ISOMETRIC ADDITIVE MAPPINGS IN GENERALIZED QUASI-BANACH SPACES

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ABSTRACT. In this paper, we prove the generalized Hyers–Ulam stability of the isometric additive mappings in generalized quasi-Banach spaces, and prove the generalized Hyers–Ulam stability of the isometric additive mappings in generalized $p$-Banach spaces.

1. INTRODUCTION AND PRELIMINARIES

We recall some basic facts concerning quasi-Banach spaces and some preliminary results.

**Definition 1.1.** ([5, 43]) Let $X$ be a linear space. A *quasi-norm* is a real-valued function on $X$ satisfying the following:

1. $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.
2. $\|\lambda x\| = |\lambda| \cdot \|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$.
3. There is a constant $K \geq 1$ such that $\|x + y\| \leq K(\|x\| + \|y\|)$ for all $x, y \in X$.

The pair $(X, \| \cdot \|)$ is called a *quasi-normed space* if $\| \cdot \|$ is a quasi-norm on $X$.

A quasi-Banach space is a complete quasi-normed space.

A quasi-norm $\| \cdot \|$ is called a *$p$-norm* ($0 < p \leq 1$) if

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p$$

for all $x, y \in X$. In this case, a quasi-Banach space is called a *$p$-Banach space*.

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Given a $p$-norm, the formula $d(x, y) := \|x - y\|^p$ gives us a translation invariant metric on $X$. By the Aoki-Rolewicz theorem [43] (see also [5]), each quasi-norm is equivalent to some $p$-norm. Since it is much easier to work with $p$-norms than quasi-norms, henceforth we restrict our attention mainly to $p$-norms.

In [26], the author generalized the concept of quasi-normed spaces.

**Definition 1.2.** Let $X$ be a linear space. A **generalized quasi-norm** is a real-valued function on $X$ satisfying the following:

1. $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.
2. $\|\lambda x\| = |\lambda| \cdot \|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$.
3. There is a constant $K \geq 1$ such that $\|\sum_{j=1}^{\infty} x_j\| \leq \sum_{j=1}^{\infty} K\|x_j\|$ for all $x_1, x_2, \ldots \in X$.

The pair $(X, \|\cdot\|)$ is called a **generalized quasi-normed space** if $\|\cdot\|$ is a generalized quasi-norm on $X$. The smallest possible $K$ is called the **modulus of concavity** of $\|\cdot\|$.

A **generalized quasi-Banach space** is a complete generalized quasi-normed space.

A generalized quasi-norm $\|\cdot\|$ is called a **$p$-norm** ($0 < p \leq 1$) if

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p$$

for all $x, y \in X$. In this case, a generalized quasi-Banach space is called a **generalized $p$-Banach space**.

Let $X$ and $Y$ be metric spaces. A mapping $f : X \rightarrow Y$ is called an isometry if $f$ satisfies

$$d_Y(f(x), f(y)) = d_X(x, y)$$

for all $x, y \in X$, where $d_X(\cdot, \cdot)$ and $d_Y(\cdot, \cdot)$ denote the metrics in the spaces $X$ and $Y$, respectively. For some fixed number $r > 0$, suppose that $f$ preserves distance $r$; i.e., for all $x, y$ in $X$ with $d_X(x, y) = r$, we have $d_Y(f(x), f(y)) = r$. Then $r$ is called a conservative(or preserved) distance for the mapping $f$. Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be normed spaces. A mapping $L : X \rightarrow Y$ is called an isometry if $\|L(x) - L(y)\| = \|x - y\|$ for all $x, y \in X$. Aleksandrov [1] posed the following problem:

**Remark 1.3. Aleksandrov problem.** Examine whether the existence of a single conservative distance for some mapping $T$ implies that $T$ is an isometry.

The isometric problems have been investigated in several papers (see [3, 9, 12, 13, 19, 20, 21, 35, 39, 41, 42]).

The stability problem of functional equations originated from a question of S.M. Ulam [46] concerning the stability of group homomorphisms: Let $(G_1, \ast)$ be a group and let $(G_2, \circ, d)$ be a metric group with the metric $d(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta(\epsilon) > 0$ such that if a mapping $h : G_1 \rightarrow G_2$ satisfies the inequality

$$d(h(x \ast y), h(x) \circ h(y)) < \delta$$

for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \rightarrow G_2$ with

$$d(h(x), H(x)) < \epsilon$$
for all \( x \in G \)? If the answer is affirmative, we would say that the equation of homomorphism \( H(x + y) = H(x) \odot H(y) \) is stable. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is that how do the solutions of the inequality differ from those of the given functional equation?

D.H. Hyers [14] gave a first affirmative answer to the question of Ulam for Banach spaces. Let \( X \) and \( Y \) be Banach spaces. Assume that \( f : X \to Y \) satisfies
\[
\| f(x + y) - f(x) - f(y) \| \leq \varepsilon
\]
for all \( x, y \in X \) and some \( \varepsilon \geq 0 \). Then there exists a unique additive mapping \( T : X \to Y \) such that
\[
\| f(x) - T(x) \| \leq \varepsilon
\]
for all \( x \in X \).

Let \( X \) and \( Y \) be Banach spaces with norms \( \| \cdot \| \) and \( \| \cdot \| \), respectively. Consider \( f : X \to Y \) to be a mapping such that \( f(tx) \) is continuous in \( t \in \mathbb{R} \) for each fixed \( x \in X \). Th.M. Rassias [33] introduced the following inequality: Assume that there exist constants \( \theta \geq 0 \) and \( p \in [0, 1) \) such that
\[
\| f(x + y) - f(x) - f(y) \| \leq \theta (\| x \|^p + \| y \|^p)
\]
for all \( x, y \in X \). Th.M. Rassias [33] showed that there exists a unique \( \mathbb{R} \)-linear mapping \( T : X \to Y \) such that
\[
\| f(x) - T(x) \| \leq \frac{2\theta}{2 - 2^p} \| x \|^p
\]
for all \( x \in X \). The above inequality has provided a lot of influence in the development of what is now known as \textit{generalized Hyers–Ulam stability} of functional equations. Beginning around the year 1980 the topic of approximate homomorphisms, or the stability of the equation of homomorphism, was studied by a number of mathematicians. Gavruta [11] following Th.M. Rassias approach for the stability of the linear mapping between Banach spaces obtained a generalization of Th.M. Rassias’ Theorem. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [2, 4, 6, 7, 10, 11, 15, 16, 17, 18, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 34, 36, 37, 38, 40, 44]).

In this paper, we prove the generalized Hyers–Ulam stability of the isometric Cauchy mapping and the isometric Jensen mapping in generalized quasi-Banach spaces, and prove the generalized Hyers–Ulam stability of the isometric Cauchy mapping and the isometric Jensen mapping in generalized \( p \)-Banach spaces.

2. \textbf{Stability of the Isometric Additive Mappings in Generalized Quasi-Banach Spaces}

Throughout this section, assume that \( X \) is a generalized quasi-normed vector space with generalized quasi-norm \( \| \cdot \| \) and that \( Y \) is a generalized quasi-Banach space with generalized quasi-norm \( \| \cdot \| \). Let \( K \) be the modulus of concavity of \( \| \cdot \| \).
Theorem 2.1. Let \( r > 1 \) and \( \theta \) be positive real numbers, and let \( f : X \to Y \) be a mapping such that
\[
\|f(x + y) - f(x) - f(y)\| \leq \theta(||x||^r + ||y||^r), \tag{2.1}
\]
\[
\|f(x)|| - ||x|| \leq 2\theta||x||^r \tag{2.2}
\]
for all \( x, y \in X \). Then there exists a unique isometric Cauchy additive mapping \( A : X \to Y \) such that
\[
\|f(x) - A(x)\| \leq \frac{2K\theta}{2^r - 2}||x||^r \tag{2.3}
\]
for all \( x \in X \).

Proof. Letting \( y = x \) in (2.1), we get
\[
\|f(2x) - 2f(x)\| \leq 2\theta||x||^r \tag{2.4}
\]
for all \( x \in X \). So
\[
\|f(x) - 2f\left(\frac{x}{2}\right)\| \leq \frac{2\theta}{2^r}||x||^r
\]
for all \( x \in X \). Hence
\[
\|2^n f\left(\frac{x}{2^n}\right) - 2^n f\left(\frac{x}{2^n}\right)\| \leq \frac{2^n}{2^{rn}} \left(||x||^r + ||y||^r\right)
\]
for all \( x, y \in X \). It follows from (2.5) that the sequence \( \{2^n f\left(\frac{x}{2^n}\right)\} \) is a Cauchy sequence for all \( x \in X \). Since \( Y \) is complete, the sequence \( \{2^n f\left(\frac{x}{2^n}\right)\} \) converges. So one can define the mapping \( A : X \to Y \) by
\[
A(x) := \lim_{n \to \infty} 2^n f\left(\frac{x}{2^n}\right)
\]
for all \( x \in X \).

By (2.1),
\[
\|A(x + y) - A(x) - A(y)\| = \lim_{n \to \infty} 2^n\|f\left(\frac{x + y}{2^n}\right) - f\left(\frac{x}{2^n}\right) - f\left(\frac{y}{2^n}\right)\|
\]
\[
\leq \lim_{n \to \infty} \frac{2^n}{2^{rn}} \left(||x||^r + ||y||^r\right) = 0
\]
for all \( x, y \in X \). So
\[
A(x + y) = A(x) + A(y)
\]
for all \( x, y \in X \). Moreover, letting \( l = 0 \) and passing the limit \( m \to \infty \) in (2.5), we get (2.3).

Now, let \( A' : X \to Y \) be another Cauchy additive mapping satisfying (2.3). Then we have
\[
\|A(x) - A'(x)\| = 2^n\|A\left(\frac{x}{2^n}\right) - A'\left(\frac{x}{2^n}\right)\|
\]
\[
\leq 2^nK\left(||A\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)|| + ||A'\left(\frac{x}{2^n}\right) - f\left(\frac{x}{2^n}\right)||\right)
\]
\[
\leq \frac{2^{n+1}K^2\theta}{(2^r-2)2^{rn}}||x||^r,
\]
which tends to zero as \( n \to \infty \) for all \( x \in X \). So we can conclude that \( A(x) = A'(x) \) for all \( x \in X \). This proves the uniqueness of \( A \).

It follows from (2.2) that
\[
\left| \| 2^n f \left( \frac{x}{2^n} \right) \| - \| x \| \right| = 2^n \| f \left( \frac{x}{2^n} \right) \| - \| \frac{x}{2^n} \| \leq 2 \theta 2^n \| x \|^r,
\]
which tends to zero as \( n \to \infty \) for all \( x \in X \). So
\[
\| A(x) \| = \lim_{n \to \infty} \| 2^n f \left( \frac{x}{2^n} \right) \| = \| x \|
\]
for all \( x \in X \). Since \( A \) is additive,
\[
\| A(x) - A(y) \| = \| A(x - y) \| = \| x - y \|
\]
for all \( x, y \in X \), as desired. \( \square \)

**Theorem 2.2.** Let \( r < 1 \) and \( \theta \) be positive real numbers, and let \( f : X \to Y \) be a mapping satisfying (2.1) and (2.2). Then there exists a unique isometric Cauchy additive mapping \( A : X \to Y \) such that
\[
\| f(x) - A(x) \| \leq \frac{2K \theta}{2 - 2^r} \| x \|^r
\]
for all \( x \in X \).

**Proof.** It follows from (2.4) that
\[
\| f(x) - \frac{1}{2} f(2x) \| \leq \theta \| x \|^r
\]
for all \( x \in X \). So
\[
\frac{1}{2^l} f \left( 2^l x \right) - \frac{1}{2^m} f \left( 2^m x \right) \leq K \sum_{j=l}^{m-1} \frac{2^j \theta}{2^j} \| x \|^r
\]
(2.6)
for all nonnegative integers \( m \) and \( l \) with \( m > l \) and all \( x \in X \). It follows from (2.6) that the sequence \( \{ \frac{1}{2^n} f \left( 2^n x \right) \} \) is a Cauchy sequence for all \( x \in X \). Since \( Y \) is complete, the sequence \( \{ \frac{1}{2^n} f \left( 2^n x \right) \} \) converges. So one can define the mapping \( A : X \to Y \) by
\[
A(x) := \lim_{n \to \infty} \frac{1}{2^n} f \left( 2^n x \right)
\]
for all \( x \in X \).

The rest of the proof is similar to the proof of Theorem 2.1. \( \square \)

**Theorem 2.3.** Let \( r < 1 \) and \( \theta \) be positive real numbers, and let \( f : X \to Y \) be a mapping with \( f(0) = 0 \) satisfying (2.2) such that
\[
\| 2f \left( \frac{x + y}{2} \right) - f(x) - f(y) \| \leq \theta (\| x \|^r + \| y \|^r)
\]
(2.7)
for all \( x, y \in X \). Then there exists a unique isometric Jensen additive mapping \( A : X \to Y \) such that
\[
\| f(x) - A(x) \| \leq \frac{(3 + 3^r)K^2 \theta}{3 - 3^r} \| x \|^r
\]
for all \( x \in X \).
Proof. Letting \( y = -x \) in (2.7), we get
\[
\| - f(x) - f(-x) \| \leq 2\theta \| x \|^r
\]
for all \( x \in X \). Letting \( y = 3x \) and replacing \( x \) by \(-x \) in (2.7), we get
\[
\| 2f(x) - f(-x) - f(3x) \| \leq (3^r + 1)\theta \| x \|^r
\]
for all \( x \in X \). Thus
\[
\| 3f(x) - f(3x) \| \leq K(3^r + 3)\theta \| x \|^r
\]  
(2.8)
for all \( x \in X \). So
\[
\| \frac{1}{3^n} f(3^n x) - \frac{1}{3^m} f(3^m x) \| \leq K^2 \frac{3^r + 3}{3} \sum_{j=l}^{m-1} \frac{3^j \theta}{3^j} \| x \|^r
\]  
(2.9)
for all nonnegative integers \( m \) and \( l \) with \( m > l \) and all \( x \in X \). It follows from (2.9) that the sequence \( \{ \frac{1}{3^n} f(3^n x) \} \) is a Cauchy sequence for all \( x \in X \). Since \( Y \) is complete, the sequence \( \{ \frac{1}{3^n} f(3^n x) \} \) converges. So one can define the mapping \( A : X \to Y \) by
\[
A(x) := \lim_{n \to \infty} \frac{1}{3^n} f(3^n x)
\]
for all \( x \in X \).

The rest of the proof is similar to the proof of Theorem 2.1. \( \Box \)

**Theorem 2.4.** Let \( r > 1 \) and \( \theta \) be positive real numbers, and let \( f : X \to Y \) be a mapping with \( f(0) = 0 \) satisfying (2.7) and (2.2). Then there exists a unique isometric Jensen additive mapping \( A : X \to Y \) such that
\[
\| f(x) - A(x) \| \leq \frac{(3^r + 3)K^2\theta}{3^r - 3} \| x \|^r
\]
for all \( x \in X \).

Proof. It follows from (2.8) that
\[
\| f(x) - 3f(\frac{x}{3}) \| \leq \frac{K(3^r + 3)\theta}{3^r} \| x \|^r
\]
for all \( x \in X \). So
\[
\| 3^l f(\frac{x}{3^l}) - 3^m f(\frac{x}{3^m}) \| \leq K^2 \frac{3^r + 3}{3^r} \sum_{j=l}^{m-1} \frac{3^j \theta}{3^j} \| x \|^r
\]  
(2.10)
for all nonnegative integers \( m \) and \( l \) with \( m > l \) and all \( x \in X \). It follows from (2.10) that the sequence \( \{ 3^n f(\frac{x}{3^n}) \} \) is a Cauchy sequence for all \( x \in X \). Since \( Y \) is complete, the sequence \( \{ 3^n f(\frac{x}{3^n}) \} \) converges. So one can define the mapping \( A : X \to Y \) by
\[
A(x) := \lim_{n \to \infty} 3^n f(\frac{x}{3^n})
\]
for all \( x \in X \).

The rest of the proof is similar to the proof of Theorem 2.1. \( \Box \)
3. Stability of the isometric additive mappings in generalized \( p \)-Banach spaces

Throughout this section, assume that \( X \) is a generalized quasi-normed vector space with generalized quasi-norm \( \| \cdot \| \) and that \( Y \) is a generalized \( p \)-Banach space with generalized quasi-norm \( \| \cdot \| \).

The following two results except for isometries are given by Tabor [45]. The proofs of isometries are similar to the proof of Theorem 2.1.

**Theorem 3.1.** (45) Let \( r > 1 \) and \( \theta \) be positive real numbers, and let \( f : X \to Y \) be a mapping satisfying (2.1) and (2.2). Then there exists a unique isometric Cauchy additive mapping \( A : X \to Y \) such that

\[
\| f(x) - A(x) \| \leq \frac{2\theta}{(2^{pr} - 2^p)^{\frac{1}{p}}} \| x \|^{r}
\]

for all \( x \in X \).

**Remark 3.2.** The result for the case \( K = 1 \) in Theorem 2.1 is the same as the result for the case \( p = 1 \) in Theorem 3.1.

**Theorem 3.3.** (45) Let \( r < 1 \) and \( \theta \) be positive real numbers, and let \( f : X \to Y \) be a mapping satisfying (2.1) and (2.2). Then there exists a unique isometric Cauchy additive mapping \( A : X \to Y \) such that

\[
\| f(x) - A(x) \| \leq \frac{2\theta}{(2^p - 2^{pr})^{\frac{1}{p}}} \| x \|^{r}
\]

for all \( x \in X \).

**Remark 3.4.** The result for the case \( K = 1 \) in Theorem 2.2 is the same as the result for the case \( p = 1 \) in Theorem 3.3.

**Theorem 3.5.** Let \( r < 1 \) and \( \theta \) be positive real numbers, and let \( f : X \to Y \) be a mapping with \( f(0) = 0 \) satisfying (2.7) and (2.2). Then there exists a unique isometric Jensen additive mapping \( A : X \to Y \) such that

\[
\| f(x) - A(x) \| \leq \frac{K(3 + 3^r)\theta}{(3^p - 3^{pr})^{\frac{1}{p}}} \| x \|^{r}
\]

(3.1)

for all \( x \in X \).

**Proof.** It follows from (2.8) that

\[
\| f(x) - \frac{1}{3} f(3x) \| \leq \frac{K(3^r + 3)\theta}{3} \| x \|^{r}
\]

(3.2)

for all \( x \in X \). Since \( Y \) is a generalized \( p \)-Banach space,

\[
\| \frac{1}{3^j} f(3^j x) - \frac{1}{3^m} f(3^m x) \|^p \leq \sum_{j=l}^{m-1} \| \frac{1}{3^j} f(3^j x) - \frac{1}{3^{j+1}} f(3^{j+1} x) \|^p \leq \frac{K^p(3^r + 3)^p\theta^p}{3^p} \sum_{j=l}^{m-1} \frac{3^{pj}}{3^{pj}} \| x \|^{pr}
\]

(3.3)
for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in X$. It follows from (3.3) that the sequence \( \{ \frac{1}{3^n} f(3^n x) \} \) is a Cauchy sequence for all $x \in X$. Since $Y$ is complete, the sequence \( \{ \frac{1}{3^n} f(3^n x) \} \) converges. So one can define the mapping $A : X \to Y$ by

\[
A(x) := \lim_{n \to \infty} \frac{1}{3^n} f(3^n x)
\]

for all $x \in X$.

By (2.7),

\[
\|2A(\frac{x + y}{2}) - A(x) - A(y)\|
= \lim_{n \to \infty} \frac{1}{3^n} \|2f(3^n \cdot \frac{x + y}{2}) - f(3^n x) - f(3^n y)\|
\leq \lim_{n \to \infty} \frac{3^{rn}}{3^n} \theta(||x||^r + ||y||^r) = 0
\]

for all $x, y \in X$. So

\[
2A(\frac{x + y}{2}) = A(x) + A(y)
\]

for all $x, y \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.3), we get (3.1).

Now, let $A' : X \to Y$ be another Jensen additive mapping satisfying (3.1). Then we have

\[
\|A(x) - A'(x)\|^p = \frac{1}{3^{pn}} \|A(3^n x) - A'(3^n x)\|^p
\leq \frac{1}{3^{pn}} (\|A(3^n x) - f(3^n x)\|^p + \|A'(3^n x) - f(3^n x)\|^p)
\leq 2 \cdot \frac{3^{pn}}{3^{pn}} \cdot \frac{K^p(3^r + 3)^p \theta^p}{3^r - 3^r} ||x||^{pr},
\]

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $A(x) = A'(x)$ for all $x \in X$. This proves the uniqueness of $A$.

The rest of the proof is similar to the proof of Theorem 2.1. \(\square\)

**Remark 3.6.** The result for the case $K = 1$ in Theorem 2.3 is the same as the result for the case $p = 1$ in Theorem 3.5.

**Theorem 3.7.** Let $r > 1$ and $\theta$ be positive real numbers, and let $f : X \to Y$ be a mapping with $f(0) = 0$ satisfying (2.7) and (2.2). Then there exists a unique isometric Jensen additive mapping $A : X \to Y$ such that

\[
\|f(x) - A(x)\| \leq \frac{K(3^r + 3)\theta}{(3^r - 3^r)^{\frac{1}{r}}} ||x||^r
\]

for all $x \in X$.

**Proof.** It follows (3.2) that

\[
\|f(x) - 3f(\frac{x}{3})\| \leq \frac{K(3^r + 3)\theta}{3^r} ||x||^r
\]
for all \( x \in X \). Since \( Y \) is a generalized \( p \)-Banach space,
\[
\| 3^l f \left( \frac{x}{3^l} \right) - 3^m f \left( \frac{x}{3^m} \right) \|^p \leq \sum_{j=l}^{m-1} \| 3^j f \left( \frac{x}{3^j} \right) - 3^{j+1} f \left( \frac{x}{3^{j+1}} \right) \|^p 
\]
\[
\leq \frac{K^p (3^r + 3)^p \theta^p}{3^{pr}} \sum_{j=l}^{m-1} \frac{3^{pj}}{3^{prj}} |x|^p 
\]
(3.4)

for all nonnegative integers \( m \) and \( l \) with \( m > l \) and all \( x \in X \). It follows from (3.4) that the sequence \( \{ 3^n f \left( \frac{x}{3^{n}} \right) \} \) is a Cauchy sequence for all \( x \in X \). Since \( Y \) is complete, the sequence \( \{ 3^n f \left( \frac{x}{3^{n}} \right) \} \) converges. So one can define the mapping \( A : X \rightarrow Y \) by
\[
A(x) := \lim_{n \to \infty} 3^n f \left( \frac{x}{3^n} \right)
\]
for all \( x \in X \).

The rest of the proof is similar to the proofs of Theorems 2.1 and 3.5. \( \square \)

Remark 3.8. The result for the case \( K = 1 \) in Theorem 2.4 is the same as the result for the case \( p = 1 \) in Theorem 3.7.

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