Landscapes of Investigation

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Abstract: According to many observations, traditional mathematics education falls within the exercise paradigm. This paradigm is contrasted with landscapes of investigation serving as invitations for students to be involved in processes of exploration and explanation. The distinction between the exercise paradigm and landscapes of investigation is combined with a distinction between three different types of reference which might provide mathematical concepts and classroom activities with meaning: references to mathematics; references to a semi-reality; and references to a real-life situation. The six possible learning milieus are illustrated by examples.

Moving away from the exercise paradigm and in the direction of landscapes of investigation may help to abandon the authorities of the traditional mathematics classroom and to make students the acting subjects in their learning processes. Moving away from reference to pure mathematics and in the direction of real life references may help to provide resources for reflection on mathematics and its applications. My hope is that finding a route among the different milieus of learning may provide new resources for making the students both acting and reflecting and in this way providing mathematics education with a critical dimension.

Kurzreferat: Der traditionelle Mathematikunterricht fällt vielen Beobachtungen nach in das Übungsparadigma. Dieses Paradigma wird den "landscapes of investigation" gegenüber gestellt, einer Einladung an Schüler, sich auf den Prozess des Entdeckens und der Erklärung einzulassen. Übungsparadigma und "landscapes of investigation" werden zusammen mit dem Unterschied zwischen drei verschiedenen Formen der Bezugnahme betrachtet, die mathematische Begriffen und Unterrichtsaktivitäten mit Bedeutung erfüllen sollen: Bezug zur Mathematik; Bezug zur Semi-Realität; Bezug zu Situationen des realen Lebens. Die sechs möglichen Lernmilieus werden mit Beispielen illustriert. Weg vom Übungsparadigma und hin zu den "landscapes of investigation" kann dazu beitragen, die Autoritäten des traditionellen Mathematikunterrichts zu verlassen und die Schüler zu den aktiv Handelnden in ihrem Lernprozess zu machen. Weg vom Bezug zur reinen Mathematik und hin zum realen Lebensbezug kann helfen das Nachdenken über Mathematik und ihre Anwendungen zu fördern. Meine Hoffnung ist, dass ein Weg zwischen den verschiedenen Lernumgebungen neue Möglichkeiten schafft, Schüler zum aktiven Handeln und zum Nachdenken anzuregen und damit den Mathematikunterricht um eine kritische Dimension zu bereichern.

ZDM-Classification: C30

In his observations in English classrooms, Cotton (1998) has noticed that a mathematics lesson is divided into two parts, first, the teacher presents some mathematical ideas and techniques, then the students work with selected exercises. He has also noticed that there are variations of the same pattern, reaching from a full-lesson teacher presentation to a full-lesson student occupation with exercises. According to this and many other observations, traditional mathematics education falls within the *exercise paradigm*. Most often, the mathematical textbook represents a 'given' for the classroom practice. Exercises are formulated by an authority external to the classroom. This means that the justification of the relevance of the exercises is not part of the mathematics lesson itself. Furthermore, a central premise of the exercise paradigm is that one and only one answer is correct.

The exercise paradigm can be contrasted with an *investigative approach*. Such an approach can take many forms, one example being project work, as described for primary and secondary school education in Nielsen, Patronis and Skovsmose (1999) and in Skovsmose (1994), and for university studies in Vithal, Christiansen and Skovsmose (1995). In general, project work is located in a 'landscape' which provides resources for making investigations. Project work represents a learning milieu, different from the exercise paradigm.

My interest in the investigative approach is related to critical mathematics education, which can be characterised in terms of different concerns,[1] one of which is the development of *mathemacy*, seen as a competence similar to literacy, as characterised by Freire. Mathemacy refers not only to mathematical skills, but also to a competence in interpreting and acting in a social and political situation structured by mathematics. Critical mathematics education includes a concern for developing mathematics education in support of democracy, implying that the micro-society of the mathematics classroom must also show aspects of democracy. Critical mathematics education emphasises that mathematics as such is not simply a subject to be taught and learnt (no matter whether the processes of learning are organised according to a constructivist or a sociocultural approach). Mathematics itself is a topic which needs to be reflected upon, as mathematics is part of our technology-based culture, and it exercises many functions, which may best be characterised by a slight reformulation of 'Kranzberg's First Law': What mathematics is doing is neither good nor bad, nor is it neutral (see Kranzberg 1997). D'Ambrosio (1994) has used as more harsh formulation emphasising that mathematics makes part of our technological, military, economic and political structures, and as such it becomes a resource for wonders as well as for horrors.[2] Making a critique of mathematics as part of mathematics education is a concern of critical mathematics education. Such concerns seem better taken care of outside the exercise paradigm.

The following presentation is partly based on my work on project-based mathematics education, and it is related to my work with teachers, with whom I have discussed these ideas - teachers working in very different political, economic and cultural contexts in Colombia, South Africa, Brazil, England and Denmark. I always start with an example.

An example

A landscape which can support investigative work, I call a *landscape of investigation*.[3] We take a look at the good

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1 See Skovsmose and Nielsen (1996).

2 See also D'Ambrosio (1998) and Skovsmose (1998a, 1999b, 2000).

old table of numbers, which has certainly decorated the walls of many mathematics classrooms and served as basis for a variety of exercises. We concentrate on a rectangle drawn on the table. If the numbers in the corners of the rectangle are labelled *a*, *b*, *c* and *d*, it is possible to calculate the value of *F* determined by

$$F = ac - bd$$

The rectangle can then be translated to another position, and the value of $F = ac - bd$ can be calculated again.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	...						

Figure 1: The good old table of numbers.

For instance (see Figure 1), we observe that 22·34 - 24·32 = -20, and that 37·49 - 39·47 = -20. Let us try to translate the rectangle to a different position and again calculate the value of *F*. By the way, what will happen if we rotate the rectangle 90^0 and make the same calculation? Well, ... What is going to happen, if we choose a bigger rectangle and make a similar translation? What will now be the value of $F = ac - bd$? How does the value of *F* depend on the size of the rectangle?

Naturally, it is possible to investigate translations of other figures. What will happen if we calculate the values $F = ac - bd$, and *a*, *b*, *c*, and *d* refer to the numbers determined by the corners of the shapes shown in Figure 2? Which of these figures can be ‘translated’ within the table of numbers without the value of *F* being changed?

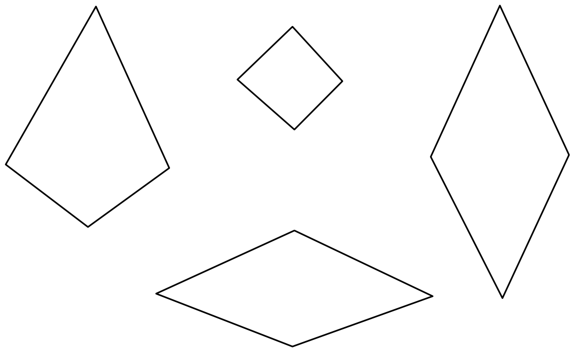

Figure 2: Other figures to be translated.

Why not investigate a function different from *F*? For instance, what will happen if we permute the operations ‘subtraction’ and ‘multiplication’ and instead of $F = ac - bd$ calculate:

$$G = (a - c)(b - d)$$

(*a*, *b*, *c*, and *d* still refer to the corners of a rectangle)? Would *G* be constant under translation? What about the other figures shown at Figure 2? Do other functions exist which are rectangle-translatable (meaning that the value of the function is kept constant during a translation)? Yes, of course a function *H* defined as $H = 0a + 0b + 0c + 0d$. But do more ‘interesting’ rectangle-translatable functions exist? If we succeed in finding such a function, would it, then, also be rhombus-translatable? Would, in fact, any rectangle-translatable function be rhombus-translatable? In more general terms: what functions make which figures translatable?

What if we consider negative numbers? Thus, the number table from Figure 1 could be extended adding numbers to the left and to the right of each line, so that we have to do with number lines placed on top of each other. We could then consider translations which bring the figures into areas with negative numbers. Incidentally, what would happen if the table was set up as shown at the Figure 3?

It must also be possible to carry out the calculation in a different number base. Would the quality of ‘translatability’ depend on which number base we are considering?

Naturally, we need not concentrate on configurations of numbers determined by the corners of a figure with four corners. We could consider any configuration of numbers, $a_1, \ldots, a_n$, and any function, $F = F(a_1, \ldots, a_n)$. The question would then be: what functions defined on a configuration of numbers are constant with respect to translation of the configuration? And why not consider rotation as well? Or any other movement of the figure? Furthermore, up to now we have concentrated on a particular property of the function *F*, being constant or not, but we could observe many other properties of the function *F*. This leads to the question: What functions

[3] The following example is inspired by Ole Einar Torkildsen’s lecture at the NOMUS-Conference in Aalborg (Denmark) in 1996.

defined on a configuration of numbers exhibit ‘nice’ properties under translation?

1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30	31	32	33	34	35
36	37	38	39	40	41	42
43	44	45	...			

Figure 3: A different set up for the table of numbers.

What if ...?

We imagine that this example has occupied some students and a teacher for a while. We have been observing their conversation. The teacher has asked “What if ...?”, and later we hear again his or her “What if ....?” The students might be surprised by some of the mathematical properties indicated by the questions. Mumbling is heard all around. Later it becomes possible to hear students’ voices more clearly “What if ...? ... “Yeah, what if ....?” Maybe the teacher asks “Why is it that ...?”, which leads to more mumbling and, maybe, longer periods of silence. Later on some of the students voices can be heard “Yes, why is it that...?”

A landscape of investigation invites students to formulate questions and to look for explanations. The invitation is symbolised by the teacher’s “What if ...?” The students’ acceptance of the invitation is symbolised by their “Yes, what if ...?” In this way the students become involved in a process of *exploration*. The teacher’s “Why is it that ...?” provides a challenge, and the students’ “Yes, why is it that ...?” illustrates that they are facing the challenge and that they are searching for *explanations*. When the students in this way take over the process of exploration and explanation, the landscape of investigation comes to constitute a new learning milieu. In a landscape of investigation the students are in charge.

Does the example about the translation of figures then, in fact, function as a landscape of investigation? Maybe, maybe not, because a landscape only function as a landscape of investigation if the students *do* accept the invitation. Serving as a landscape of investigation is a relational property. Acceptance of the invitation depends on the nature of the invitation (the possibility of exploring and explaining pure mathematical properties of a number table might not appear so attractive to many students). It depends on the teacher (an invitation can be presented in many ways, and to some students an invitation from a teacher might sound like a command). And it depends certainly on the students (they might have other priorities for the time being). What might serve perfectly well as a landscape of investigation for one group of students in one particular situation might not provide any invitation to another group of students. The question whether a certain landscape might support an investigative approach or not is an empirical question which has to be answered through an experimental educational practice by the teacher and students involved.

Milieus of learning

Classroom practices based on landscapes of investigation contrast strikingly with the exercise paradigm. The distinction between the two can be combined with a different distinction which has to do with the ‘references’ which might provide the mathematical concepts and the classroom activities with some meaning to the children.

In philosophy, many attempts have been made to clarify the notion of meaning in terms of reference. Such attempts have inspired mathematics educators to discuss meaning in terms of possible references of mathematical concepts. For instance, the notion of fraction can be introduced by referring to the division of pizzas, while, later, the meaning of ‘fraction’ can be developed further by introducing different sets of references. However, meaning can also be seen, first of all, as a characteristic of actions, and not just as a characteristic of concepts. In my interpretation, references also include motives for actions; in other words, it includes the context for locating an aim of an action (performed by the student in a mathematics classroom). When, in what follows, I talk about different types of references, I will generally be alluding to meaning-production in mathematics education.[4]

Different types of reference are possible. First, mathematical questions and activities can refer to mathematics and to mathematics only. Second, it is possible to refer to a semi-reality - not a reality that we actually observe, but a reality constructed by, for instance, an author of a mathematical textbook.[5] Finally, students and teachers can work with tasks referring to real-life situations.

Combining the distinction between the three types of reference and the distinction between two paradigms of classroom practices, one gets a matrix showing six different types of *learning milieus* (Figure 4). What I mean by learning milieu, I shall try to clarify further by commenting on the different types of milieus suggested by the matrix.

4 For an analysis of meaning production in mathematics education, see Lins (2001).

5 Christiansen (1997) refers to a ‘virtual reality’ as a reality which is established by the mathematical exercise itself. I use the notion ‘semi-reality’ in a similar way.

	Tradition of exercises	Landscapes of investigation
References to pure mathematics	(1)	(2)
References to a semi-reality	(3)	(4)
Real-life references	(5)	(6)

Figure 4: Milieus of learning.

Type (1) is positioned in a context of 'pure mathematics' as well as in the tradition of exercises. This learning milieu is dominated by exercises, which can be of the form:

(27a - 14b) + (23a + 5 b) - 11a =

(16·25) - (18·23) =

(32·41) - (34·39) =

Type (2) is characterised as a landscape of investigation located in numbers and geometric figures. The introductory example about the translation of geometric figures in a number table illustrates this type of milieu.

The type (3) milieu is located in the paradigm of exercises with references to a semi-reality. The nature of such a semi-reality can be illustrated by the following example:

> Shopkeeper A sells dates for 85p per kilogram. B sells them at 1.2 kg for £1. (a) Which shop is cheaper? (b) What is the difference between the prices charged by the two shopkeepers for 15 kg of dates?

Certainly there is talk about dates, shops and prices. But I do not suppose that the person who constructed this exercise made any empirical investigation of how dates are sold or interviewed a person in order to find out under what circumstances it would be relevant to buy 15 kg of dates. The situation is artificial. The exercise is located in a semi-reality. This example is presented in Dowling's book, *The Sociology of Mathematics Education: Mathematical Myths/ Pedagogical Texts*, in which he describes the 'the myth of references'. It is certainly a myth that such an exercise refers to any reality. But, as I see it, it has a reference: a semi-reality imagined by the author of the problem.

It might be a reference which can support some students in solving the problem. However, the practice of mathematics education has established specific standards for how to operate in such a semi-reality. If, for instance, a student asks the teacher about the distance between the shops and the home of the person who is going to buy the dates, and if the student wants to figure out how long distance it is possible to carry a bag of 15 kg. by making an experiment in the school yard, and if the student asks whether both shops can be expected to deliver the dates or not, and whether it can be assumed that the qualities of the dates from the two shops are the same, then the teacher would most likely regard the student as trying to obstruct the whole mathematics lesson.

Certainly, such questions generate obstruction considering the general 'agreement' between teacher and students operating in the exercise paradigm. Solving exercises with reference to a semi-reality is an elaborated competence in mathematics education, based on a well-specified contract between teacher and students. Some of the principles from the agreement are the following: The semi-reality is fully described by the text of the exercise. No other information is relevant in order to solve the exercise. Further information is totally irrelevant; the sole purpose of presenting the exercise being to solve it. A semi-reality is a world without sense impressions (to ask for the taste of the dates is out of the question), only the measured quantities are relevant. Furthermore, all quantitative information is exact, as the semi-reality is *defined* in terms of these measures. Thus, the question whether it is relevant to negotiate the price or to buy, say, a little less than15 kg of dates is devoid of meaning. The exactitude of the measurement combined with the assumption that the semi-reality is fully described by the information provided, makes it possible to maintain the one-and-only-one-answer-is-correct assumption. The metaphysics of the semi-reality makes sure that this assumption can be maintained, not only when references are made exclusively to numbers and geometric figures, but also when references in exercises are made to 'shops', 'dates', 'kilograms', 'prices', 'distances' as well as to other seemingly empirical entities.[6] In particular, this metaphysics has structured the communication between teacher and students.

It has been considered irrelevant to make actual observations of how mathematics is operating in real-life situations to being able to construct exercises of type (3). But recently, much more careful studies of mathematical practices in different work-situations have been carried out.[7] Real-life based exercises provide a learning milieu of type (5). For instance, figures concerning unemployment can be presented as part of the exercise, and based on such figures questions can be asked about the decrease or the increase of the employment, comparisons can be made between different periods of time, different countries, etc.[8] All figures referred to are real-life figures, and this provides a different condition

[6] If it is not realised that the way mathematics fits the semi-reality has nothing to do with the relationship between mathematics and reality, then the ideology of certainty has found a habitat. For a discussion of the ideology of certainty, see Borba and Skovsmose (1997).

[7] See, for instance, Wedege (1999).

[8] See, for instance, Frankenstein (1989) for exercises of this type.

for the communication between teacher and students, as it now makes sense to question and to supplement the information given by the exercise. Still, the activities are settled in the exercise paradigm.

Like milieu (3), milieu (4) also contains references to a semi-reality, but now this semi-reality is not used as a resource for a production of exercises, but as an invitation for the students to make explorations and explanations. The 'big horse race' can serve as an example. The racecourse is drawn on the blackboard, and eleven horses: 2, 3, 4, ..., 12, are ready for start. Two dice are thrown, the sum of the number of spots shown is calculated and a cross is made at the diagram. As Figure 5 shows, the sum 6 came out three times before any of the other sums. Horse 6, therefore, became the lucky winner, followed by horse 7 and horse 10.

				X						
				X	X			X		
	X	X		X	X	X	X	X		X
2	3	4	5	6	7	8	9	10	11	12

Figure 5: The terrain of the horse race.

This horse race can be developed into a greater classroom activity. Imagine that we have to do with children, about 11 years old. Two bookmaker agencies are set up behind desks in corners of the classroom. A small group of students run each agency. Independent of each other, the agencies announce their odds. The rest of the class, the very wealthy gamblers, make their bets: "Look, agency A pays back 8 times the money on horse number 9. But look at agency B! They pay back 40 times for horse number 10!" Placing the bets has to be done in a hurry, as the next race is soon going to start. Another group of children are in charge of the race, they ring the bell, and (a kind of) silence enters the classroom. The dice are thrown, the sums are calculated, crosses are made, the horses race towards the goal line. Some of the gamblers show big smiles.

Agency A has only few customers. Their odds seem far less favourable than those provided by agency B. However, a new race is going to start. New odds are suggested. The gamblers become surprised: "What marvellous odds this agency A is now offering!" New bets, new races, new winners, new losers. The horses are not anonymous any more, and horse number 2 is called the turtle. Suddenly, one agency losers its whole fortune. Anyway, a new millionaire sets up a new agency.

The teacher suggests that it is time for a derby. Up to now the races have had the length of 3 units, but a derby must be of 5 units, at least. Odds are produced by the agencies. Some of the gamblers are wearing paper hats. After the second derby, some of the gamblers start wondering: Could horse number 7 possible be particularly fit for derbies?

Even after several races, there is no smell of horses in the classroom. The great horse race takes place in a semi-reality, but not in the educational paradigm defined by exercises. And the many remarks about the abilities of the different horses ("Horse number 11 needs some vitamin pills") are not perceived as obstructions. The strict logic governing the semi-reality of learning milieu number (3) is no longer in operation. The whole activity is located in a landscape of investigation. Many discoveries are waiting for the children. Strategies are to be produced and improved. And, as I have chosen to describe the activity, the children certainly accepted the invitation to participate in the big horse race.[9]

Another Example

Naturally, it is possible to develop landscapes of investigation with a greater degree of reality involved than the big horse race. In *Towards a Philosophy of Critical Mathematics Education* I have discussed some examples organised as project work, and such work can illustrate the learning milieu (6).

The project "Energy" concentrated on input-output models for energy. As an introduction, the students calculated how much energy certain types of breakfast contained (energy was measured in kJ). Then it was calculated, using formulas from sports research, how much energy was used during a certain trip on a bike. Formulas expressed the use of energy as a function of different parameters like speed, length of the trip, type of bike, and the 'frontal area' of the cyclist. How to measure this area? The students found a method and completed their calculations. In this way they were introduced to the idea of making an input-output model for energy.

Then the project concentrated on input-output models for farming. The students investigated a specific farm, not far away from the school. First, it was calculated how much energy in terms of, for instance, petrol was used in preparing a certain field during a year. Sitting in the barn, the students listened to the farmer explaining the methods of preparing the field. The students then measured the breadth of the different tools: the plough, the harvest, etc. This made it possible for them to estimate how many kilometres the farmer, on an annual basis, had to drive the tractor in preparing the field. On this field barley was grown, and it was calculated how much energy the harvested barley contains. In these calculations, research statistics about farming were used. According to the students' calculations, the input-output figures were very profitable: the harvested barley contained about 6 times the energy that was 'supplied' to the field, the reason, of

[9] I made this description while I was a visiting scholar in England. Had I stayed in Denmark, I would most likely have described the big cycling race.

course, being that the sun is a great supplier of energy. This result could be compared to the official statistics in Denmark, which reveals that the factor is somewhat smaller, one reason being that the students did not consider all the relevant types of transports which are necessary in order to manage the farming.

In this particular farm the barley was used as food for pigs, and the students could set up a new input-output model. They collected information about how much pigs were eating depending on their weight, and about the time needed before they were brought to the bacon factory. It was then possible to set up a new input-output model, and the factor was calculated to be about 0.2. Only one-fifth of the energy contained in the food supplied to the pigs is contained in the meat. Meat production, thus, seems to be a bad ‘economy’ in terms of energy.

Were these findings characteristic of the chosen farm only? The official statistics about Danish farming could report that also in this case the students’ results were similar to official results. From an energy point of view, turning barley into meat costs a lot of energy. In this sense, the students’ investigations became exemplary, and this is an essential element in project work. The discussion can be carried on. Is Danish farming doing things in a particular bad way, from an energy point of view? Not particularly bad. As statistics can tell, not as bad as, for instance, the U.S. farming which demonstrates a most problematic energy account.

This project illustrates different aspects of learning milieu (6). The references are real, and they provide the activities (and not only the concepts) with meaning. The students are making calculations related to real farming. This means that ‘authorities’ which were exercising their power in the exercise paradigm are eliminated; thus, the assumption one-and-only-one-answer-is-correct does not make sense any longer. When such a project is running, textbooks can rest safely in a corner of the classroom. The teacher comes to serve as a supervisor, and new inquiry-oriented discussions emerge: How in fact to calculate the front area of a cyclist? The problems now become setting up models for the input-output calculations, and it becomes important to reflect on the result of the calculations. Are the results reliable? Did we consider the relevant factors? Well, we can compare to official statistics. But are these results correct? Critical reflection on mathematics and on mathematical modelling gets a new significance.

In Denmark the official curriculum is no hindrance for students and teacher to operate in learning milieu (6). No exams after each school year, where the students can fail, determine specific classroom activities. Only after the 9th year will the students face a national exam in mathematics. Marks will be given, but everybody will pass. Furthermore, this exam supports an investigative approach since, in its written part, it does not presuppose any memorised knowledge and, in its oral part, it concentrates on groups of students making mathematical investigations. Nevertheless, the exercise paradigm also finds a strong support in this corner of the world.

Moving between different learning milieus

Naturally, the matrix in Figure 4 represents a strong simplification. The vertical line separating the exercise paradigm from landscapes of investigation is certainly a very ‘broad’ line, representing a huge terrain of possibilities. Some exercises can provoke problem solving activities, which might turn into genuine mathematical investigations. Problem posing means a further step into landscapes of investigation, although problem posing activities can be very different from project work. No doubt the horizontal lines are also “fluffy”. My point, of course, is not to try to provide any clear-cut classification, but to elaborate the notion of milieus of learning in order to facilitate discussions about making changes in mathematics education.

A good deal of mathematics education is switching between the milieus (1) and (3). In this sense the exercise paradigm provides a foundation for ‘tradition’ in mathematics education. Many studies in mathematics education have been carried out providing a desolate picture of what is going on in a traditional classroom. Some of these studies, however, do not acknowledge that other possible learning milieus do exist in mathematics education, and that the observations are linked to a particular organisation of the mathematics classroom, although a most typical one. [10] A differentiation between ‘the school mathematics tradition’ and ‘the inquiry mathematics tradition’ has been suggested by Richard (1991). This differentiation certainly also fits into the matrix. Exercises are a defining element of the school mathematics tradition.

In Denmark a challenge to the school mathematics tradition has been presented by the type (6) learning milieu. However, I find it important that the challenges are organised in terms of learning milieus of types (2) and (4) as well as of (6). I do not want to make the claim that milieu (6) is the only essential alternative to the exercise paradigm. In fact, I do not want to suggest that a particular learning milieu can become designated to represent the ultimate goal for mathematics education, critical or not.

I support a mathematics education moving between the different milieus as presented in the matrix. In particular, I do not regard it as an aim to abandon exercises from mathematics education altogether. It might make good sense after, say, the big horse race to use a period for ‘consolidation’ in which the students work with exercises related to the notion of probability. It is important that students and teacher together find their route among the different milieus of learning. The ‘optimal’ route cannot be determined in advance but has to be decided upon by students and teacher. The matrix of learning milieus can also be used as an analytic tool. For instance, it is possible for the students and the teacher to reconsider last year’s route: Which learning milieus did we experience? Have we spent all the time in one or two milieus? In which milieu did we experience a particular success? Did some moves from one milieu to another caused difficulties? Many considerations of planning can be

[10] See, for instance, Walkerdine (1988).

referred to the matrix.

Long ago, I was engaged in a mathematical project involving young children, about 7 years old. The main aim of the project was to plan and to construct a playing ground outside the windows of the classroom where there was a small piece of ground available for the class. Certainly, this activity took place in a learning milieu of type (6), and, as a result of the project, a small playground was in fact set up outside the windows of the classroom with the active help of parents during a few weekends. Before that, however, much activity had taken place. As the first thing, the children visited other playing grounds in order to test which one was a 'good' one. Children, 7 years old, are experts in carrying out this kind of test. More difficult, however, was specifying the exact quality of the good playing ground. How tall are the swings? How much sand is needed? Etc. Many things have to be measured, and in order not to forget such measures it becomes important to make notes about the observations. Not an easy task!

Such periods of intense activity are very fruitful and important, but other more relaxed types of activities are important as well, both for the teacher and for the children. As part of the project about the playground (which lasted for a few months), there were organised periods of 'office work', which actually looked like an excursion into the learning milieu of type (1). The children were organised in small groups working in their 'offices'. As in any public office, voices were low. The children had juice or lemonade in plastic cups standing on their desks which, by some magic, now looked like real office desks. Sometimes the office workers nibbled at a cookie while they added up numbers. Sometimes the radio poured out low, soft, music. Sometimes the teacher played the guitar. The papers scattered around the desks contained first of all exercises in adding and subtraction. The point is that the children during the more intensive periods of project work had recognised the importance of being able to add numbers, and to add them correctly. During office hours, this kind of skills could then be consolidated, and reasons for doing such office work were found in the previous periods of the project work. The actual set-up of 'office work' broke the pattern of the normal exercise paradigm, although the activity as such was of type (1). This illustrates that the route between the different milieus might help to provide the students' activities with new meaning. The office work did not take place in an atmosphere of the school mathematics tradition, although it took place in the exercise paradigm. In particular, the communication between teacher and students in the office, was not governed by the same logic as the communication between teacher and students adjusted to the school mathematics tradition.

The consolidation provided by office work also serves as a preparation for being engaged in a new project. To create harmony between project work and course work has been a big challenge to the project-based mathematics education – no matter whether we have to do with project based university studies in mathematics or with elementary school mathematics.

Sometimes, in discussions with teachers, it has been suggested to me that before trying to investigate any landscape the students should be equipped with some understanding and techniques which, most efficiently, can be produced within an exercise paradigm. The big horse race illustrates why, in my opinion, this is not generally the case. Had the children, before the race, been introduced to some basic notions of probability illustrated by the canonical diagram: the number of eyes of the red dice is shown at the x-axes, the number of eyes of the blue dice is shown at the y-axes, and the sum ..., then the fascination of the game could be lost. However, an opposite route is relevant in many cases, i.e. the route from (4) to (3). When the game has been tried out and the children have become familiar with the strengths and weaknesses of the different horses and they have got an idea about the reliability of odds, then the children and the teacher can start making particular observations and finding explanations. And exercises can be used as a means of stabilising some experiences.

Generalisation: Culture of the Mathematics Classroom

The six types of learning milieus have been specified in terms of references (to pure mathematics, to a semi-reality or to real-life situation) and forms of organising tasks (exercises or investigations), but many other elements can be considered in specifying learning milieus.

A learning milieu is certainly also determined by the nature of the stratification of the students which may take place. By a stratification, I mean a way of providing an order of the students according to ability. A stratification can be made in the most brutal way, as for instance in many schools in England, by a public streaming of the students into A, B, C and D levels. An evaluation of a student can be communicated to the student, and to the student only. But explaining to a student where he or she is 'positioned' in terms of performance is very different from making a stratification public. Such stratification is a most powerful way of disciplining students. And when a stratification is integrated in exam systems of passing and failure, then the stratification takes the form of school violence towards students. Public stratification contradicts the concern of developing the mathematics classroom as a micro-society where democratic values can be experienced. The nature of the communication between teacher and students definitely depends on the stratification in operation.

Many other aspects are important for characterising a learning milieu: forms of communication which can vary from being fixed by the exercise-discourse to being a dialogue; use of information and communication technologies; economic resources of the school; the students' background; political conflicts represented in the classroom; the students future possibilities in life; etc.[11] Taken together all such aspects influence the learning milieu, and they establish the *culture of the classroom*. But in order to give this notion a full value,

[11] For a discussion of communication in the mathematics classroom see Alrø and Skovsmose (1996a, 1996b, 1998).

we have to consider all such aspects.[12] Thus, in my opinion, *The Culture of the Mathematics Classroom*, edited by Seeger, Voigt and Waschescio is limited in its conception of culture.[13] In the index of the book we do not even find a reference to exam systems or project work. Having said this, I shall now return to a discussion of the matrix, i.e. to a discussion of milieus of learning, admitting that my discussion touches on only a small corner of the topic: the culture of the mathematics classroom.

The Risk Zone

French research in mathematics education has paid much attention to the notion of *didactical contract*.[14] With reference to the notion of learning milieu, a didactical contract can be defined in terms of 'balance in a learning milieu'. Thus, a didactical contract refers to an established harmony between the parameters of the learning milieu, i.e. a harmony between the way meaning is produced, the tasks are organised, the textbook is structured, the communication are carried out, etc. And, furthermore, this harmony must be recognised and accepted by both teacher and students. That a didactical contract is established does not, however, reveal anything about the quality of the learning milieu. It first of all indicates that teacher and students have a shared understanding and acceptance of the priorities of the learning milieu. Their interaction is not problematic as long as both parties recognise the contract.

A didactical contract can be broken in many ways, for instance when students start asking about details of a semi-reality, as described previously. The contract can be broken if the evaluation is drastically changed. In general, improvement of mathematics education is closely linked to breaking the contract. And when, initially, I suggested a challenge to the paradigm of exercise, it can also be seen as a suggestion for breaking the contract of the school mathematics tradition.

From the teacher's perspective this might appear as moving from a comfort zone into a *risk zone*. This notion has been introduced by Penteado (manus) in her study of teacher's experiences in a new learning environment where computers play a crucial role.[15] Moving between the different possible learning milieus, and paying special attention to landscapes of investigation, will cause a great deal of uncertainty. My point is that uncertainty is not to be eliminated. The challenge is to face uncertainty.

Computers in the mathematics classroom have helped to establish new landscapes of investigation (although some closed programmes try to eliminate uncertainties by adjusting the activities to the exercise paradigm).[16] The computer will immediately challenge the authority of the (traditional) mathematics teacher. Students working with, say, dynamic geometry will easily come to face situations and experience possibilities not foreseen by the teacher as part of his or her planning of the lesson. A student's eager clicking on the mouse might quickly lead to an unknown corner of the programme: What to do now? How to get out of here? The teacher must always be ready to face questions which cannot easily be answered. The traditional teacher-authority can be broken within seconds, and nobody knows which time next. Certainly not the teacher. The degree of unpredictability is high. One epistemological reason for this is that computers are not simply a tool which "extend" our way of thinking; instead computers reorganise our way of thinking.[17] The whole idea of 'reorganisation' links closely to the idea of 'risk zone'.

When students are exploring a landscape of investigation, the teacher cannot predict what questions may come next. One way of eliminating this risk is, for the teacher, to try to guide everybody back into the exercise paradigm and into the comfort zone. Thus, the whole exploration of translatability of geometric figures in the number table could be reorganised as a sequence of exercises. And instead of letting the students play around with the programme of a dynamic geometry, the teacher could specify each step to be taken: "First you select a point. Yes, all of you! This point we call A. Then you select another point. This other point we call B..." By organising the activities by means of such orders, the teacher can bring about (almost) the same picture on all the screens in the classroom. When students in this way are moving slowly forward like soldiers in columns and rows, the teacher can prevent the occurrence of the unpredictably events and challenges. By doing so, however, many learning opportunities are lost as well.

Any landscape of investigation raises challenges to a teacher. A solution is not to rush back into the comfort zone of the exercises paradigm, but to be able to operate in the new environment. The task is to make it possible for the teacher and students to operate in co-operation within a risk zone, and to make this operation a productive activity and not a threatening experience. This means, for instance, accepting that "what if ..." questions can lead the investigation into unknown territory. According to the research of Penteado, an important condition for teachers to be able to operate in a risk zone is the establishment of new forms of co-operative work in particular among teachers, but also along the line of students-parents-teachers-researcher.

However, why bother about operating in the risk zone? Why not simply accept the didactical contract of the school mathematics tradition which has been so carefully elaborated? Cobb and Yackel refer to 'intellectual autonomy' as an explicitly stated goal for their efforts to establish an inquiry mathematics tradition in contrast to a school mathematics tradition. Intellectual autonomy is characterised "in terms of students' awareness of and willingness to draw on their own intellectual capabilities when making mathematical decisions and judgements" (Cobb and Yackel, 1998, 170). Intellectual autonomy can be associated to the activities of exploration and

[12] See, for instance, Powell and Frankenstein (eds.) (1997): Valero (1999); Vithal (1999, 2000); and Volmink (1994).
[13] Compare also with Nickson (1992) and Lerman (ed.) (1994).
[14] See, for instance, Brousseau (1997).
[15] See also Penteado (1999).
[16] See, for instance, Borba (1995).
[17] For a discussion of 'reorganisation', see Borba (1999).

explanation as facilitated by landscapes of investigation. It is difficult to see this autonomy rooted in those rules which constitute the adequate behaviour when operating in a semi-reality in milieu (3). In particular, leaving the 'risk zone' in search for a comfort zone also means eliminating learning opportunities linked to computers-as-reorganisers.

Making a move in the matrix of Figure 4 away from the exercise paradigm and into the direction of landscapes of investigation may help to abandon the authorities of the traditional mathematics classroom, and make students the acting subject in their learning process. In *Towards a Philosophy of Critical Mathematics Education*, I have discussed learning as action, and emphasised the importance of establishing the students' intentions as the driving elements in the learning process. A critical subject has to be an acting subject.

Making a move in the matrix of Figure 4 away from references to pure mathematics and to real life references may help to provide resources for reflections on mathematics.[18] Thus, studies of classrooms where real-world problems were the starting point for mathematical considerations made Voigt state the following hope: "As future citizens, students will have to cope with many real-world problems that seem to be mathematically intransparent... Is the citizen competent to distinguish between necessary mathematical inferences and the suppositions of modelling that depends on interests? It could be hoped that paying more attention to the quality of the negotiation of mathematical meaning in the classroom could improve the education of the 'competent layman'." (Voigt, 1998, 195) I certainly share this hopes, and *Towards a Philosophy of Critical Mathematics Education* contains a specification of elements of a critique of mathematical modelling as being essential to the development of the competence called mathemacy. Real life references seem necessary in order to establish a detailed reflection on the way mathematics may be operating as part of our society. A critical subject is also a reflecting subject.

How to develop a mathematics education as part of our concern for democracy in a society structured by technologies that include mathematics as a constituting element?[19] How to develop a mathematics education which does not operate as a blind introduction of students to mathematical thinking, but makes students recognise their own mathematical capabilities and makes them aware of the way mathematics may operate in certain technological, military, economic and political structures? I would never dare to claim that leaving the exercises paradigm in order to explore landscapes of investigation would provide an answer to these questions. Nor would I claim that it is sufficient to build mathematics education solely on real-life references. My only hope is that finding a route among the different milieus of learning may offer new resources for making the students both acting and reflecting and in this way providing mathematics education with a critical dimension.

[18]See also Cobb, Boufi, McClain and Whitenack (1997).

[19]See, for instance, Skovsmose (1998b); Skovsmose and Valero (1999); and Valero (2001).

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