

## How to Connect School Mathematics with Students' Out-of-School Knowledge

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**Abstract:** In this paper we present an explorative study for which special cultural artifacts have been used, i.e. supermarket receipts, to try to construct with 9-year old pupils (fourth class of primary school) a new mathematical knowledge, i.e. the algorithm for multiplication of decimal numbers. Furthermore also estimation and approximation processes have been introduced, procedures that are not commonly used in ordinary teaching activity. In our study the receipts, through some modifications, have become more explicitly tools of mediation and integration between in and out-of school knowledge; so they can be utilized to create new mathematical goals, thus becoming real mathematizing tools and constituting a didactic interface between in and out-of-school mathematics. In agreement with ethnomathematical perspective we deem that it is a task for the teacher to know, in order to be able to profitably take account of the teaching, the life experienced by the pupil. Future mathematics teachers should be prepared a) to see mathematics incorporated into real world, b) to investigate mathematical ideas and practices of their pupils, and c) to look for ways to incorporate into the curriculum elements belonging to the sociocultural environment of the pupils, as a starting point for mathematical activities in the classroom. In this way the motivation, interest and curiosity of the pupils will be increased and the attitude towards mathematics of both pupils and teachers will be changed.

**Kurzreferat:** *Wie lässt sich Schulmathematik mit dem außerschulischen Wissen von Schülern verbinden?* In dieser Arbeit wird eine explorative Studie aufgezeigt, wofür spezielle kulturelle Mittel verwendet werden wie z.B. Quittungen von Supermärkten. Das Ziel ist mit 9-Jahre alten Schülern (vierte Klasse der Grundschule) neue mathematische Kenntnisse zu erarbeiten, z.B. den Algorithmus zur Multiplikation von Dezimalzahlen. Außerdem wurden Schätz- und Approximations- Prozesse eingeführt, also Methoden die im normalen Unterricht nicht behandelt werden. Die Quittungen, bzw. deren Variationen, sind in unserer Studie als Mittel benützt worden zur Vermittlung und Integration von schulischen und nicht-schulischen Kenntnissen. Auf diese Weise können sie benützt werden um neue mathematische Ziele zu formulieren und zu erreichen, sowie ein didaktisches "Interface" zu bilden zwischen schulischer und nicht-schulischer Mathematik. Im Einklang mit einer ethnomathematischen Perspektive halten wir es für die Aufgabe des Lehrers, die Lebenserfahrung des Schülers kennenzulernen, um den Unterricht optimal zu gestalten. Künftige Mathematiklehrer sollen in der Lage sein: a) Mathematik in das wirkliche Leben einzubauen; b) mathematische Ideen und Praxis ihrer Schüler zu untersuchen; c) nach Mitteln zu suchen um soziokulturelle Elemente ins Curriculum einzubauen als Ausgangspunkt für die mathematische Tätigkeit in der Schule. Auf diese Weise wird die Motivation, das Interesse und die Neugier der Schüler vergrößert und das Verhalten zur Mathematik sowohl von Schülern als von Lehrern sich ändern.

**ZDM-Classification:** C62, D32, F92

### 1. Introduction

In recent years some educators from various countries (D'Ambrosio, Gerdes, ...) have called for the recognition not only that mathematics is a cultural product, but that the ethnicity (unique sociocultural history) of students can be used in a powerful way in the learning of school mathematics, see Presmeg (1998). Connections are advocated between mathematical content and the home cultures of learners, as well as between different branches of mathematics, various disciplines in which mathematics is used, historical roots of mathematical content, and connections with the real world and the world of work (Civil 1995).

Furthermore many studies have pointed out that the local strategies developed in practice are more effective than the arithmetics algorithms, which are usually taught in school to give the students powerful general procedures that, in fact, are frequently useless in out-of-school contexts (Schliemann 1995).

The text of an important and recent italian Document about 'essential contents of basic education', after stressing that "a special and deeply innovative methodological attention has to be devoted to the teaching of mathematics ...", then goes on to point out that "research about non scholastic mathematics shows the need of teaching students to use ideas and techniques of mathematical type ...".

Our researches on problems related to teaching-learning of the meaning of decimal numbers in primary and middle schools enabled us to note a clear gap between the mathematically rich situations, mainly in the numerical field, that children experience in out-of-school and the classroom practice (Bonotto 1993, and Bonotto 1996).<sup>1</sup>

We find a significant convergence with the ethnomathematical perspective about this point. Before and outside school almost all children in the world become 'matherate' - that is to say, they develop the "capacity to use numbers, quantities, the capability of qualifying and quantifying and some patterns of inference" (D'Ambrosio 1985a). In school 'the learned' matheracy eliminates the so-called 'spontaneous' matheracy. An individual who manages perfectly well numbers, operations, geometric forms and notions, when facing completely new and formal approach to the same facts and needs creates a psychological blockage with grows as a barrier between different modes of numerical and geometrical thought.<sup>2</sup> As a consequence "the early stages of mathematics education offer a very efficient way of instilling in the children a sense of failure and dependency" (D'Ambrosio 1985a). The question which arose, then, was what to do about "this disempowering effect of schooling", Gerdes (1996), to which we refer the reader for further analysis about ethnomathematical perspective and current developments. In agreement with D'Ambrosio (1985b), we deem that the mathematics curriculum in school should incorporate elements belonging to the socio-cultural environment of the pupils and teachers, in such a way that they facilitate the acquisition of knowledge, understanding, and compatibilization of known and current popular practices, because

"cognitive power, learning capabilities and attitudes towards

learning are enhanced by keeping the learning ambiance related to cultural background ... . It is well documented the fact of children and adults performing 'mathematically' well in their out-of-school environment, counting, measuring, solving problems and drawing conclusions using the arts or techniques [tics] of explaining, understanding, coping with their environment [mathema] that they have learned in their cultural setting [ethno]" (D'Ambrosio 1995).

## 2. Our perspective

Having recognized that the usual school practice is strange to the rich experience students attain about numbers out of and before primary school, the question arises of how can we benefit from what children already know avoiding, at the same time, strengths and limitations that are typical of the everyday socio-cultural situations.

How can we design better opportunities for children to develop mathematical knowledge that is wider than what they would developed outside of schools, but that preserves the focus on meaning found in everyday situations, as suggested by Schliemann (1995)?

Starting from the school year 1993/94 one has been exploring and implementing some elementary school activities more relatable to the experiential worlds of the pupils, and consistent with a sense-making disposition. Although mathematics learning and practice in school and out-of-school differ in some significant ways, see Lave (1988), and Resnick (1987), we deem that those conditions that often make extra-school learning more effective can and must be re-created, at least partially, in classroom activities. Indeed while some differences between the two contexts may be inherent, many differences can be narrowed by creating classroom situations that promote learning processes more close to the learning processes emerging in the out-of-school mathematics practices.

Furthermore we stress that bringing real world situations into school mathematics is a necessary, although not sufficient, condition to foster "a positive attitude towards mathematics, intended both as an effective device to know and critically interpret reality, and as a fascinating thinking activity", as is stressed for example in the italian programs for the primary school. We contend that this educational objective can be completely fulfilled only if we can get the students to bring mathematics into reality.<sup>3</sup>

In other words besides 'mathematize everyday experience' it would be necessary 'everydaying mathematics'.

That can be implemented in a classroom by encouraging the children to analyze some '*mathematical facts*' that are embedded in opportune '*cultural artifacts*', and which for brevity we might call "*cultural*" or "*social mathfacts*".

Cultural artifacts besides embody the intellectual history of a culture also incorporate many theories the users accept, albeit unconsciously. "Artifact and conventions are cultural forms that have been created over the course of social history which also figure into the goals that emerge in cultural practices" (Saxe et al. 1996). Their use mediates the intellectual activities, and –

at the same time – enables and constrains the human thinking. Through these subtle processes social history is brought into any individual act of cognition (Cole 1985). Thus learning mathematics is not excluded.

An artifact is hence a representative or witness to the society we live in, of the culture we belong to, of the means and ways of communication that are peculiar to our age. We believe that an important goal, to be targeted at least by compulsory education, is that of teaching students to interpret, even critically, the reality they live in, to understand its codes and messages so as not to remain excluded or be misled.

The cultural artifacts we have introduced in classroom activities in primary school are: receipts, bottle and can labels, railway schedules, a ski-race time table, the weather forecast presented in a newspaper, and others, see Bonotto 1999a. They offered the children with the opportunity of making connections between the mathematics utilized in the real world situations and the school mathematics, because they are really part of the children's experience. This enables children to keep their reasoning processes meaningful, to monitor their inferences. As consequence, they can off-load their cognitive space and free cognitive resources to develop more knowledge (Arcavi 1994).

On the other hand, it is well known that in common teaching practice the habit of connecting mathematics classroom activities with out-of-school experience is still substantially delegated to 'wor(l)d' problems. Recent studies have documented a clear tendency of children to neglect realistic considerations and to exclude real-world knowledge from their mathematical problem solving, see e.g. Reusser & Stebler (1997).

"Rather than functioning as realistic contexts that invite or even force pupils to use their commonsense knowledge and experience about the real world, school arithmetic word problems have become artificial, puzzle-like tasks that are perceived as being separate from the real world. Thus, pupils learn that relying on commonsense knowledge and making realistic considerations about the problem context - as one typically does in real-life problem situations encountered outside school - is harmful rather than helpful in arriving at the 'correct' answer of a typical school word problem" (Verschaffel, De Corte, Borghart, 1997).

Also Freudenthal's position on this regard too is very clear:

"The context [of the butcher problem, author's note] is the textbook, rather than reality proper, or in other words, it portrays a world of pseudo-isomorphisms. In the textbook context each problem has one and only one solution: there is no access for reality, with its unsolvable and multiply solvable problems. The pupil is supposed to discover the pseudo-isomorphisms envisaged by the textbook author and to solve problems, which look as though they were tied to reality, by means of these pseudo-isomorphisms. Wouldn't it be worthwhile investigating whether and how this didactic breeds an anti-mathematical attitude and why the children's immunity against this deformation is so varied?" (Freudenthal 1991).

If we wish to modify this situation we have to change the classroom activities in which we delegate the process of creating an interplay between school mathematics and out-of-school knowledge.

On the other hand the relationship between mathematics and reality has always been both intricate and intriguing, as much complicated as interesting to be dealt with, and maybe we will never be able to analyse it completely and thoroughly.<sup>4</sup> As joke, we might say that it is a relationship of 'hate and love' since mathematics, although nourishing from real world, detaches from it as soon as possible, due to its special nature, to come back to real experience in due time to pick up new problems and examples or to find new applications. As to didactics, the fact that this relationship is sometimes denied and at other times stressed, without any explanation of the reasons for these choices, makes it difficult for students to know whether or not it is permissible for them to exploit their everyday knowledge in approaching mathematical problems. Furthermore in the teaching of mathematics there often prevails the habit of emulating the practice of academic mathematicians in exhibiting only the finished product of mathematical research rather than illuminating the process of creation. In other words, the results are presented as if the audience consisted of expert colleagues expecting a refined and elegant presentation purged of all the 'dirty work' which was necessary for producing the results. It is precisely the latter which could be illuminating and interesting for students.

### 3. About the use of cultural artifacts

In our experiences mathematics was introduced in the context of activities which could promote the movement from the situations in which it is usually utilized to the underlying mathematical structure, and back, from the mathematical concepts to the real world situations, according to the '*horizontal mathematization*' in the sense of Treffers (1987).

The double nature of cultural artifacts utilized, that of belonging to the world of everyday life and to the world of symbols, to use Freudenthal's apt expression, makes it possible.

"An essential property of artifacts, which supports their bilateral influence and offers common bases to culture and discourse, is their being ideal (conceptual) and material. They are ideal because they contain in a codified form the interactions that have mediated in the past and mediate in the present ... they are material because they are embedded in concrete artifacts" (Cole 1995).

But a different use of these artifacts supported the opportunity to favour also '*vertical mathematization*', from concepts to concepts, although only in a weak sense, given the grade level of the students. This manifested itself when symbols, embedded mathematical facts, become objects to be put in relationship, modified, manipulated, reflected upon by the children through property noticing, conjecturing, problem solving.

The use of the cultural artifacts in mathematics classroom activities can be articulated in various stages, each rich in educational potential and in content objectives. The use of some artifacts (bottle and labels, the weather forecast presented in a newspaper, ...), see Bonotto 1999a, enable the teacher to pose a great deal of possible questions, remarks, culturally and scientifically

interesting inquiries; activities and connections that can be made depend, of course, on the pupils degree of education. These artifacts can contain different codes, percentage numerical expressions, different types of quantities with their related units of measure, and hence they hold connections and links with other mathematical concepts but also to other disciplines [chemistry, biology, geography, astronomy, and so on]. It is worth saying that artifacts are as related to mathematics [like to other disciplines] as one is able to find these relationships.

First of all using adequate cultural artifacts which students may understand, analyse and interpret, we can present mathematics as a means with which to understand real world. "Viewing context as noise, apt to disturb the clear mathematical message, is wrong; the context itself is the message, and mathematics a means of decoding" (Freudenthal 1991). This allows students to become involved with mathematics and to break down their conceptions of a remote body of knowledge.

In this way we can increase the awareness of the utility of mathematics, we can encourage students to develop a positive attitude towards school mathematics and we can motivate, engage and excite curiosity of the students.

Obviously, the usefulness and pervasive character of mathematics are merely two of its many facets and can not by themselves capture its very special character, relevance and cultural value; nonetheless we deem that these two elements can be usefully exploited from the teaching point of view. The objective is also to show mathematics from a different point of view trying to change the common behaviour and attitude held while dealing with mathematics at school, both by teachers and pupils.

We ask children

- to select other cultural artifacts in their everyday life,
- to point out the embedded mathematical facts,
- to look for analogies and differences (for example, different number representations), and
- to generate problems (for example, discovering relationships between quantities).

In other words we want to encourage the children to recognize a large variety of situations as mathematical situations or more precisely as "mathematizable" situations. In contrast with the traditional classroom curriculum, children are offered endless opportunities to become acquainted with mathematics.

These artifacts are not used univocally; they can be used

- as tools to apply 'old' knowledge to 'new' contexts, thus becoming good material for 'meaningful exercises';
  - to reinforce already owned mathematical knowledge or to review it at a higher level.
- Furthermore, as we will see later on, through some modifications - for instance removing some data - these artifacts may become also real mathematizing tools, able
- to create new mathematical goals,
  - to develop new mathematical knowledge, as stepping-stone to launch, at a first stage, new concepts.
  - to provide pupils and students with a basic sense experience in mathematization.

In this new role, the cultural artifact can be used to

introduce new mathematical knowledge through those special learning processes that Freudenthal, quot., defines ‘*anticipatory learning*’ or ‘*learning by advance organizers*’. Freudenthal, quot., defines this type of learning ‘*prospective*’ as counterpart of that he defines ‘*retrospective*’ learning, that occurs when old notions are recalled to be considered at a higher level and within a broader context, typical process of adult mathematicians. And he states “prospective learning should not only be allowed but also stimulated, just as the retrospective learning should not only be organized by teaching but also activated as a learning habit”.

#### 4. Experience with receipts

Let’s now show an explorative study accompanied by the used artifacts and part of the given tasks. It has been carried out in the school year 1998/1999 in a fourth class of a primary school institute (Trebaseleghe, Padova), with 23 pupils, by the teacher Milena Basso, as on-site person in charge of updating, in the presence of the official teacher of logic-mathematical area. The teaching method used in this classroom is similar to that described in this study. This was repeated in a fourth class of a primary school institute (Asiago, Vicenza), with 21 pupils, by the student teacher, Chiara Frigo, who in the course of her mathematics degree had completed her thesis on this topic. The teaching method of this classroom is more traditional.

##### 4.1 Teaching methods

With regard to the teaching methods, the fact is emphasised that before beginning the experiences certain behavioural norms are given for distinguishing the participation of the students and of the teachers involved. We are in agreement with Greer (1997), when he asserts that it is appropriate to invoke an ‘*experimental contract*’ by analogy with the concept of ‘*didactical contract*’.

With regard to the role of the students these norms may be summarised by the fact that they are expected to:

- explain and justify their arguments,
- listen and do their best to comprehend the explanations of their companions,
- indicate points and concepts not fully understood, asking for clarification and deeper explanations,
- indicate and explain the corrections made when they realise their mistakes.

With regard to the role of the teachers they must

- encourage the initiatives of the students,
- comment on and summarise the students’ contributions, emphasising the arguments offered,
- making clear eventual contradictions,
- underline whether answers were “realistic” or not.

Moreover they must emphasise other possible strategies for solving the same problem that crop up, and get students into the habit of making comparisons between these strategies. Through such activities as questioning assumptions and debating the relative merits of alternatives strategies, students can be helped to become “flexible discourse and problem comprehenders ... and adaptive rather than routine experts at solving word problems”, see Greer (1997).

Another basic principle guiding the teaching experiments is the variety of teaching methods, involving extensive use of written descriptions of each own method, individual and whole-class discussions.

An important role has been played by verbalization. As a matter of fact, we have invited children to explicit in writing their reasoning processes to make them acquainted with the writing of mathematical facts and with reflection on these facts and on their own inferences; the children can thus acquire a first level of awareness.

“The text written by a pupil has a relevant function since it not only transmits meanings to other people, either teachers or children of the same age (univocal function), but also constitutes a reflection material thus becoming a ‘thinking tool’ (dialogical function) able to generate new meanings” (Basso/Bonotto 1996).

The influence of Vygotskian perspective makes knowledge be perceived as “something which is constructed and modified through interaction, and language becomes first of all not so much the means of expression of thought but the means of its development ...” (Dodman 1995).

The relevance given to the observation of children’s reasoning and used procedures has enabled the teacher to identify the different levels of cognitive competence and of difficulties met; it was possible to start a classroom discussion, focused on the children’s solution procedures. By asking the children to state their solving procedures, would be they correct or not, and to compare them with those used by their classmates, a further reflection on one’s own and the others’ reasoning processes is fostered. During this stage, children have identified similarities and differences between their own strategies and those adopted by the other classmates. Often this type of recognition indicates a further step onward from a cognitive point of view as well as the acquisition of a second level of awareness. As a matter of fact, children can take some distance from their own reasoning to acquire and share further knowledge.

This methodological process, can thus turn out to be very important not only for pupils but also for teachers since they are enabled to recognize and analyse individual reasoning processes, not always explicit [first explorative stage corresponding to the individual written verbalization]. Then teachers can monitor the whole classroom [collective discussion stage], a basic step to consider a new knowledge as acquired, at least at a certain stage.

##### 4.2 Content objectives

When this study has been conceived we were aware of the fact that a supermarket receipt, if duly utilised, may become a real mathematizing tool. “In the ticket, which is poor in words but rich in implicit meanings, the situation is overturned with respect to the usual buying and selling problem, which is often rich in words but poor in meaningful references”, (Basso/Bonotto 1996). Word problems are stories, stylized representations of hypothetical experiences, and not portions of daily experiences. They are narratives

“about assumed general cultural knowledge that (even) children

can be expected to have. They are not about particular children's experiences with the world ... Children's intuitions about the everyday world are in fact constantly violated in situations in which they are asked to solve word problems. This discontinuity by itself may help create the division between 'real' and 'other' math by conveying the message that what children know about the real world is not valid ... " (Lave 1995).

In this new experience, it has been thought to exploit supermarket receipts to construct a new mathematical knowledge together with pupils. The objective in relation to the content of the proposals that will be illustrated in this paper was the bringing together of the formalisation of the algorithm for multiplication of decimal numbers. For an easier acquisition, it has been decided to use also estimation and approximation, two important mathematical processes that were deemed useful to reach the objective. They would allow children to think freely with an "open mind", without the need to concentrate too much really and truly on the computation, and moreover allowing a checking process of the order of magnitude of the same results, even before calculating them. This type of demands are almost absent from Italian school curricula, although highly supported also at international level, see Gravemeijer (1997).<sup>5</sup> We deem it very useful to introduce them starting from the lowest school levels.

Particular types of receipts were used moreover because they permitted us to work with the weight unit of measurement (not dealt with yet) that, given its link with out-of-school experiences leads us to think it is easier to comprehend with regard to decimal representations with 'pure' numbers.

#### 4.3 Sequence of the teaching experiment

It has been decided to sub-divide the teaching experiment into stages (six sessions each lasting about one and half or two hours).

We will present 1 to 5 sessions showing the label and the used receipts and relevant tasks. We won't submit the last session, the most traditional and abstract one, where we have tested the learning of the multiplication algorithm - through the use of 'pure' numbers only - and the degree of understanding of the concept - through more theoretical questions.

##### I session

In the first session, being the introduction of the whole work, the weight units of measurements that had not been dealt with in none of the classrooms were presented. That was necessary to enable all pupils to read receipt data being the object of the following sessions. Children already knew decimal numbers and the length units of measurement they learnt from previous lessons. The knowledge of children about kilogram, hectogram, and gram is simply due to their extra-school experience, to their everyday life. The presentation of the weight units of measurement has been carried out using a label of leaven for pastry.

The following label is presented:

## RAISING VANILLA POWDER PANE DEGLI ANGELI

### FOR PASTRY

DOSE FOR KG OF FLOUR

In its 50-year activity "PANE DEGLI ANGELI" keeps unchanged the quality and gives pastry a balanced and even leavening as well as a delicate vanilla flavour

In this lesson the students were asked to read carefully the label and underline data they deemed important for a successful preparation of the cake. Then an oral discussion started to which the whole class took part.

The basic steps of this session, whose final result was the creation of the table of weight units of measurements and of relevant relationships, were:

- the decoding of the language and symbols used in the label as well as the 'negotiation' of their meaning;
- the passage of the label language and symbols from real world to the mathematics context;
- the flour practical weighing by using a scale;
- the consideration about the universal character of the measurement;
- the introduction of different weight units of measurement and of their symbolic representations;
- relationships between weight units of measurement.

In this first session we noticed that children were capable of translating the 'kg' sign into the symbol for the kilogram; reality, in this case, not only has been of help for the introduction and the explanation of a new topic, but has even preceded such explanation. Also words such as 'hectogram' and 'gram' were already part of the children's knowledge. The transfer of symbols used in an extra-school context into the school context has thus naturally created a link between real world and mathematics.

##### II session

The following receipt is presented:

#### SPECIAL BREAD

kg	L./kg	L.
0,478	4 000	1 910
T=0,004kg		

In this lesson the students were asked to read the receipt, to interpret the symbols and numerical expressions that they are to compare, to explain them and finally to try to establish what operation the machine has made to produce the amount to be paid.

##### III session

The following receipt is presented:

#### BRESAOLA BEEF

kg	L./kg	L.
0,196	41 800	.....

The instructions with regard to this receipt are as follows:  
1) Will you spend more or less than 41,888 lire? Explain

your answer.

- 2) Without doing the calculation in columns, are you able to find a quick way, without doing exact calculations, to find the missing price?

Afterwards, the two questions have been explained, and mainly the second one; the meaning of solving a problem without using accurate calculation has been here negotiated.

#### IV session

The following receipt is presented:

BONED PARMA HAM		
kg	L./kg	L.
0,210	38 900	.....

The instructions in this case are as follows:

- 1) In your opinion, is the missing cost bigger or smaller than 38,900?
- 2) Without doing exact calculations do you know how to find the approximate value of the missing cost?
- 3) Now try to calculate the exact cost.

#### V session

Finally there is presented the following receipt:

kg	L./kg	L.
TREVISIO TARDIVO		
CHICORY		
1,200	7980	.....
PISTACCHIO		
0,730	14500	.....
SICILIAN ORANGES		
2,240	1980	.....
	<b>TOTALE</b>	.....

The instructions with regard to this receipt are as follows: "Will 20,000 lire be enough to buy these vegetables and fruit? Write how you would work out the answer without calculating in columns".

### 5. Qualitative analysis of results

A qualitative analysis of results, based on written protocols, on individual and collective discussions (all have been recorded)<sup>6</sup>, pointed out:

- the way the link with out of school experience, favoured by the use of special artifacts, contributed to give a meaning to new mathematical knowledge or to reinforce previous one;
- the influence of teaching methods on the emerging of new knowledge or on the reinforcement of previously introduced one;
- the presence of processes characterised by a prospective or retrospective learning in the Freudenthal sense;
- the presence of situations where we witnessed horizontal or vertical mathematization;
- contribution given by estimation and approximation procedures.

We discuss two aspects that have characterized our study, that of the link with out-of-school experience, which will be discussed in the next paragraph, and that

which concerns the use of estimation and approximation processes, although other aspects would certainly merit attention.

The use of the procedures of estimation and approximation allowed the children to argue more freely, to think and concentrate on the mathematical concepts rather than on the pure and simple computation that might distract from the final objectives. The surprising thing is that the actual possibility to be 'free to choose' had baffled and alarmed the majority of the students, mainly those of the Asiago school, who were used to work with well defined schemes and to expect and find a unique solution and the result exactness.<sup>7</sup> At first the fact that one can do the operations also when the numbers were not exact had moreover aroused much perplexity in the children, and also that it was possible to find approximations for whole numbers and not just decimals. The first resistance encountered was that of "the numbers can't be changed". The children were in fact strongly anchored in the structure of the natural numbers, for which they know the operations and the relevant algorithms, for which they asked themselves "why" they should approximate the calculations when they know how to do it in the exact way.

The great contribution made by the process of estimation, however, was that of helping the students to first guess and then later formalise the "rule of the decimal point", to realise that is the algorithm for multiplying out decimal numbers. The comparison between the approximate answers and those obtained by means of the chosen algorithm decisively helped the children to put the decimal point in the correct place. The use of the method of estimation allowed the pupils to predict the results and above all lead to a discussion about their own work, whenever the result obtained from the procedure used was not compatible with the predictions previously held. Thanks to this new way of proceeding the children have had the possibility to choose the strategies most appropriate to the particular possibilities.

During the experience, since the available time had to be devoted to the reaching of the main objective, and considering also the degree of education, not all of the problems connected to approximation, some of them very delicate, (allowable error margin - depending on the context- approximation interval, error propagation, etc.) have been dealt with, that in any case had emerged. In some case it would have been useful to set limits to the allowable approximations and underline that getting to far from starting values can lead to an unacceptable result in a special context. We deem it possible, however, to introduce a whole experience on these topics in a common curriculum in a more comprehensive way and connecting it to other topics such as practical measurements, estimation of results from operations such as division, etc.

In the course of a whole-class discussion it was then revealed how some knowledge could be only formal, lacking in any significance, connected only to a mechanical process, not aware, of any algorithm; thus it is possible to recall some of their aspects in filling up the emptiness 'not seen'. Indeed, children although being capable of 'solving equivalences', were surprised when

noticing that the result [in this case the price of products] was the same when using them [that is weights are expressed in kilograms, or hectograms or grams, using the relevant prices]; they have thus proved they did not understand what making an equivalence means; they have had the opportunity to use this tool and maybe for the first time to understand its meaning; 'to touch by their hands' the consequences of this tool lead them to really understand the concept.

The method used in these experiences allows us to go back and forth along the paths of knowledge, making use of different instruments, belonging to distinct sectors. Thus it is possible to systematise and consolidate concepts, making clear what is not understood and filling up the eventual gaps. Going in this direction old knowledge is revisited that becomes attached to the new, working at the same time in the past and in the future, thus *retrospective learning* is favoured, that makes it more difficult to forget what has been learnt.

### 6. On the relationship between mathematics and reality

A significant example of the relationships between mathematics and reality, or between learned matheracy and spontaneous matheracy in the D'Ambrosio sense, is provided by the children of the Asiago school. Often in their reflections, either in the individual protocols or in the collective discussions, the children have made reference to tenths, hundredths and thousandths, in reading and translating a decimal number, also when it referred to a measure of weight, that is kilograms, hectograms and so on. When this group of lessons was planned, a unit of weight was chosen to be used because we wished to retain a more concrete idea, which was easier to understand and to use than a 'pure' decimal notation. The initial preference of this class for a more 'abstract' decimal notation seemed to contradict our expectation of the fact that the link of out-of-school experience should be able to help us understand better mathematical concepts. We have then verified that the initial preference for decimal nomenclature was due to the fact that Asiago is a mountainous resort and the pupils of that class go skiing and are used to competing; they therefore are versed in a concrete, familiar image of decimal writing, determined from the temporal scansion. Therefore, in agreement with ethnomathematical perspective, we deem that it is a task for the teacher to know, at the end of being able to profitably take account of the teaching, the life experienced by the pupil; mathematics teachers should be prepared to investigate mathematical ideas and practices of their pupils and to look for ways to incorporate their findings into their own teaching, as a starting point for mathematical activities in the classroom.

The link with out-of-school experience allowed by the use of receipts enabled children to monitor their own interferences, checking the exactness of their answers. When they found that a kilo of bread could cost two million lire they immediately commented "I must have been wrong, it gives too a big result". This link had played a vital role for other reasons too. During the

experience the misunderstanding that multiplication always produces a larger result than the factors has been removed. Before this experience pupils tended to believe that multiplication meant increase, an idea still present after the introduction of decimal numbers.<sup>8</sup> When at the end of the experience, in the VI session, it was asked whether to multiply always means obtaining higher results, the answer was "it depends on numbers", followed by the reason of that. The introduction of multiplication between decimal numbers, within the receipt context, as the operation that establishes the price once the weight and the unit price are known, lead pupils to reflect on this consideration and to see numbers as possible representation of weights or prices and hence to refer to everyday elements while reasoning. They have hence made reference to the real world and to the common sense saying or reminding that "if you buy less than a kilo you will spend less than the kilo price, if you buy more than a kilo the price will be higher". We deem it would have been difficult to reach the same result simply by working through pure numbers.

On the other hand mathematics and reality are certainly not the same thing. Mainly in the higher degrees of education, the teacher has to try to overcome the limits of the simplification of mathematics that is embedded in real world situations to grasp the special features of this discipline [exactness, abstraction, generalization, formalization, ...].<sup>9</sup> During our experiences there are occasions that allow us to emphasise this fact, compatible with the degree of learning of the pupil. More often is it noted that the results obtained from the students and those present in the receipts differ and that the motive for this is to bring us back to that fact that reality and mathematics are different worlds. They enter deeply into each other but they are governed by different laws and principles. A mathematical result could be a number with a decimal point even if it indicates a price, whereas this does not happen in the world that we frequent, given the actual Italian monetary system [actually the introduction of a unique European coin, the Euro, is rapidly changing the situation]. It is therefore underlined that mathematics is dominated by precision, whereas often reality is the fruit of compromises determined by various factors, often of a practical nature. Although mathematics gives its language, its symbolism to real world situations, in the two contexts it expresses itself in different ways. The children are aware that when they go to the bakery they never ask for "0,478 kilos of bread" [children answered during a collective discussion that cannot happen since "there is too much of mathematics"!] but rather "half a kilo of bread" or "ten rolls".

If it is true that in our experiences the first attempts occurred inside real life and that this has provided good hints for the various types of mathematization and has allowed the children good control of inferences and the results obtained, it is just as true that it is necessary to abandon the link of real world, once the necessary supports are acquired, to reach abstraction and be able to work more freely with numbers. In effect once the child has become secure and has absorbed the appropriate knowledge, this enables them to move to a level of greater depth [which becomes 'common sense' of a

superior order] and they no longer need a concrete basis [or rather it is better to say a 'common sense' of an inferior order] to support their own elaborations. It is important to note that this qualitative leap varies from child to child, it is a route narrowly personal. This makes the task of the teacher particularly complex, but also more interesting and stimulating.

### 7. Some concluding remarks

As we have noticed thanks to the reported examples, the use of the cultural artifacts in mathematics education can be articulated in various stages.

In a first stage, the cultural artifacts represent indeed the out-of-school reality; children can be asked to simply recognize the mathematical facts that are embedded and codified in the artifacts.

In a second stage they can be asked to interpret and reflect on the mathematical facts, both in themselves and in connection with real world situations.

In a third stage children can be asked to put mathematical facts in relation, make conjectures about procedures, notice properties.

During the phase where cultural artifacts are deprived of some information [like in some tickets] they lose their fixed structure and no longer faithfully represent out-of-school reality, although they are strongly tied with real world situations. Now artifacts become more explicitly tools of mediation and integration between in and out-of-school knowledge and experiences, between in and out-of-school mathematics; so they can constitute a didactic interface between the two different contexts and, if correctly used, can create new mathematical goals, thus becoming *real mathematizing tools*.<sup>10</sup>

In this new role, the cultural artifact can be used to introduce new mathematical knowledge through a '*anticipatory learning*' process and favour '*vertical mathematization*' processes without leaving the situation behind.

According to our experience when children are confronted with this type of activity they also exhibit flexibility in their reasoning processes, by exploring, comparing and selecting among different strategies; these strategies are sensible to the context and the number quantities involved, and sometimes they are close to the procedures emerging in the out-of-school mathematics practice, see Bonotto (1999a). Indeed in our experiences the differences between in and out-of-school practices pointed e.g. by Masingila et al. 1996, (goals of the activity, conceptual understanding, and flexibility in dealing with constraints), have been narrowed.

Furthermore we think that activated procedures are mastered on the long term and can hence become part of the student's cultural heritage thus being re-constructed more easily in case they are forgotten. In fact, when explication and reflection are done on one's own reasoning the used procedures are consciously acquired.

We do not want to suggest that our experiences are a prototype for all the classroom activities related to mathematics. In educational practice, this type of experiences must be joined by more traditional activities, of reinforcement, of computation, standardized exercises.

"A cherished antinomy in teaching and learning mathematics is putting on one side of a deep gorge such noble ideas as insight, understanding, thinking, and on the other side such base things as rote, routine, drill, memorising, algorithms. If I were malicious, I would add another pair of opposites: theory versus practice, suggesting that learning by insight is a noble theory while base practice is learning by rote and memorisation. However, it is not that simple, and it has never been so ... firstly because the question is not which side of the gorge to choose but rather to bridge it by the learning process that I called schematising and formalising", Freudenthal (1991).

We do believe that, given their paradigmatic value, by enacting some of these experiences in classroom, the children are offered an opportunity to change their attitudes towards school mathematics. We believe that immersing students in such situations provides a site for them to deepen and broaden their understanding of the scope and usefulness of mathematics as well as learning ways of thinking mathematically that are supported by mathematizing situations. As we have already had occasion to emphasize, the usefulness and pervasive character of mathematics are merely two of its many facets and can not by themselves capture its very special character, relevance, and cultural value; nonetheless we deem that these two elements can be usefully exploited from the teaching point of view because they can change the common behaviour and attitude held both by teachers and pupils. The following letter, written of their own free will by two pupils, after a session of one of our experiences, witnesses that.

"Today I have had a lot of fun but most of all I think I have understood all of the questions. Hopefully next Wednesday we will learn even more interesting things that will be of great help when we grow up ... Thank you a lot for your patience. Thanks from Iole and Maria Elisa".

But this letter witnesses also that these practices involve children's intentions and attention, give meaning to mathematical symbols and rules introduced and favour personal understanding.

Furthermore in this way we can design better opportunities for children to develop mathematical knowledge that is wider than what they would develop outside of schools, but that preserves the focus on meaning found in everyday situations.

But is there a reverse of the coin, if the word 'reverse' can be used? The use of these artifacts is not easy or, in any case, is not of easy implementation for the teacher that has also to try to modify his/her attitude to mathematics and that is influenced by the way he/she has learned it, see Bonotto (1999a). Future mathematics teachers should themselves be prepared

- a) to see mathematics incorporated into real world,<sup>11</sup>
- b) to investigate the mathematical ideas and practices of the cultural, ethnic, linguistic communities of their pupils, and
- c) to look for ways to incorporate into the curriculum elements belonging to the sociocultural environment of the pupils, as a starting point for mathematical activities in the classroom.

Furthermore also teachers' beliefs about what counts as mathematics (and the value accorded to different kind of mathematics) must be reconsidered (Civil 1998).

Finally the teacher has to be ready *to create* and manage *open* situations, that are continuously *transforming* and of which he/she cannot foresee the final evolution or result.<sup>12</sup> As a matter of fact, these situations are sensitive to the social interactions that are established, to the students' attitudes, reactions, their ability to ask questions, to find links between school and extra-school knowledge; hence the teacher has to be able to modify on the way the contents objectives of the lesson. The teacher has to be and to feel very strong and qualified both on the mathematical contents and on the educational objectives that are potentially contained in these artifacts. In this way the class cannot be prepared in advance in all of its aspects, nor from above; it should rather plan for various 'branches' to be then drawn together through a process whose management is quite hard.

The teacher must not forget the moment of the *institutionalization* of knowledge being constructed together with the pupils; it has to be a shared moment so that these maybe interesting, stimulating and involving activities can be duly finalized to understanding of the underlying mathematical structures or processes and their potential generalisability.

Many teachers that have already experimented with our proposals have realized that through these experiences they cannot only stimulate the curiosity of their pupils but also their own curiosity and turn pupils and themselves from passive to active elements. It has to be underlined that teachers that have tried our proposals, besides their ability to re-think their role, have proved they still had a remarkable intellectual and scientific curiosity. This has enabled them to focus on both new educational and new mathematical goals so as to become researchers; in other words, school has become for teachers too a learning community (Bonotto 1999a).

## 8. Acknowledgments

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## 9. Footnotes

<sup>1</sup> In these studies, we analyzed bugs and difficulties in the mastery of the decimal number meaning, of the relationship between the fraction and the decimal representations, in ordering decimal numbers.

<sup>2</sup> At present we are analyzing the results of an experiment on measurement, in particular on the methods used for calculation the area of a rectangle, which confirms all of this.

<sup>3</sup> We use and will use the word "reality" in its "naive" meaning, linked to the ordinary use without approaching philosophical issues; but we wish to underline its double indexation referring to space and time parameters; cultural artifacts or conventions currently used in Italy were unthinkable thirty years ago and still are unthinkable in many cultural contexts distant from our culture. The school must take into consideration the link with the socio-cultural, productive and technological environment to be able to seize and understand ever changing characteristics.

<sup>4</sup> 'Mathematics' is understood here as academic mathematics, that is, the mathematics practiced by a particular cultural group, namely the professional mathematicians.

<sup>5</sup> The NTCM Standards 2000 also emphasize this theme. "All students should acquire strategies for estimating in computational situations and the inclination to judge the

reasonableness of numerical data, included computed results. Ability and inclination to estimate depends on understanding numbers - their size, position in the number system, and equivalent forms - and on the effect of operating on those numbers ... Estimation can be used to answer a question directly ... or used to evaluate the reasonableness of an answer resulting from paper and pencil or calculator computation", and also "In grades 3-5, all students should: ... develop and use computational estimation strategies based on an understanding of number concepts, properties and relationships... Students will develop computational estimation strategies at these levels and begin to regularly assess the reasonableness of problems and solutions".

<sup>6</sup> For some protocols and collective discussions see Bonotto 1999b.

<sup>7</sup> Pupils of the Asiago school were used to oral collective discussions, but not to write down their personal remarks, their reasonings, the used strategies and, more generally, they were not used to 'write about mathematics'. Their protocols were very schematic and contained only calculations; however, on an oral demand, they were able to explain the way they had been working. Pupils of the Trebaseleghe school on the contrary were used to write about mathematics and to recognise mathematical events being present in reality since they have already had experiences with other cultural artifacts; however, in both classrooms, from a mathematical point of view, the same types of reasoning and of used procedures have emerged.

<sup>8</sup> Several researchers have closely looked at the influence of the "multiplication makes bigger, division makes smaller" misconception in relation to the solution of word problems involving decimal numbers (smaller than 1), see e.g. Verschaffel, L.; Corte, E. de; Coillie, V. van (1988).

<sup>9</sup> In out-of-school mathematics practice, persons may generalize procedures within a context but may not be able to generalize to another context since problems tend to be context specific (Masingila et al., 1996). Generalization, which is an important goal in school mathematics, is not usually a goal in out-of-school mathematics practice.

<sup>10</sup> As to the analysis of the triple role of the ticket, *as utlural artifact, as mental organizer and as mathematizing tool*, see Basso & Bonotto 1996.

<sup>11</sup> It is worth saying that the artifacts used are as related to mathematics as one is able to find these relationships.

<sup>12</sup> We believe that both the students and the teachers should be left with the feeling that the teaching matter arose while teaching, that it was born during the lesson.

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