HYERS–ULAM–RASSIAS STABILITY OF HOMOMORPHISMS IN QUASI-BANACH ALGEBRAS

CHUN-GIL PARK

This paper is dedicated to Professor Themistocles M. Rassias.

Submitted by P. K. Sahoo

Abstract. Let $q$ be a positive rational number and $n$ be a nonnegative integer. We prove the Hyers–Ulam–Rassias stability of homomorphisms in quasi-Banach algebras and of generalized derivations on quasi-Banach algebras for the following functional equation:

$$\sum_{i=1}^{n} f \left( \sum_{j=1}^{n} q(x_i - x_j) \right) + nf \left( \sum_{i=1}^{n} qx_i \right) = nq \sum_{i=1}^{n} f(x_i).$$


1. INTRODUCTION AND PRELIMINARIES

Ulam [30] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

We are given a group $G$ and a metric group $G'$ with metric $\rho(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies

$$\rho(f(xy), f(x)f(y)) < \delta$$


Date: Received: 3 March 2007; Accepted: 16 October 2007.

2000 Mathematics Subject Classification. Primary 39B52; Secondary 46B03, 47Jxx.

for all $x, y \in G$, then a homomorphism $h : G \rightarrow G'$ exists with $\rho(f(x), h(x)) < \epsilon$ for all $x \in G$?

By now an affirmative answer has been given in several cases, and some interesting variations of the problem have also been investigated. We shall call such an $f : G \rightarrow G'$ an approximate homomorphism.

Hyers [11] considered the case of approximately additive mappings $f : E \rightarrow E'$, where $E$ and $E'$ are Banach spaces and $f$ satisfies Hyers inequality
\[
\|f(x + y) - f(x) - f(y)\| \leq \epsilon
\]
for all $x, y \in E$. It was shown that the limit
\[
L(x) = \lim_{n \rightarrow \infty} f\left(\frac{2^n x}{2^n}\right)
\]
exists for all $x \in E$ and that $L : E \rightarrow E'$ is the unique additive mapping satisfying
\[
\|f(x) - L(x)\| \leq \epsilon.
\]
No continuity conditions are required for this result, but if $f(tx)$ is continuous in the real variable $t$ for each fixed $x \in E$, then $L$ is linear, and if $f$ is continuous at a single point of $E$ then $L : E \rightarrow E'$ is also continuous.

Th.M. Rassias [24] provided a generalization of Hyers' Theorem which allows the Cauchy difference to be unbounded.

**Theorem 1.1.** (Th.M. Rassias). Let $f : E \rightarrow E'$ be a mapping from a normed vector space $E$ into a Banach space $E'$ subject to the inequality
\[
\|f(x + y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p)
\]
for all $x, y \in E$, where $\epsilon$ and $p$ are constants with $\epsilon > 0$ and $p < 1$. Then the limit
\[
L(x) = \lim_{n \rightarrow \infty} \frac{f(2^n x)}{2^n}
\]
exists for all $x \in E$ and $L : E \rightarrow E'$ is the unique additive mapping which satisfies
\[
\|f(x) - L(x)\| \leq \frac{2\epsilon}{2 - 2p}\|x\|^p
\]
for all $x \in E$. If $p < 0$ then inequality (1.1) holds for $x, y \neq 0$ and (1.2) for $x \neq 0$. Also, if for each $x \in E$ the mapping $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E$, then $L$ is linear.

Th.M. Rassias [25] during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. Gajda [9] following the same approach as in Th.M. Rassias [24], gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [9], as well as by Th.M. Rassias and Šemrl [28] that one cannot prove a Th.M. Rassias' type Theorem when $p = 1$. The counterexamples of Gajda [9], as well as of Th.M. Rassias and Šemrl [28] have stimulated several mathematicians to invent new definitions of approximately additive or approximately linear mappings, cf. Găvruta [10], Czerwik [7], who among others studied the Hyers–Ulam stability of functional equations. The inequality (1.1) that was introduced for the first time by Th.M. Rassias [24] provided a lot of influence in the development of

Beginning around the year 1980, the topic of approximate homomorphisms and their stability theory in the field of functional equations and inequalities was taken up by several mathematicians (cf. [12] and the references therein).


$$\sum_{i=1}^{n} r_i Q \left( \sum_{j=1}^{n} r_j (x_i - x_j) \right) + \left( \sum_{i=1}^{n} r_i \right) Q \left( \sum_{i=1}^{n} r_i x_i \right) = \left( \sum_{i=1}^{n} r_i \right) \sum_{i=1}^{n} r_i Q(x_i).$$

In this paper we introduce the following functional equation

$$\sum_{i=1}^{n} L \left( \sum_{j=1}^{n} q(x_i - x_j) \right) + nL \left( \sum_{i=1}^{n} qx_i \right) = nq \sum_{i=1}^{n} L(x_i).$$

The purpose of the present paper is to study the Hyers–Ulam–Rassias stability of the functional equation (1.3).

We recall some basic facts concerning quasi-Banach spaces and some preliminary results.

**Definition 1.2.** ([6, 29]) Let $X$ be a real linear space. A quasi-norm is a real-valued function on $X$ satisfying the following:

1. $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.
2. $\|\lambda x\| = |\lambda| \cdot \|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$.
3. There is a constant $K \geq 1$ such that $\|x + y\| \leq K(\|x\| + \|y\|)$ for all $x, y \in X$.

The pair $(X, \| \cdot \|)$ is called a quasi-normed space if $\| \cdot \|$ is a quasi-norm on $X$. The smallest possible $K$ is called the modulus of concavity of $\| \cdot \|$. Obviously the balls with respect to $\| \cdot \|$ define a linear topology on $X$. By a quasi-Banach space we mean a complete quasi-normed space, i.e. a quasi-normed space in which every $\| \cdot \|$-Cauchy sequence in $X$ converges. This class includes Banach spaces and the most significant class of quasi-Banach spaces which are not Banach spaces are the $L_p$ spaces for $0 < p < 1$ with the quasi-norm $\| \cdot \|_p$.

A quasi-norm $\| \cdot \|$ is called a $p$-norm $(0 < p \leq 1)$ if

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p$$

for all $x, y \in X$. In this case, a quasi-Banach space is called a $p$-Banach space.
Given a $p$-norm, the formula $d(x, y) := \|x - y\|^p$ gives us a translation invariant metric on $X$. By the Aoki–Rolewicz theorem [29] (see also [6]), each quasi-norm is equivalent to some $p$-norm. Since it is much easier to work with $p$-norms than quasi-norms, henceforth we restrict our attention mainly to $p$-norms.

**Definition 1.3.** ([2]) Let $(A, \| \cdot \|)$ be a quasi-normed space. The quasi-normed space $(A, \| \cdot \|)$ is called a quasi-normed algebra if $A$ is an algebra and there is a constant $K > 0$ such that $\|xy\| \leq K\|x\| \cdot \|y\|$ for all $x, y \in A$.

A quasi-Banach algebra is a complete quasi-normed algebra.

If the quasi-norm $\| \cdot \|$ is a $p$-norm then the quasi-Banach algebra is called a $p$-Banach algebra.

In this paper, assume that $A$ is a quasi-normed algebra with quasi-norm $\| \cdot \|_A$ and that $B$ is a $p$-Banach algebra with $p$-norm $\| \cdot \|_B$. Let $K$ be the modulus of concavity of $\| \cdot \|_B$.

This paper is organized as follows: In Section 2, we prove the Hyers–Ulam–Rassias stability of homomorphisms in quasi-Banach algebras. In Section 3, we investigate isomorphisms between quasi-Banach algebras. In Section 4, we prove the Hyers–Ulam–Rassias stability of generalized derivations on quasi-Banach algebras.

### 2. Stability of homomorphisms in quasi-Banach algebras

Let $q$ be a positive rational number. For a given mapping $f : A \to B$, we define $Df : A^n \to B$ by

$$Df(x_1, \cdots, x_n) : = \sum_{i=1}^{n} f \left( \sum_{j=1}^{n} q(x_i - x_j) \right)$$

$$+ n f \left( \sum_{i=1}^{n} q x_i \right) - nq \sum_{i=1}^{n} f(x_i)$$

for all $x_1, \cdots, x_n \in X$.

We prove the Hyers–Ulam–Rassias stability of homomorphisms in quasi-Banach algebras.

**Theorem 2.1.** Assume that $r > 2$ if $nq > 1$ and that $0 < r < 1$ if $nq < 1$. Let $\theta$ be a positive real number, and let $f : A \to B$ be an odd mapping such that

$$\|Df(x_1, \cdots, x_n)\|_B \leq \theta \sum_{j=1}^{n} \|x_j\|_A^r,$$ \hspace{1cm} (2.1)

$$\|f(xy) - f(x)f(y)\|_B \leq \theta(\|x\|_A^r + \|y\|_A^r)$$ \hspace{1cm} (2.2)

for all $x, y, x_1, \cdots, x_n \in A$. If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then there exists a unique homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \leq \frac{\theta}{(nq)^r - (nq)^{\frac{r}{p}}} \|x\|_A^r$$ \hspace{1cm} (2.3)

for all $x \in A$. 
Proof. Letting $x_1 = \cdots = x_n = x$ in (2.1), we get
\[
\|nf(nqx) - n^2qf(x)\| \leq n\theta\|x\|_A^r
\]
f for all $x \in A$. So
\[
\|f(x) - nqf\left(\frac{x}{nq}\right)\| \leq \frac{\theta}{(nq)^r}\|x\|_A^r
\]
f for all $x \in A$. Since $B$ is a $p$-Banach algebra,
\[
\|(nq)^lf\left(\frac{x}{(nq)^l}\right) - (nq)^mf\left(\frac{x}{(nq)^m}\right)\|^p_B
\]
\[
\leq \sum_{j=l}^{m-1} \|(nq)^j f\left(\frac{x}{(nq)^j}\right) - (nq)^{j+1} f\left(\frac{x}{(nq)^{j+1}}\right)\|^p_B
\]
(2.4)
\[
\leq \frac{\theta^p}{(nq)^{pr}} \sum_{j=l}^{m-1} \|(nq)^{pj} f\left(\frac{x}{(nq)^j}\right)\|_A^{pr}
\]
for all nonnegative integers $m$ and $l$ with $m > l$ and all $x \in A$. It follows from (2.4) that the sequence $\{(nq)^lf\left(\frac{x}{(nq)^l}\right)\}$ is Cauchy for all $x \in A$. Since $B$ is complete, the sequence $\{(nq)^lf\left(\frac{x}{(nq)^l}\right)\}$ converges. So one can define the mapping $H : A \to B$ by
\[
H(x) := \lim_{d \to \infty} (nq)^d f\left(\frac{x}{(nq)^d}\right)
\]
for all $x \in A$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (2.4), we get (2.3).

It follows from (2.1) that
\[
\|DH(x_1, \cdots, x_n)\| \leq \lim_{d \to \infty} (nq)^d \|Df\left(\frac{x_1}{(nq)^d}, \cdots, \frac{x_n}{(nq)^d}\right)\|_B
\]
\[
\leq \lim_{d \to \infty} \frac{\theta^p}{(nq)^{dr}} \sum_{j=1}^{n} \|x_j\|_A^{pr} = 0
\]
for all $x_1, \cdots, x_n \in A$. Thus
\[
DH(x_1, \cdots, x_n) = 0
\]
for all $x_1, \cdots, x_n \in A$. By Lemma 2.1 of [21], the mapping $H : A \to B$ is Cauchy additive.

By the same reasoning as in the proof of Theorem of [24], the mapping $H : A \to B$ is $\mathbb{R}$-linear.

It follows from (2.2) that
\[
\|H(xy) - H(x)H(y)\|_B
\]
\[
= \lim_{d \to \infty} (nq)^{2d} \|f\left(\frac{xy}{(nq)^d}\right) - f\left(\frac{x}{(nq)^d}\right)f\left(\frac{y}{(nq)^d}\right)\|_B
\]
\[
\leq \lim_{d \to \infty} \frac{(nq)^{2d}\theta}{(nq)^{dr}} (\|x\|_A^{pr} + \|y\|_A^{pr}) = 0
\]
for all $x, y \in A$. So
\[
H(xy) = H(x)H(y)
\]
for all \( x, y \in A \).

Now, let \( T : A \to B \) be another mapping satisfying (2.3). Then we have

\[
\| H(x) - T(x) \|_B = (nq)^d \| H(\frac{x}{(nq)^d}) - T(\frac{x}{(nq)^d}) \|_B \\
\leq (nq)^d K(\| H(\frac{x}{(nq)^d}) - f(\frac{x}{(nq)^d}) \|_B + \| T(\frac{x}{(nq)^d}) - f(\frac{x}{(nq)^d}) \|_B) \\
\leq 2 \cdot (nq)^d K \theta \frac{\| x \|^r_A}{((nq)^p - (nq)^p)^{\frac{1}{p}}}.
\]

which tends to zero as \( n \to \infty \) for all \( x \in A \). So we can conclude that \( H(x) = T(x) \)
for all \( x \in A \). This proves the uniqueness of \( H \). Thus the mapping \( H : A \to B \)
is a unique homomorphism satisfying (2.3).

\[\square\]

**Theorem 2.2.** Assume that \( 0 < r < 1 \) if \( nq > 1 \) and that \( r > 2 \) if \( nq < 1 \). Let \( \theta \) be a positive real number, and let \( f : A \to B \) be an odd mapping satisfying (2.1) and (2.2). If \( f(tx) \) is continuous in \( t \in \mathbb{R} \) for each fixed \( x \in A \), then there exists a unique homomorphism \( H : A \to B \) such that

\[
\| f(x) - H(x) \|_B \leq \frac{\theta}{((nq)^p - (nq)^p)^\frac{1}{p}} \| x \|^r_A
\]

for all \( x \in A \).

**Proof.** It follows from (2.1) that

\[
\| f(x) - \frac{1}{nq} f(nqx) \|_B \leq \frac{\theta}{nq} \| x \|^r_A
\]

for all \( x \in A \). Since \( B \) is a \( p \)-Banach algebra,

\[
\| \frac{1}{(nq)^d} f((nq)^d x) - \frac{1}{(nq)^m} f((nq)^m x) \|_B^p \\
\leq \sum_{j=l}^{m-1} \frac{1}{(nq)^j} f((nq)^j x) - \frac{1}{(nq)^j+1} f((nq)^j+1 x) \|_B^p
\]

\[
\leq \frac{\theta^p}{(nq)^p} \sum_{j=l}^{m-1} (nq)^{prj} \| x \|^{pr} A
\]

for all nonnegative integers \( m \) and \( l \) with \( m > l \) and all \( x \in A \). It follows from (2.6) that the sequence \( \{ \frac{1}{(nq)^d} f((nq)^d x) \} \) is Cauchy for all \( x \in A \). Since \( B \) is complete, the sequence \( \{ \frac{1}{(nq)^d} f((nq)^d x) \} \) converges. So one can define the mapping \( H : A \to B \) by

\[
H(x) := \lim_{d \to \infty} \frac{1}{(nq)^d} f((nq)^d x)
\]

for all \( x \in A \). Moreover, letting \( l = 0 \) and passing the limit \( m \to \infty \) in (2.6), we get (2.5).

The rest of the proof is similar to the proof of Theorem 2.1. \[\square\]
3. ISOMORPHISMS BETWEEN QUASI-BANACH ALGEBRAS

Throughout this section, assume that $A$ is a quasi-Banach algebra with quasi-norm $\| \cdot \|_A$ and unit $e$ and that $B$ is a $p$-Banach algebra with $p$-norm $\| \cdot \|_B$ and unit $e'$. Let $K$ be the modulus of concavity of $\| \cdot \|_B$.

We investigate isomorphisms between quasi-Banach algebras.

**Theorem 3.1.** Assume that $r > 2$ if $nq > 1$ and that $0 < r < 1$ if $nq < 1$. Let $\theta$ be a positive real number, and let $f : A \to B$ be an odd bijective mapping satisfying (2.1) such that
\[
f(xy) = f(x)f(y)
\]
for all $x, y \in A$. If $\lim_{d \to \infty} (nq)^d f(\frac{e}{(nq)^d}) = e'$ and $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then the mapping $f : A \to B$ is an isomorphism.

**Proof.** The condition (3.1) implies that $f : A \to B$ satisfies (2.2). By the same reasoning as in the proof of Theorem 2.1, there exists a unique homomorphism $H : A \to B$, which is defined by
\[
H(x) := \lim_{d \to \infty} (nq)^d f(\frac{x}{(nq)^d})
\]
for all $x \in A$. Thus
\[
H(x) = H(ex) = \lim_{d \to \infty} (nq)^d f(\frac{ex}{(nq)^d}) = \lim_{d \to \infty} (nq)^d f(\frac{e}{(nq)^d} \cdot x) \\
= \lim_{d \to \infty} (nq)^d f(\frac{e}{(nq)^d})f(x) = e'f(x) = f(x)
\]
for all $x \in A$. So the bijective mapping $f : A \to B$ is an isomorphism, as desired. □

**Theorem 3.2.** Assume that $0 < r < 1$ if $nq > 1$ and that $r > 2$ if $nq < 1$. Let $\theta$ be a positive real number, and let $f : A \to B$ be an odd bijective mapping satisfying (2.1) and (3.1). If $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$ and $\lim_{d \to \infty} (nq)^d f((nq)^d e) = e'$, then the mapping $f : A \to B$ is an isomorphism.

**Proof.** The proof is similar to the proofs of Theorems 2.1, 2.2 and 3.1. □

4. STABILITY OF GENERALIZED DERIVATIONS ON QUASI-BANACH ALGEBRAS

Recently, several extended notions of derivations have been treated in the Banach algebra theory (see [18] and references therein). In addition, the stability of these derivations is extensively studied by many mathematicians; see [1, 5, 20].

Throughout this section, assume that $A$ is a $p$-Banach algebra with $p$-norm $\| \cdot \|_A$. Let $K$ be the modulus of concavity of $\| \cdot \|_A$.

**Definition 4.1.** A generalized derivation $\delta : A \to A$ is $\mathbb{R}$-linear and fulfills the generalized Leibniz rule
\[
\delta(xy) = \delta(xy)z - x\delta(y)z + x\delta(yz)
\]
for all $x, y, z \in A$. 
We prove the Hyers–Ulam–Rassias stability of generalized derivations on quasi-Banach algebras.

**Theorem 4.2.** Assume that \( r > 3 \) if \( nq > 1 \) and that \( 0 < r < 1 \) if \( nq < 1 \). Let \( \theta \) be a positive real number, and let \( f : A \to A \) be an odd mapping satisfying \((2.1)\) such that

\[
\|f(xyz) - f(xy)z + xf(y)z - xf(yz)\|_A \\
\leq \theta (\|x\|_A^r + \|y\|_A^r + \|z\|_A^r)
\]

for all \( x, y, z \in A \). If \( f(tx) \) is continuous in \( t \in \mathbb{R} \) for each fixed \( x \in A \), then there exists a unique generalized derivation \( \delta : A \to A \) such that

\[
\|f(x) - \delta(x)\|_A \leq \frac{\theta}{((nq)^p - (nq)^p)^{\frac{1}{p}}} \|x\|_A^r
\]

for all \( x \in A \).

**Proof.** By the same reasoning as in the proof of Theorem 2.1, there exists a unique \( \mathbb{R} \)-linear mapping \( \delta : A \to A \) satisfying \((4.2)\). The mapping \( \delta : A \to A \) is defined by

\[
\delta(x) := \lim_{d \to \infty} (nq)^d f\left(\frac{x}{(nq)^d}\right)
\]

for all \( x \in A \).

It follows from \((4.1)\) that

\[
\|\delta(xyz) - \delta(xy)z + x\delta(y)z - x\delta(yz)\|_A \\
= \lim_{d \to \infty} (nq)^{3d} \left\| f\left(\frac{xyz}{(nq)^{3d}}\right) - f\left(\frac{xy}{(nq)^{2d}}\right) \right\|_A \\
+ \frac{x}{(nq)^d} f\left(\frac{y}{(nq)^d}\right) - \frac{x}{(nq)^d} f\left(\frac{yz}{(nq)^{2d}}\right) \right\|_A \\
\leq \lim_{d \to \infty} \frac{(nq)^{3d} \theta}{(nq)^{dr}} \left( \|x\|_A^r + \|y\|_A^r + \|z\|_A^r \right) = 0
\]

for all \( x, y, z \in A \). So

\[
\delta(xyz) = \delta(xy)z - x\delta(y)z + x\delta(yz)
\]

for all \( x, y, z \in A \). Thus the mapping \( \delta : A \to A \) is a unique generalized derivation satisfying \((4.2)\). \( \square \)

**Theorem 4.3.** Assume that \( 0 < r < 1 \) if \( nq > 1 \) and that \( r > 3 \) if \( nq < 1 \). Let \( \theta \) be a positive real number, and let \( f : A \to A \) be an odd mapping satisfying \((2.1)\) and \((4.1)\). If \( f(tx) \) is continuous in \( t \in \mathbb{R} \) for each fixed \( x \in A \), then there exists a unique generalized derivation \( \delta : A \to A \) such that

\[
\|f(x) - \delta(x)\|_A \leq \frac{\theta}{((nq)^p - (nq)^p)^{\frac{1}{p}}} \|x\|_A^r
\]

for all \( x \in A \).
Proof. By the same reasoning as in the proof of Theorem 2.2, there exists a unique \( \mathbb{R} \)-linear mapping \( \delta : A \to A \) satisfying (4.3). The mapping \( \delta : A \to A \) is defined by

\[
\delta(x) := \lim_{d \to \infty} \frac{1}{(nq)^d} f((nq)^d x)
\]

for all \( x \in A \).

The rest of the proof is similar to the proof of Theorem 4.2. \( \Box \)

References


¹ Department of Mathematics, Hanyang University, Seoul 133-791, South Korea.

E-mail address: baak@hanyang.ac.kr