HARDY INEQUALITY OF FRACTIONAL ORDER

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This paper is dedicated to Professor Josip E. Pečarić

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ABSTRACT. We prove optimality of power-type weights in the Hardy inequality of fractional order.

1. INTRODUCTION AND THE MAIN RESULT

In [3] the following theorem was proved.

**Theorem 1.1.** Let $1 \leq p < \infty$, $\delta \in (0, 1) \cup (1, p)$ and $u$ be a locally integrable function on $[0, \infty)$. Let

(i) either $0 < \delta < 1$ and \( \lim_{t \to \infty} \frac{1}{t} \int_0^t u = 0 \),

(ii) or $1 < \delta < p$ and \( \lim_{t \to 0^+} \frac{1}{t} \int_0^t u = 0 \).

Then

\[
\int_0^\infty |u(x)|^p x^{-\delta} \, dx \leq C \int_0^\infty \int_0^\infty \frac{|u(x) - u(y)|^p}{|x - y|^\delta + 1} \, dx \, dy,
\]

where $C = (1 + p/|\delta - 1|)^p/2$.

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It is known that the restriction \( \delta \in (0, 1) \cup (1, p) \) is essential. Indeed, if either \( \delta \leq 0 \) or \( \delta \geq p \), then the integral on the right-hand side of (1.1) diverges for each nonzero function \( u \in C^\infty_0(0, \infty) \). If \( p > 1 \) and \( \delta = 1 \), then there is no finite constant \( C \) such that inequality (1.1) holds for all functions in question. Indeed, inserting the functions 

\[
u_\varepsilon(t) = \frac{t - \varepsilon}{\varepsilon} \chi_{[\varepsilon, 2\varepsilon)}(t) + \chi_{[2\varepsilon, 1/2)}(t) + 2(1 - t)\chi_{(1/2, 1)}(t)\]

into (1.1) and letting \( \varepsilon \to 0_+ \), we obtain that the constant \( C \to \infty \). (See [3, Remark 6].) Here the symbol \( \chi_I \) stands for the characteristic function of an interval \( I \subset \mathbb{R} \).

The aim of this paper is to show that power-type weights in inequality (1.1) are optimally chosen. This follows from the next result.

**Theorem 1.2.** Let \( 1 \leq p < \infty \). Suppose that \( \delta \in (0, 1) \cup (1, p) \), \( \eta \in (0, p) \) and there is a positive constant \( C \) such that the inequality

\[
\int_0^\infty |u(x)|^p x^{-\delta} \, dx \leq C \int_0^\infty \int_0^\infty \frac{|u(x) - u(y)|^p}{|x - y|^\eta+1} \, dx \, dy \tag{1.2}
\]

holds for all locally integrable functions \( u \) satisfying one of conditions (i), (ii) of Theorem 1.1. Then \( \eta = \delta \).

The proof of Theorem 1.2 is based on some ideas developed in [1] and [2].

To prove Theorem 1.2 we need several lemmas.

**Lemma 2.1.** Let \( 0 < p < \infty \) and \( w \) be a measurable nonnegative even function. Then

\[
\int_0^\infty \int_0^\infty |g(x) - g(y)|^p \, w(x - y) \, dx \, dy = 2 \int_0^\infty \left( \int_0^\infty |g(y + h) - g(y)|^p \, dy \right) w(h) \, dh, \tag{2.1}
\]

provided that the left-hand side of the equality makes sense.

**Proof.** Using the change of variables \( x = y + h \) in the inner integral and applying the Fubini theorem, we obtain

\[
\int_0^\infty \int_0^\infty |g(x) - g(y)|^p \, w(x - y) \, dx \, dy = \int_0^\infty \left( \int_{-y}^\infty |g(y + h) - g(y)|^p \, w(h) \, dh \right) \, dy
\]

\[
= \int_0^\infty \left( \int_0^\infty |g(y + h) - g(y)|^p \, dy \right) w(h) \, dh
\]

\[
+ \int_{-\infty}^0 \left( \int_h^\infty |g(y + h) - g(y)|^p \, dy \right) w(h) \, dh. \tag{2.2}
\]
In the second term we replace \( h \) by \( k \) and \( y \) by \( z \), then we make two changes of variables \( h = -k \) and \( z - h = y \) and use the fact that \( w(-h) = w(h) \), to arrive at

\[
\int_{-\infty}^{0} \left( \int_{-h}^{\infty} |g(y + h) - g(y)|^p dy \right) w(h) \, dh = \int_{0}^{\infty} \left( \int_{0}^{\infty} |g(y + h) - g(y)|^p dy \right) w(h) \, dh.
\]

Together with (2.2), it gives (2.1).

In what follows we write \( A \lesssim B \) (or \( A \gtrsim B \)) if \( A \leq cB \) (or \( cA \geq B \)) for some positive constant \( c \) independent of appropriate quantities involved in the expressions \( A \) and \( B \). For \( p \in [1, \infty] \), the conjugate number \( p' \) is defined by

\[
\frac{1}{p} + \frac{1}{p'} = 1 \quad \text{with the convention that} \quad \frac{1}{\infty} = 0.
\]

Lemma 2.2. Let \( w \) be a measurable nonnegative function, let \( p \in [1, \infty) \), \( \alpha \in (1, \infty) \) and \( \alpha' := \alpha / (\alpha - 1) \). Then

\[
\int_{0}^{\infty} \left( \int_{0}^{\infty} |g(y + h) - g(y)|^p dy \right) w(h) \, dh 
\lesssim \int_{0}^{\infty} \left( \int_{0}^{2h} |g(y)|^p dy \right) w(h) \, dh + \int_{0}^{\infty} \left( \int_{h}^{\infty} |g'(y)|^p dy \right) h^p w(h) \, dh
\tag{2.3}
\]

and

\[
\int_{0}^{\infty} \left( \int_{0}^{\infty} |g(y + h) - g(y)|^p dy \right) w(h) \, dh 
\lesssim \int_{0}^{\infty} \left( \int_{0}^{h} \left( \int_{y}^{\infty} |g'(\tau)|^\alpha d\tau \right)^{p/\alpha} dy \right) h^{p/\alpha'} w(h) \, dh 
+ \int_{0}^{\infty} \left( \int_{h}^{\infty} |g'(y)|^p dy \right) h^p w(h) \, dh
\tag{2.4}
\]

for all locally absolutely continuous functions \( g \) on \([0, \infty)\).

Proof. Let \( h > 0 \). Then

\[
\int_{0}^{\infty} |g(y + h) - g(y)|^p dy 
= \int_{0}^{h} |g(y + h) - g(y)|^p dy + \int_{h}^{\infty} |g(y + h) - g(y)|^p dy 
=: N_1(h) + N_2(h).
\tag{2.5}
\]

First, we estimate \( N_1 \):

\[
N_1(h) = \int_{0}^{h} |g(y + h) - g(y)|^p dy 
\lesssim \int_{0}^{h} |g(y + h)|^p dy + \int_{0}^{h} |g(y)|^p dy = \int_{0}^{2h} |g(y)|^p dy.
\tag{2.6}
\]
For the alternative estimate, we use the Hölder inequality with the exponents $\alpha$ and $\alpha'$ to get, for all $y > 0$,

$$
|g(y + h) - g(y)| = \left| \int_y^{y+h} g'(\tau) \, d\tau \right|
\leq h^{1/\alpha'} \left( \int_y^{y+h} |g'(\tau)|^\alpha \, d\tau \right)^{1/\alpha} \leq h^{1/\alpha'} \left( \int_y^\infty |g'(\tau)|^\alpha \, d\tau \right)^{1/\alpha}.
$$

Consequently,

$$
N_1(h) \leq h^{p/\alpha'} \int_0^h \left( \int_0^\infty |g'(\tau)|^\alpha \, d\tau \right)^{p/\alpha} \, dy. \quad (2.7)
$$

Now, we estimate the second term $N_2$. We use the estimate $|g(y + h) - g(y)| \leq h \int_0^1 |g'(y + \tau h)| \, d\tau$, then the Hölder inequality, the Fubini theorem and the change of variables $y + \tau h = z$ to obtain

$$
N_2(h) = \int_h^\infty |g(y + h) - g(y)|^p \, dy \leq \int_h^\infty h^p \left( \int_0^1 |g'(y + \tau h)| \, d\tau \right)^p \, dy
\leq h^p \int_h^\infty \left( \int_0^1 |g'(y + \tau h)| \, d\tau \right) \, dy = h^p \int_0^1 \left( \int_{h(1+\tau)}^\infty |g'(z)|^p \, dz \right) \, d\tau
\leq h^p \int_0^1 \left( \int_h^\infty |g'(z)|^p \, dz \right) \, d\tau = h^p \int_h^\infty |g'(y)|^p \, dy. \quad (2.8)
$$

Estimate $(2.3)$ follows from $(2.5)$, $(2.6)$ and $(2.8)$, estimate $(2.4)$ is a consequence of $(2.5)$, $(2.7)$ and $(2.8)$.

Take $R \in (0, \infty)$ and put

$$
u_R(x) := \varphi_R(x) \int_0^x \chi_{(R,2R)}(t) \, t^{-2} \, dt, \quad x \in (0, \infty), \quad (2.9)
$$

where $\varphi_R \in C^\infty(0, \infty)$ is a cut-off function such that

$$
\text{supp} \varphi_R \subset [0, 4R], \quad 0 \leq \varphi_R \leq 1,
\varphi_R(x) = 1 \text{ for } x \in [0, 3R], \quad \varphi_R(x) = 0 \text{ for } x \in [4R, \infty],
|\varphi'_R| \lesssim R^{-1} \chi_{[3R,4R]}.
$$

Obviously,

$$
\lim_{t \to \infty} \frac{1}{t} \int_0^t \nu_R(x) \, dx = 0 \quad \text{and} \quad \lim_{t \to 0^+} \frac{1}{t} \int_0^t \nu_R(x) \, dx = 0, \quad (2.10)
|\nu_R'(x)| \lesssim R^{-2} \chi_{[3R,4R]}(x) + \chi_{(R,2R)}(x) x^{-2} \quad \text{for all } x \in (0, \infty). \quad (2.11)
$$

**Lemma 2.3.** Let $1 \leq p < \infty$ and $\delta \in (0, 1) \cup (1, p)$. Assume that $\nu_R$ is given by $(2.9)$. Then

$$
\int_0^\infty |\nu_R(x)|^p x^{-\delta} \, dx \gtrsim R^{1-p-\delta} \quad \text{for all } R \in (0, \infty). \quad (2.12)
$$
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Proof. Since

\[ u_R(x) = \begin{cases} 
0 & \text{if } x \in [0, R], \\
1/R - 1/x & \text{if } x \in (R, 2R], \\
1/(2R) & \text{if } x \in (2R, 3R), 
\end{cases} \quad (2.13) \]

we obtain

\[ \int_0^\infty |u_R(x)|^p x^{-\delta} \, dx \geq \int_{2R}^{3R} \left( \frac{1}{(2R)} \right)^p x^{-\delta} \, dx \approx R^{1-p-\delta} \]

and (2.12) is verified. \qed

Lemma 2.4. Suppose that \( 1 \leq p < \infty \) and \( \eta \in (0, p) \). Let \( u_R \) be given by (2.9).

Then

\[ \int_0^\infty \int_0^\infty \frac{|u_R(x) - u_R(y)|^p}{|x - y|^\eta+1} \, dx \, dy \lesssim R^{1-p-\eta} \quad \text{for all } R \in (0, \infty). \quad (2.14) \]

Proof. We start with some auxiliary estimates. If \( \beta \in [1, \infty) \), then, by (2.11),

\[ \int_h^\infty |u_R'(t)|^\beta \, dt \lesssim \begin{cases} 
R^{1-2\beta} & \text{if } h \in [0, 4R], \\
0 & \text{if } h \in (4R, \infty). 
\end{cases} \quad (2.15) \]

Using this estimate with \( \beta = p \), the facts that \( p \in [1, \infty) \) and \( \eta \in (0, p) \), we obtain

\[ \int_0^\infty \left( \int_h^\infty |u_R'(t)|^p \, dt \right) h^{p-\eta-1} \, dh \lesssim \int_0^\infty R^{1-2p} \chi_{(0,4R]}(h) h^{p-\eta-1} \, dh 
= R^{1-2p} \int_0^{4R} h^{p-\eta-1} \, dh \approx R^{1-p-\eta} \quad \text{for all } R \in (0, \infty). \quad (2.16) \]

If \( \eta \in (1, p) \), we use (2.13) to get

\[ \int_0^\infty \left( \int_0^{2h} |u_R(t)|^p \, dt \right) h^{-\eta-1} \, dh \lesssim \int_0^{2h} \left( \int_0^{R/2} R^{-p} \, dt \right) h^{-\eta-1} \, dh 
\approx R^{-p} \int_0^{2h} h^{-\eta} \, dh \approx R^{1-p-\eta} \quad \text{for all } R \in (0, \infty). \quad (2.17) \]

Now, assume that \( \eta \in (0, 1) \) and \( \alpha \in (1, \infty) \) is such that \( \alpha' > p/\eta \). Then

\[ 0 > p/\alpha' - \eta > -1. \quad (2.18) \]

Using (2.15) with \( \beta = \alpha \), we get

\[ \int_0^h \left( \int_y^\infty |u_R'(t)|^\alpha \, dt \right)^{p/\alpha} \, dy \lesssim \int_0^h \left( R^{1-2\alpha} \chi_{(0,4R]}(y) \right)^{p/\alpha} \, dy 
\leq R^{(1-2\alpha)p/\alpha} \min\{h, 4R\} \quad \text{for all } h, R \in (0, \infty). \quad (2.19) \]
Thus, if $\eta \in (0, 1]$, then \((2.19)\) and \((2.18)\) imply that
\[
\int_0^\infty \left( \int_0^h \left( \int_y^\infty \left| u'_R(\tau) \right|^\alpha d\tau \right)^{p/\alpha} dy \right) h^{p/\alpha'-\eta - 1} dh \\
\lesssim R^{(1-2\alpha)p/\alpha} \int_0^{4R} h^{p/\alpha'-\eta} dh + R^{(1-2\alpha)p/\alpha+1} \int_4^{\infty} h^{p/\alpha'-\eta - 1} dh \\
\approx R^{1-p-\eta} \quad \text{for all } R \in (0, \infty).
\]

Now, we are able to prove \((2.14)\). To this end, we distinguish two cases.

(i) Let $\eta \in (1, p)$. Then, \((2.1)\) with $\omega(h) := |h|^{-\eta-1}$, \((2.3)\), \((2.17)\) and \((2.16)\) yield
\[
\int_0^\infty \int_0^\infty \frac{|u_R(x) - u_R(y)|^p}{|x-y|^{\eta+1}} dx dy \\
\lesssim \int_0^\infty \left( \int_0^{2h} |u_R(y)|^p dy \right) h^{-\eta-1} dh + \int_0^\infty \left( \int_y^\infty |u'_R(y)|^p dy \right) h^{p-\eta-1} dh \\
\lesssim R^{1-p-\eta} \quad \text{for all } R \in (0, \infty).
\]

(ii) Let $\eta \in (0, 1]$. Choose $\alpha \in (1, \infty)$ such that $\alpha' > p/\eta$. Then, \((2.1)\) with $\omega(h) := |h|^{-\eta-1}$, \((2.4)\), \((2.20)\) and \((2.16)\) imply that
\[
\int_0^\infty \int_0^\infty \frac{|u_R(x) - u_R(y)|^p}{|x-y|^{\eta+1}} dx dy \\
\lesssim \int_0^\infty \left( \int_0^h \left( \int_y^\infty |u'_R(\tau)|^\alpha d\tau \right)^{p/\alpha} dy \right) h^{p/\alpha'-1-\eta} dh \\
+ \int_0^\infty \left( \int_h^\infty |u'_R(y)|^p dy \right) h^{p-1-\eta} dh \lesssim R^{1-p-\eta} \quad \text{for all } R \in (0, \infty).
\]

Now, we can prove Theorem 1.2

Proof of Theorem 1.2

By \((2.10)\), the test function $u_R$ satisfies both of conditions \([i], [ii]\) of Theorem 1.1. We obtain from \((1.2)\), \((2.12)\) and \((2.14)\) that
\[
R^{1-p-\delta} \lesssim C R^{1-p-\eta} \quad \text{for all } R \in (0, \infty).
\]

Since the constant $C$ is independent of $R$, the last estimate implies that $\eta = \delta$. \qed

References

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