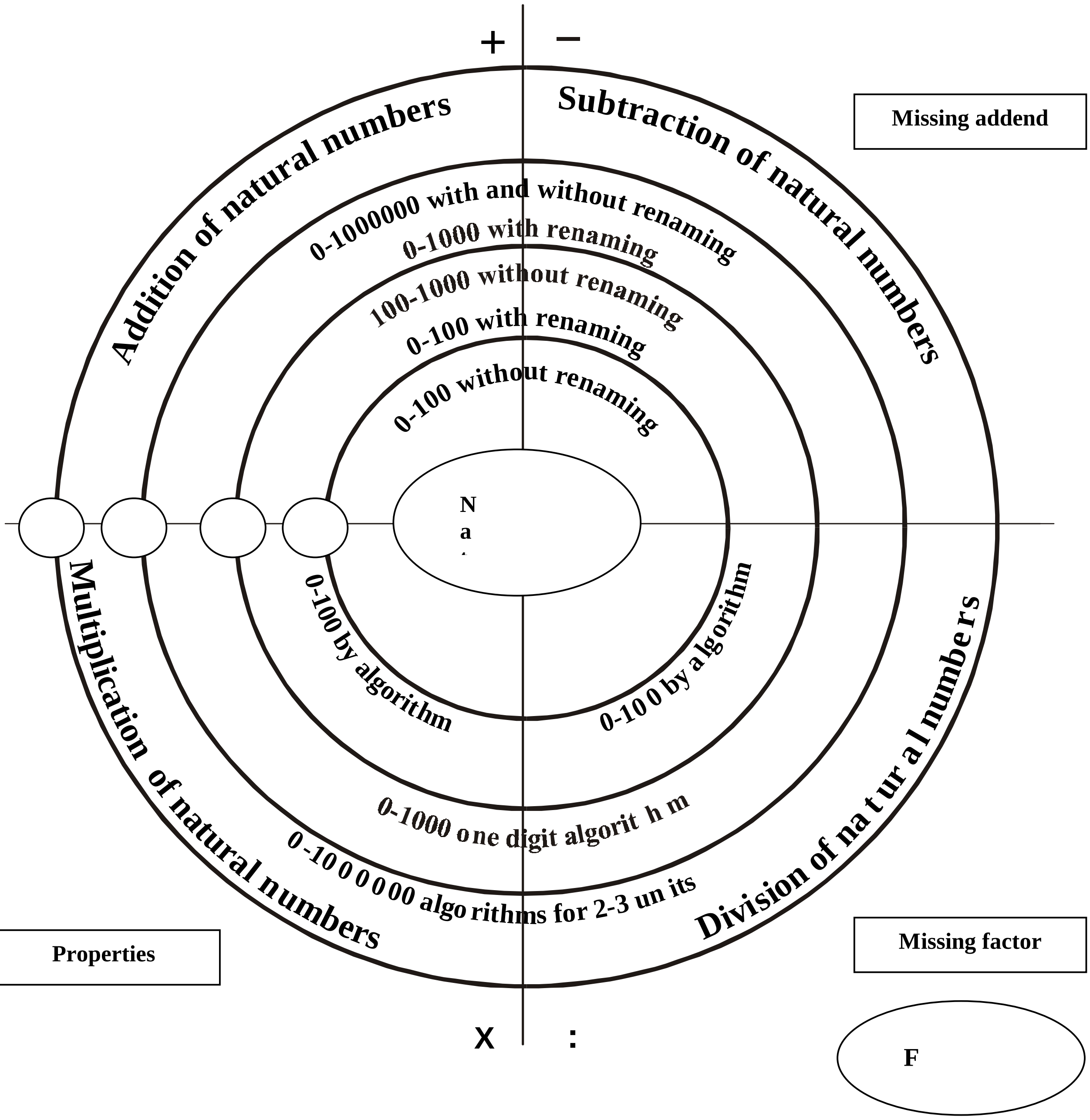

Fig. 3. A representation for the traditional curriculum in primary education

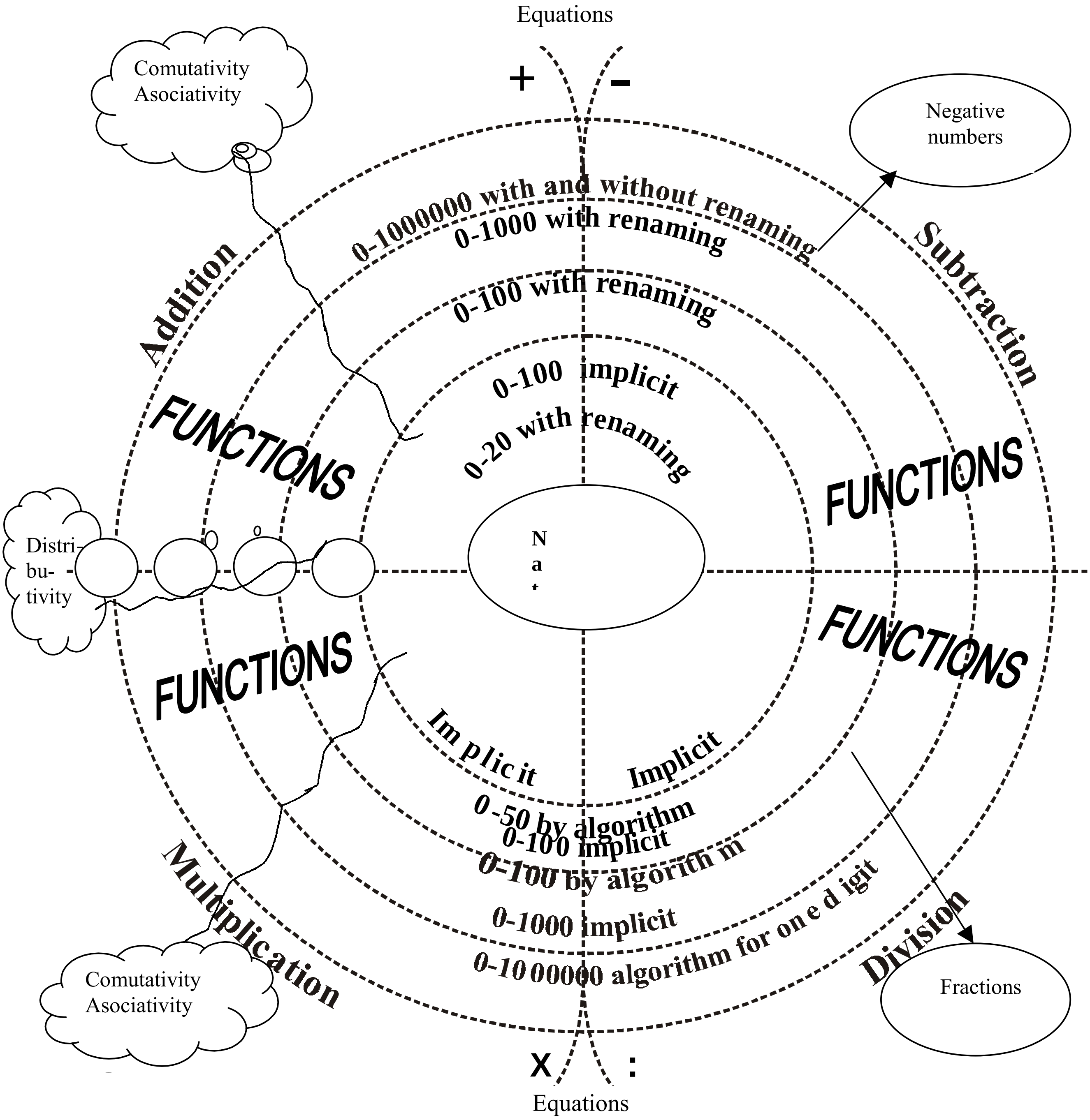

Fig. 4. A representation for the structural cognitive model of the new curriculum in primary education

The diagrams in figures 3 and 4 contain synthetic similar descriptions for arithmetic and algebra in primary education in the Romanian curriculum. To make things clearer we shall compare the content's organization, as it is appearing in the traditional curriculum and in the new one, built upon the SCL approach.

First of all, a short description for the meaning of the representations: The inner part of each concentric circle represents the curriculum content for grade I, grade II, grade III, and grade IV. The two perpendicular axes separate the plan of each circle in four zones and each zone symbolizes the "territory" of an arithmetical operation, respectively: addition, subtraction, multiplication, and division. At the border between addition and subtraction there are the equations of the type $a + x = b$, where a and b are natural numbers belonging to the set of numbers the children already studied, $a < b$, and x is an unknown variable. At the border between multiplication and division there are the equations of the type $a \bullet x = b$, where a and b are natural numbers belonging to the set of numbers children already studied, b is a multiple of a, and x in an unknown variable. The equations are significant ways of passing from the fundamental operations (addition, multiplication) to their corresponding inverse operations (subtraction, division). This idea of connecting the arithmetic operations through equations is not enough developed in the teaching practice.

Another connection between operations is valued by the graphical "up-down" disposition. More precisely, it is about the connection between multiplication and addition (multiplication is a repeating addition) and between division and subtraction (a division is equivalent with a repeating subtraction till zero). The connections are very significant in a natural explanation for "the priorities" of operations in a chain of exercises.

Outside the circles, because these must not be formally learnt, there are the properties of operations: associativity and commutativity are used in primary schools in an implicit way for both addition and multiplication. At the border between addition and multiplication is "situated" the property of distributivity of multiplication against addition.

We shall now analyze the two models by comparison. First of all, the similarities: the information taught in each school year is quite the same and also the sequence is similar. Further, there are the differences. In the traditional curriculum, the content units are strictly and clearly separated, meanwhile the structural cognitive version supposes the free passage of concepts from one level to another. That means, for instance that in grade I students study addition by counting but there is also an implicit training for place value, preparing thus the transfer to the next level. Also, in grade I only addition and subtraction are studied but there are also practiced lots of activities concerning implicit multiplication and division, which are studied later on. Also, during each school year there is a "come and go" process of reviewing from different perspective things which have been already studied.

The open structure of this curriculum permits to create the perspective and to tackle with mathematical topics that cross the formal boundaries, such as the notions of *equation* and *function*. Nevertheless, the opportunity of introducing in a natural way the *fractions* and the *negative numbers* is given.

A fundamental objective in organizing the content is the *focus on concept*. For example, if the notion of *function* is introduced in grade one, in the absence of a formal apparatus already assimilated, the child is in the situation to deeply understand the concept using intuitive concrete models. In this way, the child can internalize this fundamental notion of mathematics through appropriate methods to his or her age, in such a natural way in which the mother tongue is enriched and internalized. On the other hand, he or she gets enough time to organically complete the created mental structure with new facets of the concept. This fact makes the introduction of formal description of function in the seventh (or eighth or ninth) grade able to give more consistency to the abstract meaning of function. Instead of reducing it to a label, as it frequently happens while teaching according to the traditional curriculum, in this new way, the concept of function is developed on a very strong intuitive basis and becomes a natural acquisition in the child's mind.

Meanwhile, the model also aims at easily transferring knowledge from one domain to another; it has also in view the clarity of the interconnections between the elements of the same domain, as well as between the mathematical domains.

The closed structure of the traditional curriculum divides the subject matter into learning domains that are disconnected in-between, domains which are difficult to assimilate, where learning is made with important waste of intellectual energy and, many times, in the form of label-algorithms; the child does not have the possibility of transferring the knowledge from one area to another.

What is interesting and in the same time significant from the viewpoint of the present theory is the fact that these issues, easily noticeable by observing the children's behavior in the class, precede by figuring out the way the traditional curriculum has been designed. More or less explicitly, countries that are targeting high technology development ("The Asian tigers", for example) adopted for primary education a type of model quite similar to a SCL approach.

5. Two principles for one purpose: Learning Thinking

In common learning, after studying a certain subject, the pupil rarely reaches (and this happens only for the gifted ones) the logical model of the assimilated information (that means the didactical model, because the model of that subject as a scientific domain is impossible to be reached in school). The "output" means disconnected information, with inevitable discontinuities, difficult to be mobilized and used in new situations. This is, in fact, a failure from the very beginning in the case of those disciplines that naturally presuppose a strong and explicit connection between the assimilated notions (mathematics, physics, chemistry, foreign languages, etc.).

How to avoid this situation? In principle, by "canceling the daily lesson", but this cannot be done, as school learning would become impossible. In practice, good teachers try to redo what has been previously assimilated in order to establish a long-lasting connection with the new knowledge; however, this repetition is done accidentally, unsystematically, and, as one gets along in the curriculum (or textbook), more and more of the essential information is left out. The *cognitive constructivist principle* views this process of *returning to the basic information* from the previous lessons as essential. It is not a question of just repeating the previous essential information, but also of teaching and assimilating bridges among the chunks of information in order to give the latter a certain stability and functional independence. The result of teaching is a "functional organ" in the sense this term was used by A.N. Leontiev (1981), and knowing a domain without constructing this functional organ is unconceivable. The formula for constructing this "functional organ" is a practical problem of the teacher's book, but also a matter of accuracy of the subject's logic incorporated in a certain textbook. The "functional organ" is the psychological expression of a domain's logic, but also the expression of the didactics' quality involved in its assimilation. That "functional organ" is present in the "expert" of the domain but not in the "novice". In the expert, it has such autonomy and stability that, when he or she faces a new situation, the (necessary) elements of the functional organ are quickly selected, without any disturbance of the system as a whole. Rapidity, efficiency, and effectiveness become possible.

The *constructivist principle* in didactics is maybe the most interesting didactical achievement in a psychological perspective, after the one of B.F. Skinner (1974); it could be the most original contribution in didactics, being deeply rooted in Wertheimer's (1935) and Koffka's (1940) thought - in what was once called "Gestalt psychology" -, and mostly in the thought of L.S. Vygotski (1934/1962) and Piaget (1972). Of course, the contemporary didactical constructivism is different from what these forerunners thought, but this fact is natural, since a true development is never a mere reproduction.

The practical problem is how to ensure the development of "functional organs" as effectively as possible in the case of a subject or a certain grade, what to do in the case of a certain lesson or chapter. In this case, the didactics research must be directed from the very beginning by experimental variants, working hypothesis, careful extensions, etc. For instance, how are the structures formed when repeating the information already transmitted? Nevertheless, how to repeat it and avoid the monotony? How is this feature accomplished in the "maternal school" level (that is in the family, at the pre-school age), where the child assimilates "a world of information" and starts his or her numerous "functional organs"? Those are some questions to which the SCL approach tried to answer.

The concern for structuring, for constructing the functional organs, does not satisfactorily solve the problems of effective learning. Here is where *the formative principle* (or the principle of *directly learning intellectual capacities*) interferes. In fact, those "functional organs" may be evolved or rudimentary. Their mere existence does not entail quality. Their construction is always present, but their quality may be poor or high. The "formative didactics" assumes the premise that "functional organs" are not given once forever, a certain historic evolution is obvious. Thinking, feelings, will, the whole personality represent the content of those functional organs, and their quality increases with each significant stage covered by individuals in their evolution. This finding, as simple as it seems, can revolutionize didactics. If there is a "cultural construction", a huge "functional organ", which is assimilated within the ontogenesis as a projection of the "objectified psychology" in a certain culture, then the formation of the "inside", of the "series of functional organs", that makes up the human psychic, its qualitative development is, for the post modern era, at least partially, the task of didactics, of the training efforts, and of the quality of these efforts.

The SCL process of a domain implies three distinct stages: internalizing the basic learning units, building mental structures, and practicing them. In real teaching, the limits of these stages are not so clearly marked. At the first contacts with a new field, the focus is laid on internalization, on primary development of the mental structures and on practicing them in simple situations. Later, the internalization continues, as new information is assimilated, but the focus lies on their integration in mental structures and on practicing them so that they might become extremely mobile in three directions: able to *multiply* (reproduce) on higher levels of abstraction, able to *integrate* in new structures having the same nature or different natures, and able to *mobilize* with great precision when there is a need of them in solving some practical tasks (by their appropriate reduction to the already developed intellectual models, or by their temporary changing for the purpose of solving totally new tasks). In the process of practical training, the models confront the external requirements that are identified as being known or unknown; in the second case, finding a solution implies an effort of creation. From a certain moment in learning a domain, it is compulsory to stress on practicing the already created internal structures. Amplifying them, through new information included, multiplying them to other level of abstracting, banding them in new structures, training their mobilization, practicing creative problem solving, all these are targeting to transform the already created internal structures into instruments of a new learning.

For the future, by affirming the formative principle, didactics tend to enter into a special relation with history: it could intensify the developmental processes, raise their quality and direct them on the scale of a school cohort's life. If for the curriculum there is the problem of anticipatory development of the functional organs of a certain quality, starting from objectives that include the future (from the viewpoint of anthropology), then the "building of what is deeply human" becomes exactly the school's task. The need for fashioning the human mind came into being while the information society engendered (to a great extent) its high tech, but did not create a sufficiently developed human being able to handle it and to morally dominate it; for the future, this inability becomes dangerous for the very existence of mankind. The need for superior development at intellectual and moral levels led to research in the sphere of training responsible creativity and learning thinking.

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