A Decomposition Formula of 
Idempotent Polyhedral Cones Based on 
Idempotent Superharmonic Spaces

Laurent Truffet

Ecole des Mines de Nantes, Dpt. Automatique-Productique
4, rue A. Kastler, La Chantrerie, BP 20722 44307 Nantes Cedex 3, France
e-mail: Laurent.Truffet@emn.fr

Abstract. In this paper we study the generators of idempotent polyhe-
dral cones which appear of main importance in many fields of applic-
tions such as control of discrete event systems, verification of concurrent 
systems, analysis of Petri nets. We give an explicit formula for the set 
of generators. This formula makes clearly appear the role of the data 
used to describe the idempotent polyhedral cone. This formula is based 
on the Develin-Sturmfels cellular decomposition. From this formula we 
provide an algorithm which could be easily partially parallelizable. From 
this formula we also give a bound on the number of generators and the 
expression of the necessary and sufficient condition under which the set 
of generators is reduced to the null space. We illustrate our results on 
an example of transportation network.

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1. Introduction

In this paper we consider a completed idempotent semifield \( \overline{\mathbb{K}} = (\overline{\mathbb{K}}, \oplus, \odot, 0; \leq) \) 
equipped with its natural order \( \leq \) which is assumed to be total or linear (see 
Definition 2.2 in Section 2) and we study \emph{idempotent polyhedral cones} 
over \( \overline{\mathbb{K}} \) 
defined by:

\[
C_{A,B} = \{ x \in \mathbb{k}^n | A \odot x \leq B \odot x \} \tag{1}
\]
where the inequality $A \odot x \leq B \odot x$ is defined by:
\[
\forall i = 1, \ldots, m, \oplus_{j=1}^{n} a(i, j) \odot x(j) \leq \oplus_{j=1}^{n} b(i, j) \odot x(j),
\]
(2)

$a(i, j)$ and $b(i, j)$ denote entry $(i, j)$ of matrices $A$ and $B$, respectively. $x(j)$ denotes the $j$th coordinate of vector $x$, $\leq$ denotes the total order on $\mathbb{k}$. $m$ and $n$ are given integers $\geq 1$.

The main guidance of this paper is to provide an expression of the set of generators as an explicit function of the data, i.e. the entries of the matrices $A$ and $B$.

**Motivations.** It appears that the knowledge of all elements of the set $C_{A,B}$ is of main importance in many application fields. We cite some of them hereafter.

- In economics for the synchronization of industrial processes (see e.g. [10]).
- For the control of Discrete Event Systems (DES) with synchronizations (see e.g. [20] and the example of Subsection 5.2).
- For the control and the comparison of DES (see [1]).
- For the verification of the correctness of programs (see e.g. [24], [3]).
- For the verification of time behavior of concurrent systems (see e.g. [16]) based on Difference-Bound Matrices (DBM).
- For the analysis of time Petri nets which modelize communication protocol (see e.g. [5]).

From a theoretical point of view such sets also appear as basic tools in tropical convexity (see e.g. [19] where the definition of tropical half space corresponds to a particular set $C_{A,B}$ defined over the semiring $(\mathbb{R}, \min, +, +\infty, 0; \leq)$, see also [15]), in optimization theory (see e.g. [18, and references therein], [8]), in the spectral analysis of a matrix where the eigenvalue of a matrix could be interpreted as the cycle time or the inverse of the throughput of e.g. a manufacturing system (see e.g. [4, and references therein]), in max-linear programming (see [6]).

**Contributions of the paper.** It is already known at least since the work of Butkovic and Hegedeus [7] that there exists a matrix $G$ such that the set $C_{A,B}$ can be represented by $(\oplus, \odot)$-linear combinations of the column vectors of the matrix $G$. However, to the best knowledge of the author only algorithmic construction of matrix $G$ is known. In this paper an explicit formula for the matrix $G$ is provided (see Corollary 3.1). This formula makes clearly appear the role of the data, i.e. the entries of matrices $A$ and $B$. Note that our matrix $G$ we obtain has a different form from the one obtained in [7] because it is obtained by a different approach, e.g. we do not use the assumption [7, (13) p. 207] which reduces the problem to a form that the authors called standard. Reducing our problem to the standard form does not make clearly appear the role of the data in the geometrical decomposition formula. As in [7] the matrix $G$ we obtain is not minimal in the sense that some column vectors of $G$ could be $(\oplus, \odot)$-linear combinations of some of the other columns of $G$. A procedure to delete redundant columns is already known at least since Cuninghame-Green [12]. This procedure is called $A$-test (see [12, Theorem
The formula of matrix $G$ is given in Corollary 3.1 as a consequence of the geometrical decomposition of the set $C_{A,B}$ (Theorems 3.1, 3.2 and 3.3). The geometrical decomposition is based on properties of idempotent superharmonic spaces $H^+_Q := C_{Q,I}$, where $I$ denotes the identity matrix. These properties are summarized in Proposition 2.3. The two fundamental key properties which are central to obtain the geometrical decomposition of the set $C_{A,B}$ are as follows.

A1. If $Q$ is an $n \times n$ matrix then all elements of $H^+_Q$ are $(\oplus, \odot)$-linear combinations of the column vectors of the Kleene star matrix of $Q$, i.e. $Q^* := I \oplus Q \oplus Q \odot Q \oplus \cdots$ (see the precise definition (8) in Subsection 2.3).

A2. If $Q$ and $Q'$ are two $n \times n$ matrices then

$$H^+_Q \cap H^+_{Q'} = H^+_{Q \oplus Q'}.$$ 

The idea of the decomposition used in this paper comes from the cells decomposition idea of Develin and Sturmfels (see [15]). The cells correspond to our superharmonic spaces. The property A2 is the algebraic version of [15, Corollary 11]. The cellular decomposition idea was also used by Sergeev (see [26], [27]) to study eigenspace of particular matrices and particular cones (i.e. Max cones). Independently, the property A1 is used in this paper and by Sergeev [27] where super harmonic spaces are called Kleene cones.

Even if the starting point is the same let us note that our approach is also different from the one developed in Faye et al. [17] which is not based on the use of the above property A2. Moreover, method developed in [17] only leads to an algorithmic construction of a matrix $G$ (which is also different from our matrix $G$).

Let us stress that idempotent superharmonic spaces are involved in many above mentioned fields of application. They are known as DBM in model checking theory (see e.g. [16]), zone and/or octagons (see [24, and references therein]). Idempotent superharmonic spaces are also of main importance in the development of idempotent potential theory (see e.g. [2]). Elements of $H^+_Q$ are also known as $Q$-sub-invariant and they play an important role in the study of invariant measures (i.e. existence and representation) of Bellman-Maslov chains (see e.g. [14]).

From this geometrical decomposition of the set $C_{A,B}$ we provide a formula for a bound on the number of column vectors of the matrix $G$ (see Remark 3.1). This bound is different from the one obtained by Develin and Sturmfels, which is related to Catalan numbers $(2n)!/n!$. We also derive the expression of a necessary and sufficient condition under which $C_{A,B}$ is reduced to the null space (see Subsection 5.1). Note that this problem is also treated in [7, Lemma 2] but under a different approach. Finally, an algorithm which could be easily parallelizable is provided (see Appendix C and Remark C.1). This algorithm is very similar to the approach proposed by Develin and Sturmfels to prove that any intersection of finitely generated idempotent semimodules is again finitely generated. Let us note that the number of processors required can be non-polynomial. If we want to reduce the number of redundant generators we then have to apply the sequential $A$-test procedure. Thus, the whole algorithm which provides the minimal set
of generators is in fact partially parallelizable. The part which is parallelizable corresponds to the geometrical decomposition procedure of the set $C_{A,B}$.

**Organization of the paper.** In Section 2 we introduce basic definitions and notations used in this paper. In Section 3 we establish the geometric decomposition formula of the set $C_{A,B}$. This is the main part of the paper. In Section 4 we briefly discuss some related topics. Finally, in Section 5 we give some possible applications of our work. In particular, two transportation network examples are studied.

2. Preliminaries

2.1. Basic algebraic structures

Let us define the fundamental (idempotent) algebraic structures used in this paper. Our main reference is [4, Chapters 3 and 4].

**Definition 2.1.** (Basic structures)

0. **Semigroup.** A semigroup is a set $S$ endowed with an associative operation $\oplus : S \times S \rightarrow S$.

1. **Monoid.** A monoid is a set $M = (M, \oplus, \circ)$ which is a semigroup with a neutral element $\circ$. Moreover, if $\oplus$ is commutative then $M$ is a commutative monoid.

2. **Group.** A group is a monoid $G = (G, \circ, 1)$ such that all elements are invertible, i.e. $\forall a, \exists! c s.t. a \circ c = c \circ a = 1$.

3. **Semiring.** A semiring is a set $S = (S, \oplus, \circ, 0, 1)$ with $0 \neq 1$ such that $(S, \oplus, 0)$ is a commutative monoid, $\circ : S \times S \rightarrow S$ is associative and its neutral element is $1$, $\circ$ has $0$ as absorbing element, $\circ$ distributes over $\oplus$.

4. **Semifield.** A semifield is a set $K = (k, \oplus, \circ, 0, 1)$ such that $(k, \oplus, 0, 1)$ is a semiring and $(k \setminus \{0\}, \circ, 1)$ is a group.

Semigroup, monoid, group, semiring, semifield are said to be idempotent when $\oplus$ is idempotent (i.e. $\forall a, a \oplus a = a$). Semirings are also known as **dioids** (see e.g. [4]).

2.2. Order properties of idempotent semirings

Let $S = (S, \oplus)$ be a commutative idempotent semigroup. We define the natural (or standard) partial order $\leq$ as follows:

$$x \leq y \overset{\text{def}}{\Leftrightarrow} \exists z \text{ s.t. } y = x \oplus z \Leftrightarrow y = x \oplus y \text{ (because } \oplus \text{ is idempotent).} \quad (3)$$

The notation $x \geq y$ means $y \leq x$. The relation $x < y$ means that $x \leq y$ and $x \neq y$. From now on, $\leq$ will denote the partial order defined by (3).
Remark 2.1. Let us note that if $S$ is a semigroup the binary relation $\leq$ is only transitive. If $S$ is a monoid the binary relation is a preorder (i.e., reflexive and transitive).

**Proposition 2.1.** Let $S = (S, \oplus, \odot, o, 1; \leq)$ be any naturally ordered idempotent semiring.

(i) $\oplus$ is non-decreasing, i.e.:

$$a \leq b \Rightarrow \forall c, a \oplus c \leq b \oplus c$$  \hspace{1cm} (4)

(ii) $\odot$ is non-decreasing, i.e.:

$$a \leq b \Rightarrow \forall c, a \odot c \leq b \odot c.$$  \hspace{1cm} (5)

**Remark 2.2.** The standard results of Proposition 2.1 also hold when the semiring is complete.

By definition of $\leq$ and because $\oplus$ is idempotent and commutative we have: $a \leq a \oplus b$ and $b \leq a \oplus b$. Conversely, assume that there exists $c$ such that $a \leq c$ and $b \leq c$. Then, because $\oplus$ is non-decreasing (see (4)) and idempotent we have $a \oplus b \leq c \oplus c = c$. Thus, for all $a, b$ the supremum of $a$ and $b$, $a \lor b$ exists and is $a \oplus b$. This well-known result is recalled in the next proposition.

**Proposition 2.2.** The class of all idempotent commutative semigroups coincides with the class of all sup-semilattices.

From this proposition we immediately deduce that the class of idempotent commutative monoids with neutral element $o$ coincides with the class of sup-semilattices having the bottom element $\bot = o$.

An idempotent commutative semigroup $(\overline{S}, \oplus; \leq)$ is a complete ordered set iff $\forall A \subseteq \overline{S}$, $\oplus A := \oplus_{a \in A} a$ exists in $\overline{S}$.

An idempotent semiring $\overline{S} = (\overline{S}, \oplus, \odot, o, 1; \leq)$ is complete if $(\overline{S}, \oplus; \leq)$ is a complete ordered set and $\forall B \subseteq \overline{S}$, $\forall c \in \overline{S}$: $(\oplus B) \odot c = \oplus_{b \in B} b \odot c$, $c \odot (\oplus B) = \oplus_{b \in B} c \odot b$. The top element of $\overline{S}$, $\top := \oplus \overline{S}$, verifies: $\forall a \in \overline{S}$ $a \oplus \top = \top \oplus a = \top$, and $a \odot \top = \top \odot a = o$ if $a = o$ and $\top$ otherwise. We will adopt the following notation:

$$\overline{S} := \overline{S} \setminus \{\top\}, \ S = (\overline{S}, \oplus, \odot, o, 1; \leq).$$

**Definition 2.2.** A complete idempotent semiring $\overline{K} = (\overline{K}, \oplus, \odot, o, 1; \leq)$ is said to be a completed idempotent semifield if $K$ is an idempotent semifield.

Let us remark that a completed semifield cannot be a semifield because $\top$ is not invertible (see e.g. [21, Subsection 3.3 and references therein]).
2.3. Important notations

\[ [1, k] := \{1, \ldots, k\} \text{ for all integer } k \geq 1. \]

Let \( S = (\mathbb{S}, \oplus, \odot, 1, \odot; \leq) \) be a naturally ordered semiring. The set \( \text{Mat}_{mn}(S) \) denotes the set of all matrices with \( m \) rows and \( n \) columns whose coefficients lie in the set \( S \).

Let \( A \) be any matrix. \( a(i, \cdot), a(\cdot, j) \) and \( a(i, j) \) denote the \( i \)th row, the \( j \)th column and the entry \((i, j)\) of the matrix \( A \), respectively.

Vectors are column vectors. \((\cdot)^T\) denotes transpose operator. If \( v \) is a vector \( v(i) \) denotes the \( i \)th coordinate of \( v \).

\( \odot \) denotes the null vector.

Let \( S = (\mathbb{S}, \oplus, \odot, 1, \odot; \leq) \) be a naturally ordered semiring. If \( A = [a(i, j)] \) and \( B = [b(i, j)] \) are matrices with suitable dimensions then \( A \leq B \) if \( \forall i, j, a(i, j) \leq b(i, j) \), the addition of two matrices is defined by:

\[ A \oplus B := [a(i, j) \oplus b(i, j)] \tag{6} \]

and the multiplication by:

\[ A \odot B := [\oplus_{k} a(i, k) \odot b(k, j)]. \tag{7} \]

Let \( \bar{S} = (\bar{\mathbb{S}}, \oplus, \odot, 1, \odot; \leq) \) be a complete idempotent naturally ordered semiring. If \( A \) is a matrix \( \in \text{Mat}_{mn}(\bar{S}) \) then we define:

\[ A^* := I \oplus A \oplus A^\odot 2 \oplus \cdots \oplus A^\odot n \oplus \cdots, \tag{8} \]

where \( A^\odot n = A \odot \cdots \odot A \) (\( n \)-fold) if \( n \geq 1 \) and \( I \) if \( n = 0 \). \( I \) denotes the identity element of \( \text{Mat}_{mn}(\bar{S}) \). In other words \( (\text{Mat}_{mn}(\bar{S}), \oplus, \odot, *) \) with \( \oplus, \odot \) and \( * \) respectively defined by (6), (7) and (8) is a Kleene Algebra. The properties used in this paper are the following ones. \( A^* \) has no null column and \((A \odot B)^* \geq A^* \odot B^* \).

If \( \Omega \) is an arbitrary set then \( 2^\Omega \) denotes the set of all parts of \( \Omega \), \( \Omega^c \) denotes the complementary set of \( \Omega \). If \( \Omega \) is finite then \( |\Omega| \) denotes the number of elements in \( \Omega \).

If \( P(1), \ldots, P(m) \) are \( m \) assertions then \( \text{OR}_{j \in [1,m]} P(j) \) denotes the assertion defined as follows:

\[ \text{OR}_{j \in [1,m]} P(j) := P(1) \text{ or } \ldots \text{ or } P(m). \tag{9} \]

2.4. Semimodules and idempotent superharmonic spaces

Let \( \mathcal{S} = (\mathbb{S}, \oplus, \odot, 0, 1) \) be any semiring. A left \( \mathcal{S} \)-semimodule is a commutative monoid \((\mathcal{X}, +, 0)\), equipped with an external law \( \mathcal{S} \times \mathcal{X} \to \mathcal{X}, (s, x) \mapsto s.x \) such that for all \( x, y \in \mathcal{X}, s, s' \in \mathcal{S}, s.(s'.x) = (s \odot s').x, s.(x+y) = s.x + s.y, (s \odot s').x = s.x + s'.x, o.x = 0, 1.x = x \). Right semimodule is defined symmetrically. When \( \mathcal{X} \) is both left and right semimodule we will say that \( \mathcal{X} \) is a semimodule. When \( + \) is idempotent \( \mathcal{X} \) is called an idempotent semimodule.
Let $\mathbb{K} = (\mathbb{K}, \oplus, \odot, 0, 1; \leq)$ be a completed idempotent naturally ordered semifield. We are interested in idempotent semimodules isomorphic to $\mathbb{K}^r$ for some integer $r \geq 1$ equipped with the laws:

$$\forall x, y \in \mathbb{K}, \forall s \in \mathbb{K}, (x \oplus y)(i) := x(i) \oplus y(i), \ (s \cdot x)(i) := s \cdot x(i) =: (s \odot x)(i).$$

And

$$0 := \odot$$

the null vector of $\mathbb{K}^r$.

More precisely the semimodules of interest are the following ones.

1. Semimodule generated by matrix $Q \in \text{Mat}_{mn}(\mathbb{K})$ defined as the set of all $(\oplus, \odot)$-linear combinations of the column vectors of matrix $Q$, i.e.:

$$\text{im}(Q) := Q \odot \mathbb{K}^n = \{ Q \odot x, x \in \mathbb{K}^n \}. \quad (10)$$

Note that in the literature idempotent semimodules are also known as moduloids or pseudomodules (see e.g. [28]), tropical linear spaces (see [25, Suggestion 2]), tropical convex hulls (see [19]), max-plus cone (see e.g. [3]), or simply the image by the map $x \mapsto Q \odot x$ (see e.g. [9]).

2. Idempotent superharmonic space generated by matrix $Q \in \text{Mat}_{mn}(\mathbb{K})$:

$$\mathcal{H}^+_Q := \{ x \in \mathbb{K}^n | Q \odot x \leq x \}. \quad (11)$$

3. Kernel of a matrix $Q \in \text{Mat}_{mn}(\mathbb{K})$:

$$\ker(Q) := \{ x \in \mathbb{K}^n | Q \odot x = 0 \}. \quad (12)$$

Note that our definition of kernel is the one used by e.g. Shubin [23, p. 156]. The definition of the kernel in [9] corresponds to a particular set $\mathcal{C}_{A,B}$ (see our discussion in Section 4).

In the next proposition we study useful relationship between the first two kinds of semimodules.

**Proposition 2.3.** Idempotent superharmonic spaces and semimodules defined over any completed idempotent naturally ordered semifield $\mathbb{K} = (\mathbb{K}, \oplus, \odot, 0, 1; \leq)$ satisfy the following properties.

(P0) $\forall Q \in \text{Mat}_{mn}(\mathbb{K})$:

$$\mathcal{H}^+_Q = \mathcal{H}^+_{I \odot Q}.$$

(P1) $\forall Q \in \text{Mat}_{mn}(\mathbb{K})$:

$$\mathcal{H}^+_Q = \text{im}(Q^+) \cap \mathbb{K}^n.$$

(P2) $\forall Q', Q'' \in \text{Mat}_{mn}(\mathbb{K})$:

$$\mathcal{H}^+_{Q'} \cap \mathcal{H}^+_{Q''} = \mathcal{H}^+_{(Q' \odot Q'')}.$$

(P3) $\forall Q \in \text{Mat}_{mn}(\mathbb{K})$:

$$\mathcal{H}^+_Q = \mathcal{H}^+_Q.$$ 

(P4) If $Q \in \text{Mat}_{mn}(\mathbb{K})$ is such that $Q^{\leq 2} \leq Q$ then $\mathcal{H}^+_Q = \text{im}(I \oplus Q) \cap \mathbb{K}^n.$

(P5) $\forall Q_1 \in \text{Mat}_{mn_1}(\mathbb{K}), \forall Q_2 \in \text{Mat}_{mn_2}(\mathbb{K})$:

$$\text{im}(Q_1) \cup \text{im}(Q_2) \subseteq \text{im}(Q_i, i = 1, 2)$$

where $(Q_i, i = 1, 2)$ denotes the $m \times (n_1 + n_2)$ matrix which $n_1$ first columns are the ones of $Q_1$ and the $n_2$ last columns are the ones of $Q_2$.

**Proof.** (P0), (P2) and (P3) are obvious because $\oplus = \vee$ is idempotent non-decreasing and $\odot$ is non-decreasing (see Proposition 2.1).

(P1). Let $x \in \mathcal{H}^+_Q$. Because $\oplus$ and $\odot$ are non-decreasing we deduce that $Q \odot x \leq x$ implies that $\forall n$ $Q^{\odot n} \odot x \leq x$. Thus, $\oplus^n_{i=0} Q^{\odot i} \odot x \leq x$, for all $n$, in particular:
$Q^* \odot x \leq x$. Because $I \leq Q^*$ we also have: $x \leq Q^* \odot x$. Thus by antisymmetry of $\leq$ we have: $Q^* \odot x = x$ and $x \in \mathbb{k}^n$ which obviously implies that $x \in \text{im}(Q^*) \cap \mathbb{k}^n$.

Conversely, if $x = Q^* \odot y \in \mathbb{k}^n$ for some $y \in \mathbb{k}^n$, then $Q \odot x = Q \odot Q^* \odot y \leq x$ because $Q \odot Q^* \leq Q^*$ and $\oplus$ and $\odot$ are non-decreasing. Thus, $x \in \mathcal{H}_Q^+$.

(Q4). $Q^{\odot 2} \leq Q$ implies $Q^* = I \oplus Q$. From (P1) we get $\mathcal{H}_Q^+ = \text{im}(I + Q)\mathbb{k}^n$. To conclude, we just have to remark that $x \in \mathbb{k}^n, \; Q \in \text{Mat}_{mn}(K) \Rightarrow (I \oplus Q) \odot x \in \mathbb{k}^n$.

(P5). We remark that $Q_1 \odot x_1 = (Q_1 \; Q_2) \odot \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$ and $Q_2 \odot x_2 = (Q_1 \; Q_2) \odot \begin{pmatrix} 0 \\ x_2 \end{pmatrix}$ for all $x_i \in \mathbb{k}^n$, $i = 1, 2$.

\[ \square \]

3. Decomposition formula

Let $\overline{\mathbb{k}} = (\mathbb{k}, \oplus, \odot, \circ, 1; \leq)$ be a completed naturally ordered idempotent semifield whose ordering relation is assumed to be total, i.e.:

\[ \forall x, y \in \overline{\mathbb{k}}, \; x \oplus y \in \{x, y\}. \]  \hspace{1cm} (13)

Let us consider the idempotent polyhedral cone:

\[ \mathcal{C}_{A,B} = \{x \in \mathbb{k}^n|A \odot x \leq B \odot x\} \]

for some matrices $A$ and $B \in \text{Mat}_{mn}(K)$. Noticing that $\mathcal{C}_{A,B} = \bigcap_{i=1}^m \mathcal{C}_{a(i,:),b(i,:)}$ and that $a(i,:) \leq b(i,:) \Rightarrow \mathcal{C}_{a(i,:),b(i,:)} = \mathbb{k}^n$ we assume without loss of generality (w.l.o.g) that matrices $A$ and $B$ are such that:

\[ \forall i \in [[1, m]], \; a(i,:) \nleq b(i,:). \]

The first decomposition result is stated in the next theorem.

**Theorem 3.1.** Under the following definitions and notations:

- $\forall i \in [[1, m]]$:
  \[ B_i = \text{supp}(b(i,:)) := \{j \in [[1, n]]|b(i, j) \neq 0\}, \]  \hspace{1cm} (14)

- define the following set:
  \[ \mathcal{L} := \{i \in [[1, m]]|B_i \neq \emptyset\}, \]  \hspace{1cm} (15)

- $\forall i \in \mathcal{L}, \; \forall j \in B_i$ we define:
  \[ r^{i,j} := b(i, j)^{-1} \odot (a(i,:)) \oplus b(i,:), \]  \hspace{1cm} (16)

- $\forall i \in \mathcal{L}, \; \forall j \in B_i$ we define the $n \times n$ matrix $Q^{i,j}$ by:
  \[ \forall l \in [[1, n]], \; Q^{i,j}(l,:) = \begin{cases} e_l^T r^{i,j} & \text{if } l \neq j \\ r^{i,j} & \text{otherwise} \end{cases} \]  \hspace{1cm} (17)

where $(e_i)_{i=1}^n$ denotes the canonical basis of $\mathbb{k}^n$. 

• finally define:

\[
\mathcal{K}(\mathcal{L}^c) := \begin{cases} k^n & \text{if } \mathcal{L}^c = \emptyset \\ \cap_{i \in \mathcal{L}^c} \ker(a(i, \cdot)) & \text{otherwise} \end{cases}
\]  

(18)

and

\[
\mathcal{I}(\mathcal{L}) := \begin{cases} k^n & \text{if } \mathcal{L} = \emptyset \\ \cap_{i \in \mathcal{L}} \cup_{j, i \in B_i} \mathcal{H}^+_{Q^i, j_i} & \text{otherwise,} \end{cases}
\]

(19)

we have:

\[
C_{A,B} = \mathcal{K}(\mathcal{L}^c) \cap \mathcal{I}(\mathcal{L}).
\]

(20)

Proof. See Appendix A. □

3.1. Expressions of the set \( \mathcal{K}(\mathcal{L}^c) \)

We assume that \( \mathcal{L}^c \neq \emptyset \).

**Theorem 3.2.** Let us define

\[
A_i = \text{supp}(a(i, \cdot)) := \{ j \in [1, n] | a(i, j) \neq 0 \}.
\]

(21)

The set \( \mathcal{K}(\mathcal{L}^c) \) can be expressed as:

\[
\mathcal{K}(\mathcal{L}^c) = \text{im}(e_k, k \in \cap_{i \in \mathcal{L}^c} A_i^c) \cap k^n
\]

(22a)

or, dually by:

\[
\mathcal{K}(\mathcal{L}^c) = \{ x \in k^n | x(k) = 0, k \in \cup_{i \in \mathcal{L}^c} A_i \}
\]

(22b)

with the conventions: \( \text{im}(e_k, k \in \emptyset) = \{ 0 \} \) and \( \{ x \in k^n | x(k) = 0, k \in \emptyset \} = k^n \).

Proof. By definition of the set \( \mathcal{L} \) (see (15)) we have \( \forall i \in \mathcal{L}^c b(i, \cdot) = 0^T \), thus: \( a(i, \cdot) \odot x \leq b(i, \cdot) \odot x \Leftrightarrow a(i, \cdot) \odot x = 0 \) and \( a(i, \cdot) \odot x = 0 \Leftrightarrow \forall j \in [1, n], a(i, j) \odot x(j) = 0 \). Because \( K \) is a semifield then \( (k \setminus \{ 0 \}, \odot, 1) \) is a group and:

\[
a(i, j) \odot x(j) = 0 \Leftrightarrow (a(i, j) \neq 0 \text{ and } x(j) = 0) \text{ or } (a(i, j) = 0 \text{ and } x(j) \in k).
\]

Thus,

\[
\mathcal{K}(\mathcal{L}^c) = \{ x \in k^n | x(k) = 0, k \in \cup_{i \in \mathcal{L}^c} A_i \}.
\]

Noticing that \( (\cup_{i \in \mathcal{L}^c} A_i)^c = \cap_{i \in \mathcal{L}^c} A_i^c \) we deduce that:

\[
\mathcal{K}(\mathcal{L}^c) = \text{im}(e_k, k \in \cap_{i \in \mathcal{L}^c} A_i^c) \cap k^n
\]

which ends the proof of the result. □
3.2. Decomposition formula of the set \( \mathcal{I}(L) \)

We assume w.l.o.g that \( L = [1, m'] \neq \emptyset \) with \( 1 \leq m' \leq m \). Let us denote \( A' \) (resp. \( B' \)) the submatrix of matrix \( A \) (resp. \( B \)) which corresponds to the \( m' \) first rows of matrix \( A \) (resp. \( B \)). With these notations we remark that \( \mathcal{I}(L) = \mathcal{C}_{A', B'} \). Finally, assume that:

\[
\forall a, \ 1 < a \Rightarrow \exists! \lim_{q \to \infty} a^{\circ q} = \top. \tag{23}
\]

Theorem 3.3. Under the following definitions and notations

- \( \forall i \in L: \quad B'_i := \{ j \in B_i | 1 < r^{ij}(j) \text{ and } \text{supp}(r^{ij}) = [1,n]\} \)

and

\[
\bar{B}_i := B_i \setminus B'_i. \tag{24a}
\]

- \( \bar{j} := (j_1, \ldots, j_{m'}) \in \bar{B}_1 \times \cdots \times \bar{B}_{m'} \),

- \( \forall \bar{j} \in \bar{B}_1 \times \cdots \times \bar{B}_{m'} \) define the \( n \times n \) matrix:

\[
G^L := (\oplus_{i=1}^{m'} Q^{i\bar{j}})^* \tag{25}
\]

- and finally define the \( n \times (|\bar{B}_1| \cdots |\bar{B}_{m'}|) \) matrix:

\[
G' := (G^L, \bar{j} \in \bar{B}_1 \times \cdots \times \bar{B}_{m'}), \tag{26}
\]

we have:

\[
\mathcal{I}(L) = \text{im}(G') \cap k^n.
\]

Proof. See Appendix B. \( \square \)

3.3. Explicit formula for matrix \( G \) such that \( \mathcal{C}_{A,B} = \text{im}(G) \cap k^n \)

From the previous theorems we are now in position to provide an explicit formula for a matrix \( G \) such that all elements of the set \( \mathcal{C}_{A,B} \) are expressed as \((\oplus, \odot)\)-linear combinations of the column vectors of matrix \( G \). The matrix \( G \) is simply obtained by the elimination of all column \( G'(:, k) \) such that \( G'(:, k) \not\in k^n \), i.e. \( G'(:, k) \) s.t \( 1^T \odot G'(:, k) = \top \) and if \( L^c \neq \emptyset \) by the elimination of all column \( G'(:, k) \) such that \( G'(:, k) \not\in K(L^c) \), i.e. \( G'(:, k) \) s.t. \( \exists l \in \cup_{i \in L^c} A_i: G'(l, k) \neq \emptyset \), recalling that sets \( A_i \) are defined by (21). We summarize this result in the next corollary.

Corollary 3.1. We assume that the columns of matrix \( G' \) defined by ((26), Theorem 3.3) have been numbered from 1 to \( |\bar{B}_1| \cdots |\bar{B}_{m'}| \). The matrix \( G \) such that \( \mathcal{C}_{A,B} = \text{im}(G) \cap k^n \) is defined as follows for all \( k \in [1, |\bar{B}_1| \cdots |\bar{B}_{m'}|] \):

\[
G(:, k) := \begin{cases} 
\text{a. if } 1^T \odot G'(:, k) = \top \text{ or } (L^c \neq \emptyset \text{ and } \exists l \in \cup_{i \in L^c} A_i: G'(l, k) \neq \emptyset) \\
G'(:, k), \text{ otherwise.}
\end{cases} \tag{27}
\]

Remark 3.1. From this corollary we deduce a bound of the number of elements which generate the set \( \mathcal{C}_{A,B} \). This bound is:

\[
\prod_{i=1}^{m'} |\bar{B}_i| - |\{ k | 1^T \odot G'(:, k) = \top \text{ or } (L^c \neq \emptyset \text{ and } \exists l \in \cup_{i \in L^c} A_i: G'(l, k) \neq \emptyset) \}|.
\]
4. Related topics

In this section we discuss some related works on similar sets. We consider a completed idempotent semifield totally ordered \( K = (\bar{K}, \oplus, \circ, 0, 1; \leq) \).

**Homogeneous sets.** In [7] the authors studied \((\oplus, \circ)\)-linear systems of equalities, i.e. set of the form:

\[
\mathcal{V}_{E,F} := \{ x \in \mathbb{k}^n | E \circ x = F \circ x \},
\]

where \( E \) and \( F \) are \( m \times n \) matrices.

Noticing that \( E \circ x = F \circ x \iff \begin{pmatrix} E \\ F \end{pmatrix} \circ x = \begin{pmatrix} F \\ E \end{pmatrix} \circ x \) we easily deduce that:

\[
\mathcal{V}_{E,F} = \mathcal{C}_{A,B},
\]

with \( A := \begin{pmatrix} E \\ F \end{pmatrix} \) and \( B := \begin{pmatrix} F \\ E \end{pmatrix} \). Conversely, because \( \oplus = \lor \), we have:

\[
\mathcal{C}_{A,B} = \mathcal{V}_{A \oplus B, B}.
\]

Motivated by synchronization problem of two dynamical systems the authors studied in [11] set defined as follows:

\[
\mathcal{W}_{E,F} := \{ x, y \in \mathbb{k}^n | E \circ x = F \circ y \},
\]

where \( E \) and \( F \) are \( m \times n \) matrices. Note that if \( E = F \) this set is called kernel in [9]. Noticing that \( E \circ x = F \circ y \iff \begin{pmatrix} E & \circ \\ \circ & F \end{pmatrix} \circ \begin{pmatrix} x \\ y \end{pmatrix} \leq \begin{pmatrix} \circ & F \\ F & \circ \end{pmatrix} \circ \begin{pmatrix} x \\ y \end{pmatrix} \), where \( \circ \) denotes the null matrix, we easily deduce that:

\[
\mathcal{W}_{E,F} = \mathcal{C}_{A,B},
\]

with

\[
A = \begin{pmatrix} E & \circ \\ \circ & F \end{pmatrix}, B = \begin{pmatrix} \circ & F \\ F & \circ \end{pmatrix}.
\]

Conversely, noticing that \( (A \oplus B) \circ x = B \circ x \iff \begin{pmatrix} A \oplus B \\ B \end{pmatrix} \circ x = \begin{pmatrix} I \\ I \end{pmatrix} \circ y \) we have:

\[
\mathcal{C}_{A,B} = \mathcal{W}_{E,F},
\]

with \( E = \begin{pmatrix} A \oplus B \\ B \end{pmatrix} \) and \( F = \begin{pmatrix} I \\ I \end{pmatrix} \), recalling that \( I \) denotes the identity matrix.

**Non-homogeneous sets.** In [7, Section 6] the author studied non-homogeneous \((\oplus, \circ)\)-linear systems of equalities, i.e. set of the form:

\[
\mathcal{P}_{E,e,F,f} := \{ x \in \mathbb{k}^n | E \circ x \oplus e = F \circ x \oplus f \},
\]

where \( E, F \in \text{Mat}_{m,n}(\bar{K}) \) and \( e, f \in \mathbb{k}^m \). Then, it is easy to see (see also [7, Proposition 4]) that \( x \in \mathcal{P}_{E,e,F,f} \iff \begin{pmatrix} x \\ 1 \end{pmatrix} \in \mathcal{C}_{A,B} \) with matrices \( A \) and \( B \) respectively defined as follows:

\[
A := \begin{pmatrix} E & e \\ F & f \end{pmatrix}, B := \begin{pmatrix} F & f \\ E & e \end{pmatrix}.
\]
Conversely, we easily see that \( C_{A,B} = \mathcal{P}_{A\oplus B,\emptyset,\emptyset} \).

In conclusion of this section it appears that sets of the form \( C_{A,B} \) have the same power of description than the sets of the form \( \mathcal{V}_{E,F}, \mathcal{W}_{E,F} \) and \( \mathcal{P}_{E,F,F} \). Note that observations of this kind were already mentioned in e.g. [3], [7], [11].

5. Applications

In Subsection 5.1 we provide an explicit condition under which the set \( C_{A,B} \) is reduced to the null space. This condition comes from the geometric decomposition exposed in Section 3. In Subsection 5.2 we study a numerical example inspired by a transportation network studied in e.g. [13].

5.1. Conditions under which \( C_{A,B} = \{\emptyset\} \)

**Theorem 5.1.** \( C_{A,B} = \{\emptyset\} \) if and only if

(i) \( \mathcal{L}^c \neq \emptyset \) and \( \cup_{i \in \mathcal{L}^c} A_i = [[1, n]] \) or \( \forall k \in \cap_{i \in \mathcal{L}^c} A_i^c \neq \emptyset \) \( A \odot e_k \not\leq B \odot e_k \)

or

(ii) \( \mathcal{L}^c = \emptyset \) and

\[
\forall j \in \hat{B}_1 \times \cdots \times \hat{B}_m, \quad 1^T \odot G^j = 1^T.
\]

**Proof.** The condition \( \mathcal{L}^c \neq \emptyset \) and \( \cup_{i \in \mathcal{L}^c} A_i = [[1, n]] \) implies \( \mathcal{K}(\mathcal{L}^c) = \{\emptyset\} \). By Theorem 3.1 we then have: \( C_{A,B} = \{\emptyset\} \) implies \( \mathcal{I}(\mathcal{L}) \) we have: \( C_{A,B} = \{\emptyset\} \).

The condition \( \mathcal{L}^c \neq \emptyset \) and \( \forall k \in \cap_{i \in \mathcal{L}^c} A_i^c \neq \emptyset \) \( A \odot e_k \not\leq B \odot e_k \) implies that

\[
\mathcal{K}(\mathcal{L}^c) = \text{im}(e_k, k \in \cap_{i \in \mathcal{L}^c} A_i^c \cap k^n \neq \{\emptyset\}) \text{ (see (22a), Theorem 3.2) and all vectors } e_k, k \in \cap_{i \in \mathcal{L}^c} A_i^c \text{ do not satisfy the } |\mathcal{L}| \text{ other inequalities of the system } A \odot x \leq B \odot x,
\]

which implies that \( \mathcal{K}(\mathcal{L}^c) \cap \mathcal{I}(\mathcal{L}) = \{\emptyset\} \).

If the condition (ii) is true we apply the following lemma to matrix \( G \) defined by (26, Theorem 3.3).

**Lemma 5.1.** Let \( \mathcal{K} = (\mathcal{K}, \oplus, \odot, \emptyset, 1; \leq) \) be a completed naturally ordered idempotent semifield. Let us consider \( Q \in \text{Mat}_{n_1,n_2}(\mathcal{K}) \) such that \( Q \) has no null column. Then

\[
\text{im}(Q) \cap k^{n_2} = \{\emptyset\} \iff 1^T \odot Q = 1^T
\]

where \( 1 \) denotes the vector which components are all equal to 1.

**Proof.** Assume that \( 1^T \odot Q = T \). Then, there exists a permutation \( \sigma \) on \( [1, n_2]\) such that for all \( \alpha \in k^{n_2} \) \( (Q \odot \alpha)(i) = T \odot \alpha(\sigma(i)) \odot \alpha'(i) \) for some \( \alpha'(i) \in \mathcal{K} \). If \( \exists i : \sigma(i) \neq \sigma(i) \) then \( (Q \odot \alpha)(i) = T \not\in k \). Thus, the only solution is: \( \alpha = 0 \).

Assume that \( 1^T \odot Q \neq T \). Then, there exists \( k \) such that \( 1^T \odot Q(\cdot,k) < T \). Because \( \oplus = \vee: \quad 1^T \odot Q(\cdot,k) < T \iff \forall i, \quad Q(i,k) < T \), i.e \( Q(\cdot,k) \in k^{n_2} \). Let \( e_k \) be the \( k \)th vector of the canonical basis of \( k^{n_2} \). By assumption on \( Q, Q(\cdot,k) \) is a non null vector such that: \( Q(\cdot,k) = Q \odot e_k \in \text{im}(Q) \cap k^{n_2} \) and the lemma is proved.

We have proved that (i) or (ii) implies \( C_{A,B} = \{\emptyset\} \). Let us prove now that not (i) and not (ii) implies \( C_{A,B} \neq \{\emptyset\} \).
Let us remark that not (i) and not (ii) is logically equivalent to:

(c1) \( L^c = \emptyset \) and \( \exists j \in \tilde{B}_1 \times \cdots \times \tilde{B}_m, \ 1_T \odot G^2 \neq \top^T \),

or

(c2) \( L^c \neq \emptyset \) and \( \cup_{i \in L^c} A_i \neq [1, n] \) and \( \exists k \in \cap_{i \in L^c} A_i^c \neq \emptyset \ A \odot e_k \leq B \odot e_k \),

or

(c3) \( \cup_{i \in L^c} A_i \neq [1, n] \) and \( \exists k \in \cap_{i \in L^c} A_i^c \neq \emptyset \ A \odot e_k \leq B \odot e_k \) and \( \exists j \in \tilde{B}_1 \times \cdots \times \tilde{B}_m, 1_T \odot G^2 \neq \top^T \).

If (c2) or (c3) is true then take \( e_k \) the \( k \)th vector of the canonical basis of \( \mathbb{K}^n \).

Assume (c1) is true. By definition of matrix \( G^2 \) (see (25), Theorem 3.3), \( G^2 \) has no null column and \( A \odot G^2 \leq B \odot G^2 \). Moreover, because \( 1_T \odot G^2 \neq \top^T \) we deduce using the same arguments as in the proof of Lemma 5.1 that there exists \( k \in [1, n] \) s.t \( G^2(\cdot, k) \in \mathbb{K}^n \).

Thus,

\[ \{\emptyset\} \neq \{G^2(\cdot, k)\} \subseteq C_{A,B}, \]

and the result is now proved. \( \square \)

5.2. Study of time constraints in transportation networks

We consider the simple transportation network with four stations \( n = 1, \ldots, 4 \) represented by the Figure 1 p. 328 which is inspired by [13]. We summarized the time behavior of this transportation network using [20, Section 6]. We assume that in the initial state there is a train running on each tracks. The tracks connect the following stations. Track \( d_1 \): station 1 with 2, track \( d_2 \): station 2 with 1, track \( d_3 \): station 1 with 1 via station 3, track \( d_4 \): station 1 with 1 via station 4.

The traveling time on track \( d_i \) is denoted \( t(i) \). We have: \( t(1) = 17 \), \( t(2) = 14 \), \( t(3) = 11 \), \( t(4) = 9 \). For any \( i \), denote \( r(i) \) the index of the track which a train leaves just before it enters the track \( d_i \). We have: \( r(1) = 4 \), \( r(2) = 1 \), \( r(3) = 3 \), \( r(4) = 2 \). Let \( C(i) \) denote the set of indexes of all tracks which have to provide a connection with the train which leaves on track \( d_i \). From Figure 1 we have \( C(1) = \emptyset \), \( C(2) = \{3\} \), \( C(3) = \{1, 4\} \) and \( C(4) = \{3\} \).

Let \( x_k(i) \) be the \( k \)th departure time of the train which leaves on track \( d_i \), and \( u_k(i) \) be the schedule departure for the \( k \)th train on track \( d_i \).

A train must arrive at the station before its next departure on track \( d_i \), i.e.:

\[ t(r(i)) + x_{k-1}(r(i)) \leq x_k(i). \tag{28a} \]

The trains must satisfy the demand on connection, i.e.:

\[ t(j) + x_{k-1}(j) \leq x_k(i), \quad \forall j \in C(i) \tag{28b} \]

and finally a train cannot leave a station before its scheduled departure time, i.e.:

\[ u_k(i) \leq x_k(i). \tag{28c} \]

Consider now the completed idempotent semifield \( \mathbb{R}_{\max} := (\mathbb{R} \cup \{-\infty\}, \oplus := \max, \ominus := +, \circ := -\infty, 1 := 0; \leq) \) where \( \leq \) denotes the natural order on the
set of the real numbers \( \mathbb{R} \), the top element of \( \mathbb{R} \cup \{-\infty\} \) is \( T := +\infty \). Thus, in this subsection: \( k := \mathbb{R} \cup \{-\infty\} \) and \( \overline{k} := \mathbb{R} \cup \{-\infty\} \), for any \( a \in \overline{k} \) \( a^{-1} \) will be denoted \(-a\).

Define the \( 4 \times 4 \) matrix \( \Omega = [\omega(i, j)] \) by \( \forall i, j, \omega(i, j) := t(j) \) if \( j \in C(i) \cup \{r(i)\} \) and \( + \) otherwise. Then, assuming train leaves as soon as previous conditions have been satisfied the equations (28a)–(28c) can be written in matrix form as: 
\[
x_k = \Omega \odot x_{k-1} + u_k \quad \text{where} \quad x_k := (x_k(1), \ldots, x_k(4))^T, \quad u_k := (u_k(1), \ldots, u_k(4))^T, \quad \text{and}
\]
\[
\Omega := \begin{pmatrix}
-\infty & 17 & -\infty & -\infty \\
-\infty & -\infty & 11 & 9 \\
14 & -\infty & 11 & 9 \\
14 & -\infty & 11 & -\infty
\end{pmatrix}.
\]

### 5.2.1. Time constraints: Example 1

In this first example we assume that the following time constraints are required for trains. The time between two consecutive trains on track \( d_i \) is less than a duration \( l_i \), i.e.: 
\[
x_k(i) - x_{k-1}(i) \leq l_i.
\]

The passengers coming from track \( d_i \) do not have to wait more than a duration \( m_{i,j} \) for the departure of the train which leaves station on track \( d_j \), i.e.: 
\[
x_k(j) - \omega(j, i) - x_{k-1}(i) \leq m_{i,j}.
\]

Define the matrix \( S = [s(i, j)] \) by \( \forall i, j = 1, \ldots, 4, \ s(i, j) = -\omega(j, i) \) if \( \omega(j, i) \neq -\infty \) and \( -\infty \) otherwise. Take \( l_i = l \) and \( m_{i,j} = m \), then (30a)–(30b) can be written in matrix form as: \( \forall k \geq 1 \ K \odot x_k \leq x_{k-1} \) where 
\[
K := (-l) \odot I \oplus (-m) \odot S.
\]

If we study the free dynamics of the train (i.e. \( \forall k, u_k = \emptyset \)) then the constraints that the series \( x_k \) must satisfy are:
\[
\forall k \geq 1, \ K \odot \Omega \odot x_{k-1} \leq x_{k-1}.
\]

It means that \( \forall k \geq 1, x_{k-1} \) is an element of the Difference-Bound Matrix generated by matrix \( N := [(K \odot \Omega)(i, j)] \), i.e. the polyhedral set defined by: \( x(i) - x(j) \leq n(i, j) \), \( \forall i, j = 1, \ldots, 4 \), (see e.g. [24] and the introduction of this paper).

The assertion (32) is true iff \( \forall x, (K \odot \Omega \odot x \leq x) \Rightarrow (K \odot \Omega \odot \Omega \odot x \leq \Omega \odot x) \) which is equivalent to:
\[
\mathcal{H}_{K \odot \Omega}^+ \subseteq \mathcal{C}_{K \odot \Omega \odot \Omega}.
\]

### Numerical application

Take \( m = 4 \), \( l = 15 \) (or equivalently \( m^{-1} = -4 \) and \( l^{-1} = -15 \)). Then,
\[
K = \begin{pmatrix}
-15 & -\infty & -18 & -18 \\
-21 & -15 & -\infty & -\infty \\
-\infty & -15 & -15 & -15 \\
-\infty & -13 & -13 & -15
\end{pmatrix}, \ K \odot \Omega = \begin{pmatrix}
-4 & 2 & -7 & -9 \\
-\infty & -4 & -4 & -6 \\
-1 & -\infty & -4 & -6 \\
1 & -\infty & -2 & -4
\end{pmatrix}.
\]
We have (see (P1), Proposition 2.3) \( \mathcal{H}_{K \ominus \Omega}^+ = \text{im}((K \ominus \Omega)^*) \cap (\mathbb{R} \cup \{-\infty\})^4 \), with:

\[
(K \ominus \Omega)^* = \begin{pmatrix} 0 & 2 & -2 & -4 \\ -5 & 0 & -4 & -6 \\ -1 & 1 & 0 & -5 \\ 1 & 3 & -1 & 0 \end{pmatrix}.
\]

Then, the inclusion (33) to test is equivalent to test if \( K \ominus \Omega \ominus \Omega \ominus (K \ominus \Omega)^* \leq \Omega \ominus (K \ominus \Omega)^* \) is true. Computing \( A := K \ominus \Omega \ominus \Omega \ominus (K \ominus \Omega)^* \) and \( B := \Omega \ominus (K \ominus \Omega)^* \) we have:

\[
A = \begin{pmatrix} 12 & 14 & 13 & 11 \\ 10 & 12 & 8 & 6 \\ 11 & 16 & 12 & 10 \\ 13 & 18 & 14 & 12 \end{pmatrix}, \quad B = \begin{pmatrix} 12 & 17 & 13 & 11 \\ 10 & 12 & 11 & 9 \\ 14 & 16 & 12 & 10 \\ 14 & 16 & 12 & 10 \end{pmatrix}.
\]

We easily see that \( A \nleq B \) thus the inclusion (33) is not true and so is the time constraint (32).

However we can be interested by the following question. How many times the condition \( K \ominus \Omega \ominus x_{k-1} \leq x_{k-1} \) is verified? It means that we are interested by the set: \( T := \{ x \in \mathbb{R}^n | K \ominus \Omega \ominus x \leq \Omega \ominus x, k = 1, 2, \ldots \} \). We remark that:

\( T = \bigcap_{k \geq 1} C_{K \ominus \Omega \ominus \Omega \ominus \Omega} \). Noticing that \( \text{im}(G) \cap C_{A,B} = G \ominus (C_{A,G,B \ominus G}) \) (note that this fact was intensively used in [7]) we develop the computation of \( T \) as follows.

1/ \( C_{K \ominus \Omega \ominus \Omega} = \text{im}(G_1) \cap \mathbb{k}^4 \) with \( G_1 := (K \ominus \Omega)^* \) (see (P1), Proposition 2.3). Then, \( C_{K \ominus \Omega \ominus \Omega} \cap C_{K \ominus \Omega \ominus \Omega} = G_1 \ominus C_{A,B} \), with \( A = K \ominus \Omega \ominus G_1 \) and \( B = \Omega \ominus G_1 \).

For the numerical application \( A \) and \( B \) are defined by (34) and we remark that \( a(i, \cdot) \leq b(i, \cdot) \) for \( i = 1, 2, 3 \), thus \( C_{A,B} = C_{a(4,4),b(4,4)} \). Applying algorithm of Appendix C we have \( C_{a(4,4),b(4,4)} = \text{im}(G_2) \cap \mathbb{k}^4 \) with

\[
G_2 = \begin{pmatrix} 0 & 4 & 0 & -2 \\ -\infty & 0 & -\infty & -\infty \\ -\infty & -\infty & 0 & -\infty \\ -\infty & -\infty & -\infty & 0 \end{pmatrix}.
\]

3/ Now we have \( \bigcap_{k=1}^2 C_{K \ominus \Omega \ominus \Omega \ominus \Omega} = G_2 \ominus G_1 \ominus C_{A',B'} \), with \( A' := K \ominus \Omega \ominus G_1 \ominus G_2 \) and \( B' := \Omega \ominus G_1 \ominus G_2 \), i.e.:

\[
\]

We remark that \( C_{A',B'} = C_{a'(4,4),b'(4,4)} \) but \( \forall j, a'(4,j) > b'(4,j) \) thus: \( C_{A',B'} = \{0\} \).

We conclude that the time constraint (32) holds true for \( k = 1, 2 \) (i.e. one period of time).
5.2.2. Time constraints: Example 2

The aim of this example is to apply our geometrical decomposition of idempotent polyhedral cone without too heavy computations. To do this we require that the free dynamics of the series \( x_k \) (i.e. the vector of the \( k \)th departure times of the trains) lies in the octagon defined by:

\[
\mathcal{O} := \{ x \in \mathbb{R}^4 | x(1) \geq x(2), x(3) \leq x(4) \},
\]

for \( k = 0, 1, 2 \).

We remark that the previous octagon \( \mathcal{O} \) coincides with the idempotent polyhedral cone \( \mathcal{C}_{A',B'} \) with:

\[
A' := \begin{pmatrix} -\infty & 0 & -\infty & -\infty \\ -\infty & -\infty & -\infty & 0 \\ -\infty & -\infty & 11 & 9 \\ 14 & -\infty & 11 & -\infty \\ 25 & -\infty & 22 & 20 \\ 25 & 31 & 22 & 20 \end{pmatrix}, \quad B' := \begin{pmatrix} 0 & -\infty & -\infty & -\infty \\ -\infty & -\infty & 0 & -\infty \\ -\infty & 17 & -\infty & -\infty \\ 14 & -\infty & 11 & 9 \\ -\infty & -\infty & 28 & 26 \\ 25 & 31 & 22 & 20 \end{pmatrix}.
\]

The time constraint we impose on the free dynamics of the \( x_k \)'s is equivalent to study the idempotent polyhedral cone \( \mathcal{C}_{A'',B''} \) with:

\[
A'' := \begin{pmatrix} A' \\ A' \odot \Omega \circ \odot 2 \end{pmatrix}, \quad B'' := \begin{pmatrix} B' \\ B' \odot \Omega \circ \odot 2 \end{pmatrix}.
\]

Numerical application

Taking the matrix \( \Omega \) defined by (29) we have:

\[
A'' = \begin{pmatrix} -\infty & 0 & -\infty & -\infty \\ -\infty & -\infty & -\infty & 0 \\ -\infty & -\infty & 11 & 9 \\ 14 & -\infty & 11 & -\infty \\ 25 & -\infty & 22 & 20 \\ 25 & 31 & 22 & 20 \end{pmatrix}, \quad B'' = \begin{pmatrix} 0 & -\infty & -\infty & -\infty \\ -\infty & -\infty & 0 & -\infty \\ -\infty & 17 & -\infty & -\infty \\ 14 & -\infty & 11 & 9 \\ -\infty & -\infty & 28 & 26 \\ 25 & 31 & 22 & 20 \end{pmatrix}.
\]

We remark that \( a''(i, \cdot) \leq b''(i, \cdot) \) for \( i = 4, 6 \), thus \( \mathcal{C}_{A'',B''} = \mathcal{C}_{A,B} \) with

\[
A = \begin{pmatrix} -\infty & 0 & -\infty & -\infty \\ -\infty & -\infty & -\infty & 0 \\ -\infty & -\infty & 11 & 9 \\ 25 & -\infty & 22 & 20 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -\infty & -\infty & -\infty \\ -\infty & -\infty & 0 & -\infty \\ -\infty & 17 & -\infty & -\infty \\ -\infty & -\infty & 28 & 26 \end{pmatrix}.
\]
We now apply algorithm of Appendix C to compute the matrix $G$ such that $\mathcal{L}_{A,B} = \text{im}(G) \cap k^4$.

Step 1. Compute $\mathcal{L}$ defined by (15):

$$\mathcal{L} = [[1, 4]].$$

Thus $\mathcal{L}^c = \emptyset$ and we directly go to Step 3.

Step 3. Computation of $G'$ s.t. $\mathcal{I}(\mathcal{L}) = \text{im}(G') \cap k^4$ recalling that $\mathcal{I}(\mathcal{L})$ is defined by (19):

- Compute $\tilde{B}_i$ defined by (24b), $i = 1, \ldots, 4$:
  \[ \tilde{B}_1 := \{1\}, \quad \tilde{B}_2 := \{3\}, \quad \tilde{B}_3 := \{2\}, \quad \tilde{B}_4 := \{3, 4\}. \]

- Compute $G^j$ defined by (25), (17), $j \in \tilde{B}_1 \times \cdots \times \tilde{B}_4$:
  - compute vectors $r^{i,j}$ defined by (16):
    \[ r^{1,1} = (0, 0, -\infty, -\infty) \]
    \[ r^{2,3} = (-\infty, -\infty, 0, 0) \]
    \[ r^{3,2} = (-\infty, 0, -6, -8) \]
    \[ r^{4,3} = (-3, -\infty, 0, -8) \]
    \[ r^{4,4} = (-1, -\infty, -4, 0) \]
  - compute matrices $Q^{i,j}$ defined by (17):
    \[ Q^{1,1} = \begin{pmatrix} 0 & 0 & -\infty & -\infty \\ -\infty & 0 & -\infty & -\infty \\ -\infty & -\infty & 0 & -\infty \\ -\infty & -\infty & -\infty & 0 \end{pmatrix} \]
    \[ Q^{2,3} = \begin{pmatrix} 0 & -\infty & -\infty & -\infty \\ -\infty & 0 & -\infty & -\infty \\ -\infty & -\infty & 0 & 0 \\ -\infty & -\infty & -\infty & 0 \end{pmatrix} \]
    \[ Q^{3,2} = \begin{pmatrix} 0 & -\infty & -\infty & -\infty \\ -\infty & 0 & -6 & -8 \\ -\infty & -\infty & 0 & -\infty \\ -\infty & -\infty & -\infty & 0 \end{pmatrix} \]
    \[ Q^{4,3} = \begin{pmatrix} 0 & -\infty & -\infty & -\infty \\ -\infty & 0 & -\infty & -\infty \\ -3 & -\infty & 0 & -8 \\ -\infty & -\infty & -\infty & 0 \end{pmatrix} \]
    \[ Q^{4,4} = \begin{pmatrix} 0 & -\infty & -\infty & -\infty \\ -\infty & 0 & -\infty & -\infty \\ -\infty & -\infty & 0 & -\infty \\ -1 & -\infty & -4 & 0 \end{pmatrix} \]
\[ G^{(1,3,2,3)} = (Q^{1,1} \oplus Q^{2,3} \oplus Q^{3,2} \oplus Q^{4,3})^* = \begin{pmatrix} 0 & 0 & -\infty & -\infty \\ -\infty & 0 & -6 & -8 \\ -3 & -\infty & 0 & 0 \\ -\infty & -\infty & -\infty & 0 \end{pmatrix} \]

\[ = \begin{pmatrix} 0 & 0 & -6 & -6 \\ -9 & 0 & -6 & -6 \\ -3 & -3 & 0 & 0 \\ -\infty & -\infty & -\infty & 0 \end{pmatrix} \]

\[ \text{and} \]

\[ G^{(1,3,2,4)} = (Q^{1,1} \oplus Q^{2,3} \oplus Q^{3,2} \oplus Q^{4,4})^* = \begin{pmatrix} 0 & 0 & -\infty & -\infty \\ -\infty & 0 & -6 & -8 \\ -\infty & -\infty & 0 & 0 \\ -1 & -\infty & -4 & 0 \end{pmatrix} \]

\[ = \begin{pmatrix} 0 & 0 & -6 & -6 \\ -7 & 0 & -6 & -6 \\ -1 & -1 & 0 & 0 \\ -1 & -1 & -4 & 0 \end{pmatrix} \]

\[ \bullet \quad G' := (G^j, j \in \tilde{B}_1 \times \cdots \times \tilde{B}_4): \]

\[ G' = \begin{pmatrix} 0 & 0 & -6 & -6 & 0 & 0 & -6 & -6 \\ -9 & 0 & -6 & -6 & -7 & 0 & -6 & -6 \\ -3 & -3 & 0 & 0 & -1 & -1 & 0 & 0 \\ -\infty & -\infty & -\infty & 0 & -1 & -1 & -4 & 0 \end{pmatrix}. \]

Step 4. Compute matrix \( G \) such that \( C_{A,B} = \text{im}(G) \cap \mathbb{R}^4 \) based on the result of Corollary 3.1:

\[ G = G'. \]

Recalling that the \( x_k \)'s are daters and thus their components are \( \geq 0 \) we take \( u \) such \( x_0 = G \odot u \) have all its components \( \geq 0 \). Let us take e.g. \( u = (-\infty, -\infty, 6, 6, -\infty, -\infty, -\infty, -\infty)^T \), then: \( x_0 = G \odot u = (0, 0, 6, 6)^T \). It means that the first departure of trains on tracks \( d_1 \) and \( d_2 \) will occur at instant 0 and the first departure of trains on tracks \( d_3 \) and \( d_4 \) will occur at instant 6. Computing \( x_1 \) and \( x_2 \) we have:

\[ x_1 = \Omega \odot x_0 = (17, 17, 17, 17)^T, \quad x_2 = \Omega \odot x_1 = (34, 28, 31, 31)^T, \]

which lie in the octagon \( O \) also defined as the idempotent polyhedral cone \( C_{A',B'} \).
A. Proof of Theorem 3.1

In the following lemma we obtain the decomposition formula for the idempotent half space $C_{a(i,\cdot)b(i,\cdot)}$.

**Lemma A.1.** For all $i \in [[1, m]]$:

$$C_{a(i,\cdot)b(i,\cdot)} := \{ x \in \mathbb{R}^n | a(i,\cdot) \odot x \leq b(i,\cdot) \odot x \} = \begin{cases} \cup_{j \in [i, m]} \mathcal{H}^+_{Q_{i,j}} & \text{if } B_i \neq \emptyset \\ \ker(a(i,\cdot)) & \text{otherwise.} \end{cases}$$

**Proof.** We have $B_i = \emptyset \iff b(i,\cdot) = 0^T$. Then, $a(i,\cdot) \odot x \leq b(i,\cdot) \odot x \iff a(i,\cdot) \odot x = 0 \iff x \in \ker(a(i,\cdot))$.

Now, assume that $B_i \neq \emptyset$.

$$a(i,\cdot) \odot x \leq b(i,\cdot) \odot x \iff (a(i,\cdot) \oplus b(i,\cdot)) \odot x \leq b(i,\cdot) \odot x$$

(because $\oplus$ is idempotent)

$$\iff \bigcup_{j \in [i, m]} [(a(i,\cdot) \oplus b(i,\cdot)) \odot x \leq b(i,j) \odot x(j)]$$

(because $\leq$ is total).

Case 1: $b(i,j) \neq 0$. By definition of vector $r^{i,j}$ (see (16), Theorem 3.1) we have:

$$(a(i,\cdot) \oplus b(i,\cdot)) \odot x \leq b(i,j) \odot x(j) \iff r^{i,j} \odot x \leq x_j.$$ Now, by definition of the matrix $Q^{i,j}$ (see (17), Theorem 3.1) the following equivalence relation:

$$r^{i,j} \odot x \leq x_j \iff Q^{i,j} \odot x \leq x,$$

is true. It means that $\{ x \in \mathbb{R}^n | r^{i,j} \odot x \leq x_j \} = \mathcal{H}^+_{Q_{i,j}}$.

Case 2: $b(i,j) = 0$. Then:

$$(a(i,\cdot) \oplus b(i,\cdot)) \odot x \leq b(i,j) \odot x(j) \iff (a(i,\cdot) \oplus b(i,\cdot)) \odot x \leq 0$$

$$\iff x \in \ker(a(i,\cdot) \oplus b(i,\cdot)).$$

Finally we have:

$$\{ x \in \mathbb{R}^n | \bigcup_{j \in [i, m]} [(a(i,\cdot) \oplus b(i,\cdot)) \odot x \leq b(i,j) \odot x(j)] \}$$

$$= \{ x \in \mathbb{R}^n | \bigcup_{j \in B_i} [(a(i,\cdot) \oplus b(i,\cdot)) \odot x \leq b(i,j) \odot x(j)] \} \text{ or}$$

$$\bigcup_{j \in B_i} \mathcal{H}^+_{Q_{i,j}} \cup \left\{ \begin{array}{ll} \ker(a(i,\cdot) \oplus b(i,\cdot)) & \text{if } B_i \neq \emptyset \\ \emptyset & \text{if } B_i = \emptyset. \end{array} \right\}$$

In the case where $B_i \neq \emptyset$ we have to prove that:

$$\forall j \in B_i, \ker(a(i,\cdot) \oplus b(i,\cdot)) \subseteq \mathcal{H}^+_{Q_{i,j}}.$$

First, let us note that: $\ker(a(i,\cdot) \oplus b(i,\cdot)) = \text{im}(e_k, k \notin \text{supp}(a(i,\cdot) \oplus b(i,\cdot)))$, recalling that $e_k$ denotes the $k$th vector of the canonical basis of $\mathbb{R}^n$. 

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For all \( l \in [[1, n]] \), for all \( k \not\in \text{supp}(a(i, \cdot) \oplus b(i, \cdot)) \):
\[
Q^{i,j}(l, \cdot) \circ e_k = \begin{cases} 
 e_l^T \circ e_k & \text{if } l \neq j \\
 r^{i,j}(k) & \text{if } l = j.
\end{cases}
\]
e_l^T \circ e_k = 1 \text{ if } l = k \text{ and } 0 \text{ otherwise. Because } \forall j \in B_i \text{ supp}(r^{i,j}) = \text{supp}(a(i, \cdot) \oplus b(i, \cdot)) \text{ then } \forall k \not\in \text{supp}(a(i, \cdot) \oplus b(i, \cdot)) : r^{i,j}(k) = 0. \text{ Thus, }
Q^{i,j} \circ e_k = e_k \Rightarrow Q^{i,j} \circ e_k \leq e_k \Rightarrow e_k \in \mathcal{H}_{Q^{i,j}}^+. \quad \square
\]

To conclude the proof of the theorem let us remark that \( \mathcal{C}_{A,B} = \cap_{i=1}^{m} \mathcal{C}_{a(i, \cdot), b(i, \cdot)} \). Thus, we easily deduce from Lemma A.1 that:
\[
\mathcal{C}_{A,B} = \cap_{i \in \mathcal{L}} \ker(a(i, \cdot)) \cap \cap_{i \in \mathcal{L}} \left( \cup_{j_i \in B_i} \mathcal{H}_{Q^{i,j_i}}^+ \right),
\]
with the convention \( \cap_{\emptyset} = k^n \). \quad \square

**B. Proof of Theorem 3.3**

Recall that we have the following notation convention for the set \( \mathcal{L} : \mathcal{L} = [[1, m']] \). From the definition of set \( \mathcal{I}(\mathcal{L}) \) (see (19), Theorem 3.1) and by distributivity of \( \cap \) over \( \cup \) we have:
\[
\mathcal{I}(\mathcal{L}) = \cup_{j \in B_1 \times \cdots \times B_{m'}} \cap_{i=1}^{m'} \mathcal{H}_{Q^{i,j}}^+.
\]
For a fixed \( j \in B_1 \times \cdots \times B_{m'} \), thanks to (P2), Proposition 2.3 we have: \( \cap_{i=1}^{m'} \mathcal{H}_{Q^{i,j}}^+ = \mathcal{H}_{\oplus_{i=1}^{m'} Q^{i,j_i}}^+ \). Applying (P1), Proposition 2.3 with \( G^2 := (\oplus_{i=1}^{m'} Q^{i,j_i})^* \) we have:
\[
\cap_{i=1}^{m'} \mathcal{H}_{Q^{i,j_i}}^+ = \text{im}(G^2) \cap k^n.
\]
Thus:
\[
\mathcal{I}(\mathcal{L}) = \mathcal{C}_{A', B'} = \cup_{j \in B_1 \times \cdots \times B_{m'}} \text{im}(G^2) \cap k^n.
\]
To obtain further results on this decomposition formula we need to study the properties of the matrices \( Q^{i,j} \) defined by ((17), Theorem 3.1).

**Proposition B.1.** Assume that \( \mathcal{L} \neq \emptyset \). Then for all \( i \in B_i \) we have:
\[
\forall l \in [[1, n]], \quad Q^{i,j}(l, \cdot) \circ Q^{i,j} = \begin{cases} 
 e_l^T \circ (1 \oplus r^{i,j}(j)) \circ r^{i,j} & \text{if } l \neq j \\
 r^{i,j}(k) & \text{if } l = j.
\end{cases}
\]

**Proof.** Let \( l \in [[1, n]] \). If \( l \neq j \) then \( Q^{i,j}(l, \cdot) = e_l^T \) and \( Q^{i,j}(l, \cdot) \circ Q^{i,j} = e_l^T \circ Q^{i,j} = Q^{i,j}(l, \cdot) = e_l^T \). If \( l = j \) then \( Q^{i,j}(l, \cdot) = r^{i,j} \). Let us remark that:
\[
r^{i,j}(j) \leq r^{i,j}(j) \circ r^{i,j}(j),
\]
thus: \( r^{i,j}(j) \circ r^{i,j}(j) \circ r^{i,j}(j) = (r^{i,j}(j))^\circ 2 \). From this remark and the structure of matrix \( Q^{i,j} \) we have for all \( k \in [[1, n]] \):
\[
r^{i,j} \circ Q^{i,j}(\cdot, k) = r^{i,j}(k) \circ r^{i,j}(k) \circ r^{i,j}(j)
\]
which is equivalent to: \( r^{i,j} \circ Q^{i,j} = (1 \oplus r^{i,j}(j)) \circ r^{i,j} \). The result is now proved. \quad \square

By an obvious recurrence from the previous proposition we have:
Proposition B.2.

\[ \forall q \geq 1, \forall l \in [1, n], \quad (Q^{i,j})^{\circ q}(l, \cdot) = \begin{cases} e_i^T & \text{if } l \neq j \\ (1 \oplus r^{i,j}(j))^{\circ (q-1)} \odot r^{i,j} & \text{if } l = j \end{cases} \]

with the convention that \( a^{\circ 0} = 1 \), \( \forall a \in \mathbb{k} \).

We complete the study of the matrices \( Q^{i,j} \) by the following propositions.

Proposition B.3.

\[ 1 < r^{i,j}(j) \Rightarrow \lim_{q \to \infty} (Q^{i,j})^{\circ q}(l, \cdot) = \begin{cases} e_l^T & \text{if } l \neq j \\ \top \odot r^{i,j} & \text{if } l = j \end{cases} \]

and

\[ 1 = r^{i,j}(j) \Rightarrow \forall q \geq 1, \quad (Q^{i,j})^{\circ q} = Q^{i,j} = I \oplus Q^{i,j}. \]

Proposition B.4. \( \forall i \in \mathcal{L} \) recall that \( B'_i := \{ j \in B_i | 1 < r^{i,j}(j) \text{ and } \text{supp}(r^{i,j}) = [1, n] \} \) and \( \tilde{B}_i := B_i \setminus B'_i \). Then, we have:

\[ B'_i \neq \emptyset \Rightarrow (Q^{i,j})^*(l, \cdot) = \begin{cases} e_i^T & \text{if } l \neq j \\ \top & \text{if } l = j \end{cases} \]

and:

\[ \forall j \in \tilde{B}_i \neq \emptyset, \quad (Q^{i,j})^* = Q^{i,j}. \]

We remark that \( G^2 \geq \oplus_{j=1}^{m'} (Q^{i,j})^* \). Thus, from Proposition B.4 if there exists \( i \in [1, m'] \) such that \( B'_i \neq \emptyset \) then for all \( j \) such that \( j_i \in B'_i \), \( G^2(j_i, \cdot, \cdot) = \top^T \) and \( \text{im}(G^2) \cap \mathbb{k}^n = \{ \alpha \} \). Thus, we have:

\[ \mathcal{I}(\mathcal{L}) = C_{A',B'} = \bigcup_{j \in \tilde{B}_1 \times \cdots \times \tilde{B}_{m'}} \text{im}(G^2) \cap \mathbb{k}^n. \]

Applying (P5), Proposition 2.3 we have: \( C_{A',B'} \subseteq \text{im}(G) \cap \mathbb{k}^n \). Conversely, let \( x \in \text{im}(G) \cap \mathbb{k}^n \) then \( \exists \alpha \), \( \beta \in \tilde{B}_1 \times \cdots \times \tilde{B}_{m'} \) such that:

\[ x = \bigoplus_{j \in \tilde{B}_1 \times \cdots \times \tilde{B}_{m'}} G^2 \odot \alpha. \]

By definition of matrices \( G^2 \) (see (25), Theorem 3.3) \( \text{im}(G^2) \cap \mathbb{k}^n \subseteq C_{A',B'} \) which implies that: \( A' \odot G^2 \leq B' \odot G^2 \). Thus, because \( \oplus \) and \( \odot \) are non-decreasing (see Proposition 2.1):

\[ x \in \mathbb{k}^n \text{ and } A' \odot x \leq B' \odot x, \]

and \( x \in C_{A',B'} \). Finally, \( C_{A',B'} = \text{im}(G') \cap \mathbb{k}^n \). \( \square \)
C. Algorithmic construction of matrix $G$ such that $C_{A,B} = \text{im}(G) \cap k^n$

Step 1. Compute $\mathcal{L}$ defined by (15).

Step 2. If $\mathcal{L}^c \neq \emptyset$ then compute $\mathcal{K}(\mathcal{L}^c)$ defined by (18) using (22b) else goto Step 3.

Step 3. Computation of $G'$ s.t. $\mathcal{I}(\mathcal{L}) = \text{im}(G') \cap k^n$ recalling that $\mathcal{I}(\mathcal{L})$ is defined by (19):

If $\mathcal{L} = \emptyset$ return $k^n$
else (* assume $\mathcal{L} = [1, m']$)
   - compute $\tilde{\mathcal{B}}_i$ defined by (24b), $i = 1, \ldots, m'$
   - compute $G^\mathcal{L}$ defined by (25), (17), $j \in \tilde{\mathcal{B}}_1 \times \cdots \times \tilde{\mathcal{B}}_{m'}$:
     * compute matrix $Q^{i,j_i}$ defined by (17), for $i = 1, \ldots, m'$, $j_i \in \tilde{\mathcal{B}}_i$
     * compute $G^\mathcal{L} := (\oplus_{i=1}^{m'} Q^{i,j_i})^*$ (see formula (25))
   - $G' := (G^\mathcal{L}, j \in \tilde{\mathcal{B}}_1 \times \cdots \times \tilde{\mathcal{B}}_{m'})$.

Step 4. Compute matrix $G$ such that $C_{A,B} = \text{im}(G) \cap k^n$ based on the result of Corollary 3.1, i.e.:
   - eliminate all column $G'(:, k)$ s.t $\mathbf{1}^T \odot G'(:, k) = \top$
   - if $\mathcal{L}^c \neq \emptyset$ eliminate all column $G'(:, k)$ s.t $\exists l \in \cup_{i \in \mathcal{L}^c} \mathcal{A}_i: G'(l, k) \neq 0$, recalling that sets $\mathcal{A}_i$ are defined by (21).

Remark C.1. The computations of the sets $\mathcal{K}(\mathcal{L}^c)$ and $\mathcal{I}(\mathcal{L})$ are independent. The computations of the vectors $r^{i,j}$ are independent. From this fact one deduces that we can parallelize:
   - the computation of the sets $\tilde{\mathcal{B}}_i$, $i = 1, \ldots, m'$
   - the computation of the matrices $Q^{i,j}$
   - and finally the computation of the matrices $G^\mathcal{L}$, $j \in \tilde{\mathcal{B}}_1 \times \cdots \times \tilde{\mathcal{B}}_{m'}$.

Remark C.2. The computation of the matrices $G^\mathcal{L}$ requires the computation of the Kleene star of a matrix. For details and references on the computation of the Kleene star of a matrix $A$, $A^*$, the reader is referred to e.g. [4, Chapter 3] and [22].

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References


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