

The order of theorems in the teaching of Euclidean geometry: Learning from developments in textbooks in the early 20th Century

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Abstract: The question of the order of theorems in geometry teaching is very important and it was one of the central issues in the early 20th Century in England. Employing ideas from the methodological framework proposed by Schubring (1987), the order of theorems in the geometry textbooks written by Godfrey and Siddons is analysed within their pedagogy and social context. The main foci for this analysis are *Elementary Geometry* (1903) and *A Shorter Geometry* (1912), which were widely used in secondary schools at that time. The theorems in these textbooks were arranged differently from those of Euclid's *Elements*. Godfrey claimed the order was organised from an general educational point of view. In *A Shorter Geometry*, flexibility concerning the order of theorems was recognised as a revision from *Elementary Geometry*. The analysis presented in this paper provides us with information about teaching practice at that time, for example that teachers might still be bound by examinations after 1903, and helps us to understand important aspects of dealing on the order of theorems in geometry teaching.

Kurzreferat: Die Frage nach der Reihenfolge der Theoreme im Geometrieunterricht ist von großer Bedeutung und war eine der zentralen Fragen in England zu Beginn des 20ten Jahrhunderts. Unter Berücksichtigung von Ideen der methodischen Richtlinien Schubrings (1987) wird die Reihenfolge der Theoreme in den Lehrbüchern von Godfrey und Siddons innerhalb ihrer Pädagogik und im sozialen Kontext analysiert. Im Mittelpunkt der Analyse stehen die Lehrbücher *Elementary Geometry* (1903) und *A Shorter Geometry* (1912), die zu dieser Zeit im Unterricht der Sekundarstufe weit verbreitet waren. Die Anordnung der Theoreme in diesen Büchern weicht von der in Euklids *Elementen* ab. Godfrey begründet die Anordnung mit einer allgemeinbildenden Sichtweise. In *A Shorter Geometry* wurde die Flexibilität der Reihenfolge als eine Weiterentwicklung der *Elementary Geometry* betrachtet. Die Analyse in diesem Beitrag informiert über die Unterrichtspraxis dieser Zeit und hilft uns, wichtige Aspekte beim Umgang mit der Reihenfolge von Theoremen im Geometrieunterricht zu verstehen.

ZDM-Classifikation: U20, A30, G40

1. Introduction

In a number of countries the teaching of geometry remains largely based on Euclidean geometry. For example, when we examine Japanese textbooks in secondary schools (students aged from 13 to 15), we can see that they study proof in the second (students aged from 13 to 14) and third grades (students aged from 14 to 15). The content is organised as follows: parallel lines and angles, triangles, parallelograms, and similarity for the second grade, and the Pythagorean theorem and circles for the third grade (Fujita: 1996a and 1996b). The issue I address in this paper is the origin of the

organisation of the sequence of content, which remains one of the most important tasks in designing the geometry curriculum. In particular, I focus on developments in geometry teaching in England in the early 20 Century as this is recognised as one of the major influences on today's geometry teaching based on Euclidean geometry.

According to Price, major reforms mathematics teaching in England occurred in the years around 1870, 1900, and 1960 (Price; 2001, p. 217). Of these periods, the early 20th Century can be considered important to the issue in this paper because it was particularly discussed by various people in this period (I will describe this later). Some historical studies have examined the reform of mathematics education at that time. For example, Howson (1982) studied the development of mathematics education in the early 20th Century including the contribution by Godfrey. Price (1994) also discussed the development at that time by looking the activities of the Mathematical Association and its members. However, detailed examinations of textbooks have yet to be undertaken, even though the publication of new textbooks was one of major results of the reform at that time. Furthermore, studying textbooks is very important for the historical study in mathematics education (see section 3). In this paper, therefore, I address this issue by analysing some geometry textbooks which were widely used at that time. The books are those by C. Godfrey (1873-1924) and A. W. Siddons (1876-1959), major players in the reform of the teaching of geometry in England in the early 20th Century. As a methodological framework, the scheme for historical textbook analysis proposed by Schubring (1987), which I will introduce later, is utilised. As an historical study often does, this paper attempts to provide us with an opportunity to reflect on our current teaching practice by looking at the historical developments of the teaching of geometry.

2. A brief description of the teaching of geometry in the late 19th and early 20th Centuries in England

In the 19th Century in England, the teaching of geometry in secondary schools meant usually the direct teaching of Euclid's *Elements*. The value of the *Elements* in education had been considered that it would train students' ability in logical reasoning. Furthermore, it had been considered that „every Gentleman should know Greek thought“ (Griffiths: 1998, p. 195). Whereas Euclid's *Elements* was respected as a geometry textbook in secondary schools, the problems of the direct teaching of it existed even at that time. For example, according to Jackson (1924), the Report of the Schools Inquiry Commission in 1868 summarised the causes of the difficulties of the teaching of Euclid's *Elements* as follows (Jackson: 1924, pp. 36-7):

- the lack of an introductory course
- the ban on hypothetical constructions [using geometrical figures such as an angle bisector without showing how to draw them by only a compass and ruler].
- the treatment of parallels
- the treatment of incommensurable magnitude in the fifth Book

In 1871, the Association for the Improvement of

Geometrical Teaching (the AIGT) was founded by University mathematicians and teachers from public schools to improve the teaching of geometry. To offer an alternative to Euclid's *Elements*, the methods of proof and the order of theorems were discussed by the members of this association. The AIGT published its *Syllabus of Plane Geometry* in 1875 and then, in 1884 and 1886, published a geometry textbook, *The Elements of Plane Geometry* which included proofs of the theorems contained in the syllabus of 1875. *The Elements of Plane Geometry* „whilst retaining Euclid's overall sequence, rearranged theorems within allied groups and supplied new proofs“ (Howson: 1973b, p. 158). However, these efforts failed to change the teaching of geometry, partly because A. Cayley, who was the most powerful Cambridge mathematician at that time, opposed radical reform, and more importantly, the Examination Boards such as Oxford and Cambridge were reluctant to revise their examination requirements (Siddons: 1936, p. 18, Griffiths: 1998, p. 196).

In 1901 J. Perry, Professor of Engineering at the Royal College of Science, gave an address „The Teaching of Mathematics“ at the British Association for the Advancement of Science (the BAAS) meeting in Glasgow. In the address, he denounced contemporary mathematics teaching. With regard to the teaching of geometry, he questioned the educational value of Euclidean geometry for all students, and emphasised the value of the introduction of experimental tasks in the early stages (Perry, 1902: pp. 158-81). As a consequence of this address, various movements by organisations and individuals occurred aimed at reforming the teaching of geometry. In the Annual Meeting of the Mathematical Association (the MA) (the AIGT changed its name to the Mathematical Association in 1897) in 1902, the issue „the reform of geometry“ was discussed by members. It was suggested by some members that the order in the teaching of geometry be rearranged in „a more natural order“ (MA: 1902a, pp. 129-43). They did not expand on what they meant by this nor did they propose a particular order. Suggestions from „the teaching of mathematics in public schools (the letter of the 22 school masters)“ and „the MA Report on Geometry in 1902“ were rather modest. They stated that they still respected Euclid's order (Godfrey and Siddons *et al.*: 1902, MA: 1902b, pp. 168-72). Nevertheless, in the BAAS Report on the Teaching of Mathematics, it was suggested that Euclid's order was not suitable for the teaching of geometry (BAAS: 1902, pp. 197-201).

As a result of these movements, Oxford and Cambridge Universities finally revised their examination requirements in 1903. For example, the examination syndicate of Cambridge University stated that: „in demonstrative geometry, Euclid's *Elements* shall be optional as a text-book, and the sequence of Euclid shall not be enforced. The Examiners will accept any proof of a proposition which they are satisfied forms part of systematic treatment of the subject“ (The Examination Syndicate: 1903, p. 179). Hence, the teaching of geometry was released from strict adherence to Euclid's *Elements* and teachers had the freedom to organise a different geometry curriculum for students. In the next

section, before examining the order of theorems in geometry textbooks, I outline the methodology for systematically analysing textbooks.

3. Methodology

3.1. Schubring's three-dimensional textbook analysis

To study school life in the past in more detail, Schubring proposes the study of the textbook authors and their *oeuvre*, i.e. textbooks, because school textbooks are often revised and therefore examining their revisions would give us various information about changing trends and ideas in teaching:

„In fact, investigating the personality of a schoolbook author and the totality of his *oeuvre* can enable us to get insight into the social meaning of school knowledge and of the relations between the author and his „clients“: the teachers. As schoolbooks are usually reedited [re-edited] several times and during this time undergo substantial changes, one can put in relation changes in the textbook and changes in the school structure, changes in the social appreciation of school knowledge and the active or passive reactions of the school book author to these changes. Thus, choosing the work of one schoolbook author as a basic unit may be one approach by which one can better analyse the reality of school life in former days.“ (Schubring: 1987, p. 41)

To analyse textbooks with their social context, Schubring proposed a three-dimensional scheme for the methodology of textbook analysis (Schubring: 1987, p. 45):

- the first dimension consists of analysing the changes within the various editions of one textbook chosen as starting point, say an algebra textbook or an arithmetic one;
- the next dimension consists in finding corresponding changes in other textbooks belonging to the same *oeuvre*, by studying those parts dealing with related conceptual fields, say geometrical algebra, trigonometry, etc.,
- the third dimension relates the changes in the textbooks to changes in the context: changes in the syllabus, ministerial decrees, didactical debates, evolution of mathematics, changes in epistemology, etc.

3.2. Methodology in this paper

The subject in this paper is the geometry textbooks by Godfrey and Siddons, representative textbook authors at that time (Howson: 1973b and 1982, Price: 1976, Griffiths: 1998). The major geometry textbooks by Godfrey and Siddons, which were published by Cambridge University Press, are as follows:

- *Elementary Geometry Practical and Theoretical* (1903)
- *Modern Geometry* (1908)
- *A Shorter Geometry* (1912)
- *Practical and Theoretical Geometry* (1920)

These textbooks are all for secondary schools, except *Modern Geometry*, which „covers the schedule of Modern Plane Geometry required for the Special Examination in Mathematics for the Ordinary B.A. Degree at Cambridge“ (Godfrey and Siddons: 1908, Preface). Of these textbooks, *Elementary Geometry* was especially significant. It was their first textbook and considered one of the most important in the history of the

teaching of geometry in England, as the comments below, beginning with one from Howson, demonstrate:

„One book, though, had a far from temporary effect – Godfrey and Siddons“ *Elementary geometry ... Elementary geometry* was an indication of the way in which the two authors saw mathematical education progressing.“ (Howson: 1973b, p. 259)

A review of *Elementary Geometry* (1903) by Langley was reprinted in the *Mathematical Gazette* Vol. LV. No. 392, a celebration of the centenary of the MA in 1971. From 1894 to 1931, a total of 1834 mathematics books were reviewed in the *Mathematical Gazette* (The MA: 1933). That the review of *Elementary Geometry* was chosen among only 7 to be reprinted in 1971 is an illustration of its hallowed status. It is clear, then, why it is necessary to study *Elementary Geometry* in particular.

Now, borrowing ideas from Schubring's model, in particular the second and third dimensions, I introduce the method of the analysis of geometry textbook by Godfrey and Siddons. First, I analyse *Elementary Geometry*. The order of theorems of this text is described by comparing it to Todhunter's *The Elements of Euclid* (1862), which was an English translation of Euclid's *Elements* as well as a textbook widely used in the 19th Century in England. Secondly, I examine the order in *A Shorter Geometry*, their second geometry textbook, and then describe the revisions from *Elementary Geometry*. Finally I discuss particularly two points: the pedagogical background of Godfrey and Siddons and the revisions recognised in *A Shorter Geometry* and their context. The order of theorems is an important issue, and therefore, I concentrate in this paper on the order of theorems in these two textbooks, i.e. the period between 1903 to 1912.

4. The order of theorems in the textbooks by Godfrey and Siddons

4.1. The order of theorems in Elementary Geometry

While the 20th century reform of the teaching of geometry was just beginning, Forsyth, who was the Sadleirian Professor at Cambridge, suggested that Godfrey and Siddons write a new geometry textbook for secondary schools. Historically, the proposal of Forsyth to Godfrey and Siddons had a significant meaning, because he was a successor of Cayley, who had strongly opposed the radical reform of the teaching of geometry until his death in 1895 (Price: 1994, p. 30). The following extract is from Siddons' memoirs in 1952:

„While Godfrey and I [Siddons] were at the British Association meeting at Belfast in September 1902 we received word from Forsyth that the Syndicate of the Cambridge University Press would invite us to write an Elementary Geometry and at the same time he advised us to ignore Euclid's order.“ (Siddons: 1952, p. 6)

It is this reform of Euclid's order that is particularly important because it means that authors were released from following a given structure and had the freedom to write a textbook as they felt appropriate. In 1903 *Elementary Geometry* was published and was immediately a great success: „13000 [copies] of the

complete book and 9000 of Volume I in the first ten months. A further 8000 of the complete book and 3500 of Volume I followed in the next twelve months.“ (Siddons: 1952, p. 9).

Elementary Geometry consists of two parts: Part I. 'Experimental Geometry' and Part II. 'Theoretical Geometry'. 'Experimental geometry' mainly contains experimental tasks such as measurement or drawing dealing with both plane and solid figures. The purposes of this stage are a) to make students familiar with geometrical instruments and figures and b) to lead students to discover geometrical facts (a detailed examination of the roles of experimental tasks in *Elementary Geometry* can be found in Fujita: 2001). Nevertheless, 'Theoretical geometry' consisted of propositions from Euclid's *Elements* with four continuous books: Book I: Straight lines, Book II: Areas, Book III: Circles, and Book IV Similarity. It should be noted that some experimental tasks also appeared in 'Theoretical Geometry'. Also, the use of squared paper was implemented in this textbook. The activities required in exercises were plotting points, drawing figures and similar figures, finding areas and ratios of figures, and plotting loci on squared paper.

The question considered in this paper is how the order of theorems was rearranged from the order in Euclid's *Elements*. In what follows, I particularly focus on the order of Books I&II in *Elementary Geometry*, whose content corresponds to Books I&II in Euclid's *Elements*. Tables 4.1.1. and 4.1.2. summarise the order of theorems in Books I&II in *Elementary Geometry* (in the tables, 'Euclid Book I (II)', 'Proposition in Euclid' are abbreviated 'Euc. I. (II.)' and 'Prop.' respectively)

Table 4.1.1: The order of theorems in Book I *Elementary Geometry*

Sections in <i>Elementary Geometry</i>	Euclid's <i>Elements</i> (Todhunter's edition)
Angles at a point	Euc. I. Prop. 13, 14, 15
Parallel straight lines	Euc. I. Prop. 27, 28, 29, 30
Angles of a triangle, a polygon	Euc. I. Prop. 32, 16, 17
Classification of triangles	No Euclid propositions
Congruent triangles	Euc. I. Prop. 4, 26, 5, 6, 8
Constructions	Euc. I. 22, 23, 9, 10, 11, 12
Inequalities	Euc. I. Prop. 18, 19, 20, 24, 25
Parallelograms	Euc. Prop. 34, 31
Subdivision of a straight line	No Euclid propositions (1 theorem and 1 construction)
Loci	No Euclid propositions (2 theorems)
Co-ordinates	No Euclid propositions

Table 4.1.2: The order of theorems in Book II *Elementary Geometry*

Sections in <i>Elementary Geometry</i>	Euclid's <i>Elements</i> (Todhunter's edition)
Area of parallelogram	Euc. I. 35, 36
Area of triangle	Euc. I. 37, 38
Area of polygon	Euc. I. 39, 40, 41
The theorem of Pythagoras	Euc. I. 47, 48
Illustrations of	$(a+b)k$ $(a+b)k=ak+bk$

algebraical identities by means of geometrical figures	$(a+b)(c+d)=ac+bc+ad+bd$ $(a+b)^2=a^2+b^2+2ab$ $(a-b)^2=a^2+b^2-2ab$ $a^2-b^2=(a+b)(a-b)$ (Euc. II. 1, 4, 7, 5, 6)
Extension of Pythagoras' theorem	Euc. II. 12, 13

In the next section, I describe the order of theorems in *A Shorter Geometry*, the second geometry textbook by Godfrey and Siddons.

4.2. The order of theorems in A Shorter Geometry

In 1909, the Board of Education, which was founded in 1899 and which supervised the education system in England until 1944, published *Circular 711 on the Teaching of Geometry and Graphic Algebra in Secondary School*. This circular proposed three stages in geometry, and had considerable influence on teaching practices and design of textbooks at that time (Price: 1994, pp. 94-5). In accordance with the recommendations of the *Circular 711*, Godfrey and Siddons published their second geometry textbook *A Shorter Geometry* in 1912. In this section, the focus will be the orders of theorems in this textbook. In the following, I will particularly examine the content which corresponds to that of Book I&II in Euclid's *Elements*.

A Shorter Geometry consisted of three parts: „First Stage“: introductory course, „Second Stage“: discovery of the fundamental geometrical facts by experiment and intuition, and „Third Stage“: deductive development of theorems (Godfrey and Siddons: 1912, Preface). The third stage consisted, like *Elementary Geometry*, of four continuous books: Book I: Straight lines, Book II: Areas, Book III: Circles, and Book IV Similarity. The first fifteen theorems in Book I in *Elementary Geometry* appeared in the second stage, although theorems were treated as „fundamental geometrical facts“ and the proofs of these facts were not required. Book I in *A Shorter Geometry* began with geometrical constructions, and then theorems about parallelograms were introduced. In Book II in *A Shorter Geometry*, theorems about areas and Pythagoras theorem and its applications were studied. The contents in the second stage and Books I&II in *A Shorter Geometry* almost correspond to those in Books I&II in *Elementary Geometry*. Tables 4.2.1., 4.2.2., and 4.2.3. summarise the order of theorems in *A Shorter Geometry*.

Table 4.2.1: The order of theorems in *A Shorter Geometry* (The second stage)

Sections in <i>A Shorter Geometry</i>	Euclid's <i>Elements</i> (Todhunter's edition)
Angles at a point	Euc. I. Prop. 13, 14, 15
Parallel straight lines	Euc. I. Prop. 27, 28, 29
Angles of a triangle, a polygon	Euc. I. Prop. 32, 16, 17
Congruent triangles	Euc. I. Prop. 4, 26, 8, 5, 6

Table 4.2.2: The order of theorems in *A Shorter Geometry* (The third Stage Book I)

Sections in <i>A Shorter Geometry</i>	Euclid's <i>Elements</i> (Todhunter's edition)
Constructions	Euc. I. 23, 9, 10, 11, 12

Continuous change of a figure	No Euclid propositions
Parallelograms	Euc. Prop. 34, 31
Subdivision of a straight line	No Euclid propositions (1 theorem and 1 construction)
Loci	No Euclid propositions (2 theorems)
Symmetry	No Euclid propositions

Table 4.2.3: The order of theorems in *A Shorter Geometry* (The third Stage Book II)

Sections in <i>A Shorter Geometry</i>	Euclid's <i>Elements</i> (Todhunter's edition)
Area of parallelogram	Euc. I. 35, 36
Area of triangle	Euc. I. 37, 38
Area of polygon	Euc. I. 39, 40, 41
The theorem of Pythagoras	Euc. I. 47, 48
Extension of Pythagoras' theorem	Euc. II. 12, 13
Illustrations of algebraical identities by means of geometrical figures	$(a+b)k=ak+bk$ $(a+b)(c+d)=ac+bc+ad+bd$ $(a+b)^2=a^2+b^2+2ab$ $(a-b)^2=a^2+b^2-2ab$ $a^2-b^2=(a+b)(a-b)$ (Euc. II. 1, 4, 7, 5, 6)

In the next section, I describe the order in *Elementary Geometry* and *A Shorter Geometry* to show the common features and differences in these textbooks.

4.3. The orders in Elementary Geometry and A Shorter Geometry

First, let us consider the order in *Elementary Geometry*. From tables 4.1.1. and 4.1.2., it can be seen that a) the contents of Book I in Euclid's *Elements* was divided into two books in *Elementary Geometry*, b) theorems were arranged from angles (Euc. I. 13~15) and parallel lines (Euc. I. 27~30), angles of triangles (Euc. 32, 15, 16) to triangles (Euc. I. 4, 26, 5, 6, 8, 18~20, 24~25) and parallelograms (Euc. I. 34, 31), and then from areas of polygons (Euc. I. 35~41) to Pythagoras theorem (Euc. I. 47~48) and its applications (Euc. II. 12~13) and c) constructions (Euc. I. 23, 9, 10, 11, 12) were located after some theorems.

As we can see, the order was organised differently from Euclid (the pedagogical background will be considered in section 5). First of all, it can be said that the theorems were arranged beginning with simple theorems progressing to more complex theorems. Therefore, theorems of angles and parallel lines were studied first. The sum of the angles in a triangle (Euc. I. 32) was located in the early part in this text, because this theorem was proved by using the properties of parallel lines (Euc. I. 27, 28). In relation to this theorem, Euc. I. 15 and 16, which were about the properties of exterior and interior angles in triangles, were located after Euc. I. 32. Then, the theorems of triangles, the conditions of the congruent triangles, appeared. The constructions were located after the study of triangles, because they were mainly proved by using Euc. I. 8. After triangles and constructions, the theorems of parallelograms were introduced. In the end of Book I in *Elementary Geometry*, contents which were not

included in Euclid's *Elements* were introduced: theorems of loci, sub-division of straight lines, and co-ordinates. The earlier theorems studied in Book I were applied to these new sections, which showed that geometry was not restricted to that contained only Euclid's *Elements*. The co-ordinates were introduced to make students familiar with the concepts of geometry (mainly areas). That is to say, practical tasks were included in the deductive stage in *Elementary Geometry*. However, as I have mentioned above, the activities with co-ordinates were practical, and the methods of co-ordinate geometry were not used in the proofs of theorems in these textbooks. The use of co-ordinates in the Godfrey and Siddons texts did not affect to the rearrangement of Euclid's order of propositions, or their proofs. In Book II, the theorems concerning with areas (the latter part of Euc. I and Euc. II.) were arranged together. The propositions in Euc. II 1, 4, 7, 5, 6 were treated algebraically even though they were proved without algebraic symbols in Euclid's *Elements* (Euc. II. Prop. 7 states $(a+b)^2+a^2=2(a+b)a+b^2$. If $a+b=c$, then $c^2+a^2=2ca+(c-a)^2$, and we get $(c-a)^2=c^2+a^2-2ca$. Euc. II. Prop. 5 states $ab+((a+b)/2-b)^2=((a+b)/2)^2$. If $(a+b)/2=c$, then $a(2c-a)+(a-c)^2=c^2$. Again, if $a-c=d$, then $(d+c)(2c-d-c)+d^2=c^2$, and we get $(c+d)(c-d)=c^2-d^2$. Similarly, Euc. II. Prop. 6 can be identified $(c+a)(c-a)=c^2-a^2$, see in Joyce; 1996). It should be noted that hypothetical constructions were introduced in *Elementary Geometry*. For example, Euc. I. Prop. 5 (in the triangle ABC, $AB=AC \rightarrow \angle ABC=\angle ACB$) was proved by drawing an angle bisector from a vertex to the base line of the triangle, which was avoided in Euclid's *Elements*, because this was a cause of logical circularity order (the angle bisector was studied practically in Part I „Experimental Geometry“ in *Elementary Geometry*). In other words, the order in *Elementary Geometry* sacrificed the absolute rigour of Euclid's.

One might say that the rearrangement of theorems in *Elementary Geometry* was a rather minor change from the order of Euclid, because some theorems were still kept in the order they appear in Euclid. However, when we reflect that only Euclid's order was allowed before 1903, and consider the introduction of hypothetical constructions, practical works or algebra, this reform marked a radical change in the history of teaching of geometry at that time.

As to the order in *A Shorter Geometry*, from the tables 4.2.1., 4.2.2., and 4.2.3., it can be said that the order of theorems in this textbook was almost the same as in that in *Elementary Geometry*. In summary, the following features can be recognised in the order in these two textbooks:

- from simple theorems to more complex theorems:
- from theorems to geometrical constructions:
- alliances of related theorems, e.g. theorems on congruent triangles:
- introduction of hypothetical constructions.

A few revisions from *Elementary Geometry* can be recognised in *A Shorter Geometry*, e.g. the omission of some theorems (the theorems of inequality (Euc. I. Prop. 18., 19., 20., 24., 25. For example, Euc. I. Prop. 18 states that in any triangle the greater side subtends the greater angle) were omitted, but they appeared in the Appendix

of this textbook (pp. 257-77) with their proofs). In addition to the revisions, it should be noted that Godfrey and Siddons wrote a flexible commentary on the order in *A Shorter Geometry*, i.e. that teachers could interchange the order of Book II and III:

„Many teachers (ourselves included) like to take Book III before Book II. The difficulty in arranging a text-book on these lines arises from Pythagoras' theorem, which is needed for numerical exercises in Book III. But this theorem is not referred to in the theorems in Book III: and as the fact involved must be familiar to pupils from their experimental work, there is no good reason why teachers who use this book should not interchange the order of Books II and III.“ (Godfrey and Siddons: 1912, p. vii-viii)

Overall, it can be concluded that Godfrey and Siddons did not change radically the orders of theorems between two textbooks, even though flexibility of order can be recognised in *A Shorter Geometry*. In the next sections, I first discuss why Godfrey and Siddons arranged theorems the way they did in their textbooks, and then secondly, I consider the flexibility of order in *A Shorter Geometry*.

5. Pedagogical background

In examining the pedagogical ideas of Godfrey and Siddons, the book *The Teaching of Elementary Mathematics* (1931) is particularly important. Even though this book was published in 1931, the first part was written by Godfrey in around 1911 (Godfrey and Siddons: 1931, Preface). Therefore, it is one of the most important documents for understanding their pedagogy around 1910. In this book, Godfrey stated that there would be three possible orders in the teaching of mathematics: historical, psychological, and scientific:

„There are three possible orders in which a subject can be presented: the historical order: the psychological order: and the scientific order. The historical order, the order of discovery, is the most definite of the three, and is usually scrappy and unsymmetrical. The psychological order, the order indicated by psychology as the most suitable, this we cannot know in our present state of ignorance: but it is probably more akin to the historical than the scientific. By the scientific order is understood that order which is adopted by experts for the final statement of a worked-out subject.“ (Godfrey and Siddons: 1931, p. 16)

From the extract above, it can be inferred that Godfrey preferred psychological or historical orders to the scientific order for elementary mathematics teaching, because scientific orders would be „always an afterthought: the subject did not grow in that shape“ (Godfrey and Siddons: 1931, p. 17), and that is why, briefly, they *did* change the order from Euclid's *Elements*.

Although Godfrey did not describe explicitly the psychological order, it was recognised that he regarded Herbartian psychology as an important idea (Godfrey and Siddons: 1931, p. 13). Indeed, the pedagogy of Pestalozzi, Herbart and Froebel, which focused on children's perception and interest, had already begun to have influences on teaching practices in mathematics at that time (Brock and Price: 1980). Godfrey understood

the basic concept as follows: it is „in not mere knowledge, but a mass of knowledge that hangs together and grows together: a mass of correlated ideas“ (Godfrey and Siddons: 1931, p. 13). To achieve this, he considered that it would be essential to organise the teaching of mathematics in a way which would make it possible students to apperceive knowledge. It can be said that this would be consistent with the features of the order in their textbooks. As we have seen above, the theorems in their textbooks were arranged progressively from theorems about lines and angles to theorems about polygons, i.e. from simple facts to more complex facts. Furthermore, because of the proximity of related theorems, the relationships between these theorems would be closer than in Euclid's *Elements*, i.e. students would appreciate the relationships between theorems more than with the *Elements*. In fact, Godfrey stated (presumably he noticed this through his own teaching experiences at secondary schools) that „the mental process by which the old-established ideas take the new-comers by the hand is a perfectly familiar process“ (Godfrey and Siddons: 1931, p. 13). Even though I have not given the order of Book III and IV in *Elementary Geometry* and *A Shorter Geometry*, the theorems in these books were also rearranged from this point of view.

As to the location of the constructions, they always came first in Euclid's *Elements*, because they provided the proof of the existence of geometrical figures, e.g. an angle bisector, perpendicular lines, or parallel lines. On the other hand, they were located after some theorems in the textbooks by Godfrey and Siddons. This idea had appeared in the MA report on Geometry in 1902. Godfrey also considered that students would not appreciate the significance of Euclid's constructions unless they were familiar with geometrical instruments and geometrical figures (Godfrey: 1908, p. 254). It can be concluded that these pedagogical ideas underpinned the arrangement of the order of theorems in Godfrey and Siddons' textbooks.

6. Revision of textbooks and teaching

As Schubring suggests, textbooks by an author and their changes provide us with further information about teaching practices in the past. In this section, I consider why Godfrey and Siddons advocated flexibility of order, which was one of the revisions in *A Shorter Geometry*.

As a result of the revisions of examination requirements in 1903, teachers had been given a freedom to organise the teaching of geometry since 1903. Thus, as we have seen from the examples of Godfrey and Siddons, reformers started to rearrange Euclid's theorems in what they thought were more appropriate order for elementary geometry teaching. Furthermore, various new ideas such as experimental tasks, or the introduction of algebra were being achieved by various reformers (the overall context of the teaching of mathematics has been well examined by Price in *Mathematics for the Multitude?* in 1994). The abandonment of Euclid's *Elements* was enabled by *Circular 711* from the Board of Education in 1909. In particular, this circular specified the fundamental theorems, whose formal proofs would not be necessary for the teaching of geometry, as follows: „The second

stage [in geometry], then consists in the establishment of the fundamental propositions, viz., Euclid I., 13-15, 27-29, 32: with 4, 8, 26.“ (The Board of Education: 1909, p. 5). This extract also tells us that the Board of Education agreed with the new order of theorems in geometry.

At the same time, it caused „chaos“ among teachers as a variety of textbooks with various orders of theorems made them confused (Godfrey: 1920, p. 20). Such confused teachers, who were not enthusiastic like Godfrey and Siddons, preferred a „fixed standard order“ in geometry. Furthermore, some people still felt that Euclid's *Elements* would be the best text for the teaching of geometry, and it led to a debate among teachers (Anonymous: 1912). Godfrey and Siddons might have been afraid that these situations would lead to teaching only for examinations, like before 1903, or that they would lose their freedom of teaching because of the restrictions of examinations. In fact, Godfrey and Siddons stated that they did not want to emphasise the issue of the order in the teaching of geometry too much. For example, Godfrey expressed doubts about a fixed order in geometry (Godfrey: 1907, p. 101). He also stated that the difficulty caused by a variety of orders could be solved by „plenty of riders and a reasonable amount of drawing“ (Godfrey: 1908, p. 101). Siddons also wrote an unfavourable comment on „a standard sequence“ in geometry:

„With regard to the first question [Is a „standard sequence“ desirable?], I would say that, if a teacher wants a „standard sequence“, in all probability he is a bad teacher and his aim in teaching is not to develop the child but to get him through examinations.“ (Anonymous: 1912, p. 180)

That is why, whereas they provided teachers with an order, they expressed the flexibility of orders in their textbooks, i.e. teachers could organise the teaching of geometry with a liberal spirit. According to Price, the issue of the order of theorems was argued in the 1920s again (Price: 1994, pp. 123-4). The teaching committee of the MA published a report *The Teaching of Geometry in Schools* in 1923, and it concluded that „the Committee [of the MA] is of opinion that to try to establish a fixed sequence is not desirable ...“ (The MA: 1923, p. 59). Probably this conclusion reflected in part Godfrey's opinions, because he was a member of the committee, and it can be said that he considered liberalism as important in the teaching of mathematics.

This flexibility of order also provides us with a hypothesis that Godfrey and Siddons still wanted to develop further their educational ideas. In fact, Godfrey stated that „unless all the best teachers are free to try these and other experiments, we shall not discover the best way“ (in Anonymous: 1912, p. 177). It would be possible to investigate further developments of Godfrey and Siddons by analysing *Practical and Theoretical Geometry* (1920). However, for this analysis, another study is needed.

7. Conclusion

In this paper, I have analysed geometry textbooks by Godfrey and Siddons to address the issue of the order of

theorems in the early 20th Century. As we have seen above, the orders in textbooks were arranged from simple theorems to complex theorems, sacrificing the rigour of Euclid's order. This rearrangement of theorems was a radical change at that time, and was being undertaken with various pedagogical perspectives, and with the liberal spirit of reformers. Godfrey and Siddons' pedagogy and attitude to mathematics education will be worth considering even in today's teaching. Also, the revisions of orders in the textbooks, in particular the flexibility of order in *A Shorter Geometry*, provides us with an insight into the teaching practices at that time. I have also hypothesised that Godfrey and Siddons still wanted to develop the teaching of geometry after 1912, and to address this issue is an essential task in the future study of the teaching and learning of geometry.

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References

- Anonymous. (1912). 'The question of sequence in geometry'. *School World*. May, pp. 173-82.
- Board of Education. (1909). *Circular 711. Teaching of Geometry and Graphic Algebra in Secondary Schools*. London: HMSO.
- British Association for the Advancement of Science. (1902). 'Report of the British Association Committee on the Teaching of Mathematics'. *Mathematical Association. Mathematical Gazette*. Vol. II. No. 35, pp. 197-201.
- Brock, W. H. and Price, M. H. (1980). 'Squared paper in the nineteenth century: Instrument of science and engineering, and symbol of perform in Mathematical Education'. *Educational Studies in Mathematics* 11, pp. 365-381.
- Fujita, H. (ed.). (1996a). *New Mathematics 2*. Tokyo: Toyokan publisher (in Japanese).
- Fujita, H. (ed.). (1996b). *New Mathematics 3*. Tokyo: Toyokan publisher (in Japanese).
- Fujita, T. (2001). 'The Study of *Elementary Geometry* (1903) by Godfrey and Siddons (1): Roles of Experimental Tasks in the Teaching of Geometry'. *Hiroshima Journal of Mathematics Education* Vol. 9, pp. 11-9.
- Godfrey, C. (1907). 'Is there need of a recognized sequence in geometry?'. *Mathematical Association. Mathematical Gazette*. Vol. IV., pp. 100-1.
- Godfrey, C. (1920). 'Geometry Teaching: The next step'. *Mathematical Association. Mathematical Gazette*. Vol. X., pp. 20-24.
- Godfrey, C., Siddons, A. W. *et al.* (1902). 'The teaching of mathematics in public schools'. 16 January. *Nature*, pp. 258-9.
- Godfrey, C., Siddons, A. W. (1903). *Elementary Geometry practical and theoretical*. Cambridge: Cambridge at the University Press.
- Godfrey, C., Siddons, A. W. (1908). *Modern Geometry*. Cambridge: Cambridge at the University Press.
- Godfrey, C., Siddons, A. W. (1912). *A Shorter Geometry*. Cambridge: Cambridge at the University Press.
- Godfrey, C., Siddons, A. W. (1920). *Practical and Theoretical Geometry*. Cambridge: Cambridge at the University Press.
- Godfrey, C., Siddons, A. W. (1931). *The teaching of Elementary Mathematics*. Cambridge: Cambridge University Press.
- Griffiths, H. B. (1998). 'The British experience'. Mammana, C. and Villani, V. (eds.). *Perspectives on the Teaching of Geometry for the 21st Century, An ICMI Study*. Netherlands: Kluwer Academic Publishers, pp.194-204.
- Howson, A. G. (1973a). 'Charles Godfrey (1873-1924) and the reform of mathematics education'. *Educational Studies in Mathematics*, 5, pp.157-80.
- Howson, A. G. (1973b). 'Milestone or Millstone?'. *Mathematical Association. Mathematical Gazette*. Vol. LVII, pp. 258-66.
- Howson, A. G. (1982). *A history of mathematics education in England*. Cambridge: Cambridge University Press.
- Jackson, G. B. (1924). *The Teaching of Geometry in Secondary Schools*. Manchester University, MED Thesis.
- Joyce, D. E. (1996). 'Euclid's Elements Book II'. <http://aleph0.clarku.edu/~djoyce/java/elements/bookII/bookII.html>.
- Langley, E. M. (1903). 'Review-Elementary Geometry. and A school geometry.' *Mathematical Association. Mathematical Gazette*. Vol. II., p. 369, reprinted Vol. LV. (1971), pp. 239-40.
- Mathematical Association. (1902a). 'the Annual Meeting of the Mathematical Association'. *Mathematical Association. Mathematical Gazette*. Vol. II. No. 31, pp. 129-143.
- Mathematical Association. (1902b). 'Report of the M.A. Committee on Geometry'. *Mathematical Association. Mathematical Gazette*. Vol. II. No 33, pp. 168-172.
- Mathematical Association. (1923). *The Teaching of Geometry in Schools*. London: G. Bell and Sons, LTD.
- Mathematical Association. (1933). *Index to the Mathematical Gazette Volumes I-XV April 1894 to December 1931 Nos. 1-216*. London: G. Bell and Sons, LTD.
- Perry, J. (1902). 'The teaching of mathematics'. *Educational Review*. Vol. XXIII, pp. 158-81.
- Price, M. H. (1976). 'Understanding Mathematics a perennial problem? part 3'. *Mathematics in School* 5., pp. 18-20.
- Price, M. H. (1994). *Mathematics for the Multitude? – A history of the Mathematical Association*. Leicester: The Mathematical Association.
- Price, M. H. (2001). 'England'. Louise S. Grinstein and Sally I. Lipsey (eds.). *Encyclopaedia of mathematics education*. New York: Routledge/Falmer, pp. 216-8.
- Quadling, D. (1996). 'A century of textbooks'. *Mathematical Association. Mathematical Gazette*. No. 487., pp. 119-26.
- G. Schubring. (1987). 'On the Methodology of Analysing Historical Textbooks: Lacroix as Textbook Author'. *For the Learning of Mathematics*. 7. No. 3, Canada: FLM Publishing Association.

Siddons, A. W. (1936). 'Progress'. Mathematical Association. *Mathematical Gazette*. Vol. XX, pp. 7-26.

Siddons, A. W. (1952). *Godfrey and Siddons*. Cambridge: Cambridge University Press.

Todhunter, I. (1862). *Euclid for the use of schools and colleges; comprising the first six books and portions of the eleventh and twelfth books; with notes, an appendix, and exercises*. London: Macmillan and Co.

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