The Densest Translation Ball Packing by Fundamental Lattices in Sol Space

To the Memory of Professor Benno Klotzek

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Abstract. In the eight homogeneous Thurston 3-geometries – E³, S³, H³, S²×R, H²×R, \( \widetilde{\text{SL}_2\mathbb{R}} \), Nil, Sol – the notions of translation curves and translation balls can be introduced in a unified way by initiative of E. Molnár (see [3], [7]). P. Scott in [9] defined Sol lattices to which lattice-like translation ball packings can be defined.

In our joint work [8] with E. Molnár we have studied the relation between Sol lattices and lattices of the pseudoeuclidean (or Minkowskian) plane (see [1], [2]). In the present paper the translation balls of Sol geometry are investigated, their volume is computed, and the notions of Sol parallelepiped and density of the lattice-like ball packing are defined. Moreover, the densest translation ball packing by so-called fundamental lattices, which is one (Type I/1) of the 17 Bravais-type of Sol-lattices described in [8] is determined. It turns out that the optimal arrangement has a richer symmetry group (in Type I/2) for \( N = 4 \). This density is \( \delta \approx 0.56405083 \) and the kissing number of the balls to this packing is 6. In our work we shall use the affine model of the Sol space through affine-projective homogeneous coordinates introduced by E. Molnár in [5].

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†I regret to inform the reader that Professor Benno Klotzek died on 13 June 2008 in Potsdam.
1. On Sol geometry

In this section we summarize the significant notions and denotations of Sol geometry (see [5], [9]).

Sol is defined as a real Lie group by multiplication
\[(a, b, c)(x, y, z) = (x + ae^{-z}, y + be^{z}, z + c).\]  
(1.1)

We note that the conjugacy by \((x, y, z)\) acts on the \((a, b, 0)\) plane:
\[(x, y, z)^{-1}(a, b, 0)(x, y, z) = (ae^{-z}, be^{z}, 0)\]  
(1.2)

only by its \(z\)-component, where \((x, y, z)^{-1} = (-xe^z, -ye^{-z}, -z)\). Sol geometry can also be affinely (projectively) interpreted by “right translations” on its points as the following matrix formula shows according to (1.1):
\[
(1; a, b, c) \rightarrow (1; a, b, c) \begin{pmatrix} 1 & x & y & z \\ 0 & e^{-z} & 0 & 0 \\ 0 & 0 & e^{z} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = (1; x + ae^{-z}, y + be^{z}, z + c)
\]  
(1.3)

by row-column multiplication. This defines “translations” \(L(R) = \{(x, y, z) : x, y, z \in \mathbb{R}\}\) on the points of Sol = \{(a, b, c) : a, b, c \in \mathbb{R}\}. These translations are not commutative, in general. Here we consider \(L\) as projective collineation group with right actions in homogeneous coordinates as usual in classical affine-projective geometry. We will use the Cartesian homogeneous coordinate simplex \(E_0(e_0), E_1^\infty(e_1), E_2^\infty(e_2), E_3^\infty(e_3), \{e_i\} \subset V^4\) with unit point \(E(e = e_0 + e_1 + e_2 + e_3)\) which is distinguished by an origin \(E_0\) and by the ideal points of coordinate axes, respectively. Thus Sol can be visualized in the affine 3-space \(A^3\) (so in \(E^3\)) as well [3].

In this affine-projective context E. Molnár has derived in [5] the usual infinitesimal arc-length square at any point of Sol by pull back translation as follows
\[(ds)^2 := e^{2z}(dx)^2 + e^{-2z}(dy)^2 + (dz)^2.\]  
(1.4)

Hence we get infinitesimal Riemann metric by the symmetric metric tensor field \(g\) on Sol by components as usual.

The full isometry group of Sol, leaving the metric (1.4) invariant, has eight components, since the stabilizer of the origin is isomorphic to the dihedral group \(D_4\).

2. Translation curves and balls

We consider a Sol curve \((1, x(t), y(t), z(t))\) with a given starting tangent vector at the origin \(O(1, 0, 0, 0)\)
\[u = \dot{x}(0), \ v = \dot{y}(0), \ w = \dot{z}(0).\]  
(2.1)
For a translation curve let its tangent vector at the point \((1, x(t), y(t), z(t))\) be defined by the matrix (1.3) with the following equation:

\[
(0, u, v, w) \begin{pmatrix}
1 & x(t) & y(t) & z(t) \\
0 & e^{-z(t)} & 0 & 0 \\
0 & 0 & e^{z(t)} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} = (0, \dot{x}(t), \dot{y}(t), \dot{z}(t)).
\] (2.2)

Thus, translation curves in \(\text{Sol}\) geometry (see [3] and [7]) are defined by the first order differential equation system

\[
\dot{x}(t) = u e^{-z(t)}, \quad \dot{y}(t) = v e^{z(t)}, \quad \dot{z}(t) = w,
\]

whose solution is the following:

\[
\begin{align*}
x(t) &= -\frac{u}{w}(e^{-wt} - 1), \\
y(t) &= \frac{v}{w}(e^{wt} - 1), \\
z(t) &= wt,
\end{align*}
\]

if \(w \neq 0\) and

\[
\begin{align*}
x(t) &= ut, \\
y(t) &= vt, \\
z(t) &= z(0) = 0
\end{align*}
\]

if \(w = 0\). (2.3)

Remark 2.1. The geodesic curves in \(\text{Sol}\) geometry can be determined by a second order differential equation system whose solution with elliptic integral cannot be expressed in terms of elementary functions, in general (see [3]). This topic is having been studied by my colleagues on the base of [5]. Translation curves and spheres (balls) seem to be simpler and more natural than geodesical ones in \(\text{SL}_2\mathbb{R}, \text{Nil}\) and \(\text{Sol}\) geometry. In the other 5 geometries the situation is simpler [7].

We assume that the starting point of a translation curve is the origin, because we can transform a curve into an arbitrary starting point by translation (1.3), moreover, unit velocity translation can be assumed:

\[
\begin{align*}
x(0) &= y(0) = z(0) = 0; \\
u &= \dot{x}(0) = \cos\theta \cos\phi, \\
v &= \dot{y}(0) = \cos\theta \sin\phi, \\
w &= \dot{z}(0) = \sin\theta;
\end{align*}
\]

\[-\pi \leq \phi \leq \pi, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}.\] (2.4)

Definition 2.1. The translation distance \(d^t(P_1, P_2)\) between the points \(P_1\) and \(P_2\) is defined by the arc length of the above translation curve from \(P_1\) to \(P_2\).

Definition 2.2. The sphere of radius \(r > 0\) with centre at the origin (denoted by \(S^t_O(r)\)) with the usual longitude and altitude parameters \(\phi\) and \(\theta\), respectively, by (2.4), is specified by the following equations:

\[
S^t_O(r) : \begin{cases}
x(\phi, \theta) = -\cot\theta \cos\phi(e^{-r\sin\theta} - 1), \\
y(\phi, \theta) = \cot\theta \sin\phi(e^{r\sin\theta} - 1), \\
z(\phi, \theta) = r \sin\theta.
\end{cases}
\] (2.5)

Definition 2.3. The body of the translation sphere of centre \(O\) and of radius \(r\) in the \(\text{Sol}\) space is called translation ball, denoted by \(B^t_O(r)\), i.e. \(Q \in B^t_O(r)\) iff \(0 \leq d^t(O, Q) \leq r\).
Remark 2.2. The translation sphere is a simply connected surface without self-intersection in $\text{Sol}$ space.

We obtain the volume formula of the translation ball $B^r_O(r)$ of radius $r$ by the Jacobian of (2.5) and a careful Maple computation by the following integral:

**Theorem 2.1.**

\[
\text{Vol}(B^r_O(r)) = \int_V \, dx \, dy \, dz
\]

\[
= \int_0^r \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos \theta}{\sin^2 \theta} (e^{r \sin \theta} + e^{-r \sin \theta} - 2) \, d\phi \, d\theta \, d\rho \tag{2.6}
\]

\[
= 4\pi \int_0^r \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos \theta}{\sin^2 \theta} (\cosh(r \sin \theta) - 1) \, d\theta \, d\rho.
\]

An easy power series expansion with substitution $\rho \sin \theta =: z$ can also be applied, no more detailed.

From the equation of the translation spheres $S^r_O(r)$ (see (2.5)) it follows that the plane sections of following spheres, given by parameters $\theta$ and $r$, parallel to $[x, y]$ plane are ellipses by the equations (see Figure 3, $r = 3$):

\[
\frac{x^2}{k_1^2} + \frac{y^2}{k_2^2} = 1 \quad \text{where}
\]

\[
k_1^2 = (-\cot \theta(e^{-r \sin \theta} - 1))^2, \quad k_2^2 = (\cot \theta(e^{r \sin \theta} - 1))^2. \tag{2.7}
\]
3. The discrete translation group $\Gamma(\Phi)$

**Definition 3.1.** Let $\Gamma < \text{L}(\mathbb{R})$ be a subgroup, generated by three translations $\tau_1(t_1^1, t_1^2, t_1^3)$, $\tau_2(t_2^1, t_2^2, t_2^3)$, $\tau_3(t_3^1, t_3^2, t_3^3)$ with $\mathbb{Z}$ (integer) linear combinations. Here upper indices indicate the corresponding $(e_1, e_2, e_3)$ coordinates of basis translations. $\Gamma$ is called discrete translation group or lattice of $\text{Sol}$, if its action is discrete (by the induced orbit topology), i.e. there is a compact fundamental parallelepiped” (with side face identifications on its bent side faces) $	ilde{F} = \mathbb{L}/\Gamma$ (Figure 5).

**Theorem 3.1.** Each lattice $\Gamma$ of $\text{Sol}$ has a group presentation (see [8], [9], [4])

$$
\Gamma = \Gamma(\Phi) = \{\tau_1, \tau_2, \tau_3 : [\tau_1, \tau_2] = 1, \tau_3^{-1}\tau_1\tau_3 = \tau_1\Phi^T, \tau_3^{-1}\tau_2\tau_3 = \tau_2\Phi^T\},
$$

(3.1)

where $t_1^3 = t_2^3 = 0$ holds, and this implies $[\tau_1, \tau_2] = \tau_1^{-1}\tau_2^{-1}\tau_1\tau_2 = 1$, that means, the commutator of $\tau_1, \tau_2$ is the identity, as a first condition of discrete action of $\Gamma$. Moreover there exists $\Phi = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ with $\text{tr}(\Phi) = p + s > 2$, $ps - qr = 1$, such that for above $\tau_1(t_1^1, t_1^2, 0)$, $\tau_2(t_2^1, t_2^2, 0)$ the matrix $T = \begin{pmatrix} t_1^1 & t_2^1 \\ t_1^2 & t_2^2 \end{pmatrix} \in \text{GL}_2(\mathbb{R})$ satisfies the following: $T^{-1}\Phi T =: \Phi^T = \begin{pmatrix} e^{-t_3^3} & 0 \\ 0 & e^{t_3^3} \end{pmatrix}$ is just a hyperbolic rotation fixed by $t_3^3$ in $\tau_3$ above. Namely, $\tau_1\Phi^T = (t_1^1 e^{-t_3^3}, t_1^2 e^{t_3^3})\begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$, $\tau_2\Phi^T = (t_2^1 e^{-t_3^3}, t_2^2 e^{t_3^3})\begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$ hold in the commutative basic vector plane of $\text{Sol}$, spanned by...
e_1 and e_2. These basis vectors are just the eigenvectors of Φ^T to eigenvalues e^{-t_3^1} and e^{t_3^2}, respectively.

Remark 3.1. This theorem shows some new aspects to P. Scott’s discussion in [9]. The “discreteness conditions” provide new possibilities for attacking our problem. E.g. t_1^3 = t_2^3 = 0 above is not necessary condition yet, but it can be assumed later on. See also our new publication [8] on the classification of Sol lattices.

Remark 3.2. We refer only to 
\[
[τ_1, τ_2] = τ_1^{-1}τ_2^{-1}τ_1τ_2 =
\begin{pmatrix}
1 & t_2^1(1-e^{-t_1^1}) + t_1^1(e^{-t_2^1} - 1) & t_2^2(1-e^{t_1^3}) + t_1^2(e^{t_2^3} - 1) & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]
that means [τ_1, τ_2] is in the base plane, and the same is true for [τ_2, τ_3] and [τ_3, τ_1]. Thus t_1^3 = 0 = t_2^3 can be taken, and so [τ_1, τ_2] = 1 the identity.

Definition 3.2. A Sol point lattice Γ_P(Φ) is a discrete orbit of point P in the Sol space generated by an arbitrary lattice Γ(Φ) above. For visualizing a point lattice we have chosen the origin as starting point, by the homogeneity of Sol.

In the next section we shall mention some aspects to Theorem 3.1.

4. On Sol lattices

We consider Sol translations defined in (1.1) and (1.3). We assume in the following that t_1^3 = t_2^3 = 0 in Definition 3.1 (see Remark 3.2.) and choose two such
translations \((\tau_1, \tau_2)\) for the plane lattice \(\Gamma^0 < \Gamma\), as normal subgroup (given by its orthonormal coordinate system \((O, e_1, e_2)\), see formula (1.4) and Figure 4). The third basis translation \(\tau_3\) has the crucial component \(t^3_3\) to the third \(e_3\) direction. The coordinate plane \([x, y]\) and the parallel planes \([x, y]^{(\tau_3)k}\), \(k \in \mathbb{Z}\) produced by \(\tau_3\), contain congruent integer lattices generated by \(\tau_1\) and \(\tau_2\). These lattices are denoted by \(\Gamma^{k\tau_3}\) \((k \in \mathbb{Z})\). We repeat for explanations

\[
\begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} \Phi^T = \begin{pmatrix} t^1_1 & t^1_2 \\ t^2_1 & t^2_2 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \Phi^T = \begin{pmatrix} t^1_1 & t^1_2 \\ t^2_1 & t^2_2 \end{pmatrix} \begin{pmatrix} e^{-t^3_3} & 0 \\ 0 & e^{t^3_3} \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \in \Gamma^0(\tau_1, \tau_2),
\]

i.e.

\[
\begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} \Phi^T = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} t^1_1 & t^1_2 \\ t^2_1 & t^2_2 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \tag{4.2}
\]

Here stands

\[
\Phi = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \in SL_2(\mathbb{Z}), \text{ i.e. } p, q, r, s \in \mathbb{Z}, \text{ and } ps - qr = 1, \text{ so that }
\]

\[
0 \neq \det \begin{pmatrix} t^1_1 & t^1_2 \\ t^2_1 & t^2_2 \end{pmatrix} = t^1_1 t^2_2 - t^2_1 t^1_2 = D \Rightarrow \begin{pmatrix} t^1_1 & t^1_2 \\ t^2_1 & t^2_2 \end{pmatrix}^{-1} = \frac{1}{D} \begin{pmatrix} t^2_1 & -t^2_2 \\ -t^1_2 & t^1_1 \end{pmatrix},
\]

thus

\[
\begin{pmatrix} p & q \\ r & s \end{pmatrix} = \frac{1}{D} \begin{pmatrix} t^1_1 & t^1_2 \\ t^2_1 & t^2_2 \end{pmatrix} \begin{pmatrix} e^{-t^3_3} & 0 \\ 0 & e^{t^3_3} \end{pmatrix} \frac{t^2_1 & -t^2_2 \\ -t^1_2 & t^1_1} = \frac{1}{D} \begin{pmatrix} t^1_1 e^{-t^3_3} t^2_2 - t^2_1 e^{t^3_3} t^1_1 & -t^1_1 e^{-t^3_3} t^2_1 + t^2_1 e^{t^3_3} t^1_1 \\ t^2_1 e^{-t^3_3} t^2_2 - t^2_1 e^{t^3_3} t^1_1 & -t^2_1 e^{-t^3_3} t^2_1 + t^2_1 e^{t^3_3} t^1_1 \end{pmatrix} = \begin{pmatrix} e^{-t^3_3} - \frac{2t^1_1 t^2_2}{D} \sinh t^3_3 & \frac{2t^1_1 t^2_2}{D} \sinh t^3_3 \\ \frac{2t^1_1 t^2_2}{D} \sinh t^3_3 & e^{t^3_3} + \frac{2t^1_1 t^2_2}{D} \sinh t^3_3 \end{pmatrix} \tag{4.3}
\]

holds for later arguments. E.g. \(N = p + s = 2 \cosh t^3_3\), and the equalities

\[
\frac{t^1_2}{t^1_1} = \frac{N - 2p - \sqrt{N^2 - 4}}{2q}, \quad \frac{t^2_2}{t^2_1} = \frac{N - 2p + \sqrt{N^2 - 4}}{2q} \tag{4.3'}
\]

hold by [8].

**Remark 4.1.** From this we read:

1. The role of matrices \(T\) and \(\Phi\) in Theorem 3.1.

2. The matrix equation above shows, how to determine the possible matrices \(T\), so \(\tau_1\) and \(\tau_2\) for given \(p, q, r, s\), i.e. above \(\Phi\). This will express also the affine equivalence of our fundamental lattices, see also Definitions 4.1, 5.2.

Namely \(D = \omega^2\), \(T = \begin{pmatrix} \omega \cosh \mu & \omega \sinh \mu \\ \omega \sinh \mu & \omega \cosh \mu \end{pmatrix}\) can be solved for \(\omega\) and \(\mu\) by formulas (4.3).
3. The main invariant of our lattice \( \Gamma^0(\Phi)(\tau_1, \tau_2) \) is \( p + s = \text{tr}(\Phi) = 2 \cosh t_3 \), we also denote it by \( p + s = \mathcal{N} \in \mathbb{N} \) (natural numbers). Different traces \( \mathcal{N} \) surely lead to affinely non-equivalent Sol-lattices.

If we ask for a \( \Phi \)-invariant bilinear form or scalar product

\[
\langle x, y \rangle = \langle x^i e_i, y^j e_j \rangle = x^i \langle e_i, e_j \rangle y^j =: x^i b_{ij} y^j
\]

(by Einstein-Schouten index conventions for indices 1, 2) for the typical lattice transformation \( \Phi^T \) as a hyperbolic rotation, then we get the signature \((-\,+,+\,+)\) for \( b_{ij} \). Thus, we say that the plane lattice \( \Gamma^0(\Phi) \) is a pseudoeuclidean or Minkowskian lattice. In a new basis

\[
\begin{pmatrix}
e_1 \\
e_2
\end{pmatrix} = \begin{pmatrix}
\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\
-\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}
\end{pmatrix} \begin{pmatrix}e_1 \\
e_2
\end{pmatrix},
\]

we get

\[
\begin{pmatrix}e_1 \\
e_2
\end{pmatrix} \Phi^T = \begin{pmatrix}\cosh t_3 & \sinh t_3 \\
\sinh t_3 & \cosh t_3
\end{pmatrix} \begin{pmatrix}e_1 \\
e_2
\end{pmatrix}
\]

for the hyperbolic rotation

\[
\begin{pmatrix}e_1 \\
e_2
\end{pmatrix} \Phi^T = \begin{pmatrix}e^{-t_3} & 0 \\
0 & e^{t_3}
\end{pmatrix} \begin{pmatrix}e_1 \\
e_2
\end{pmatrix}.
\]

Figure 4.

The so-called regular lattices of the pseudoeuclidean or Minkowskian plane were investigated by K. Alpers and E. Quaisser in [1] (see also [2]). The pseudoeuclidean plane, denoted by \( E^2 \), is an affine plane over the field \( \mathbb{R} \) of real numbers together with a symmetric bilinear form just of signature \((-\,+,+)\). In [1] these \( E^2 \) plane lattices were classified, by the following definition:

**Definition 4.1.** A lattice \( \Gamma_1 \) in pseudoeuclidean plane is called (affinely) equivalent to a lattice \( \Gamma_2 \) \((\Gamma_1 \sim \Gamma_2)\) if there is an affine transformation \( \alpha \) with \( \Gamma_2 = \Gamma_1 \alpha \), i.e. any symmetry \( S^\alpha \) of \( \Gamma_2 \) is derived by a symmetry \( S \) of \( \Gamma_1 \) with conjugacy \( S^\alpha = \alpha^{-1} S \alpha \). A plane lattice \( \Gamma^0 \) is called regular if there is a hyperbolic rotation \((\Phi \sim \Phi^T = T^{-1} \Phi T \text{ in our Theorem 3.1})\) as a symmetry of \( \Gamma^0 \).
Remark 4.2. The homothety $\Omega(O, \omega)$ with centre $O$ and ratio $\omega$ maps the lattice $\Gamma$ onto an equivalent lattice $\Gamma_{\Omega}$, moreover $\Gamma \sim \Gamma_{h}$ where $h = h(O, \mu)$ is an arbitrary hyperbolic rotation with hyperbolic angle $\mu$ (in Remark 4.1.2).

According to K. Alpers and E. Quaisser [1] we are interested now in Sol lattices where its plane sublattices $\Gamma^0$ are regular i.e. contain nonidentical hyperbolic rotations (see (4.4)). The regular plane lattice classes $\Gamma(\Phi)(\tau_1, \tau_2)$ can be given by Definition 4.1 with the following standard basis (see Theorem 3.1 and Remark 4.1–2) in the orthonormal coordinate system $\{O, e_1, e_2\}$:

$$
\tau_1 = (1, 0), \quad \tau_2 = \frac{1}{2q}(N - 2p, \sqrt{N^2 - 4}). \quad (4.5)
$$

Definition 4.2. An above lattice $\Gamma(\Phi)(\tau_1, \tau_2)$, where translations $\tau_1 = (1, 0)$ and $\tau_2 = \frac{1}{2q}(N - 2p, \sqrt{N^2 - 4})$ in coordinate system $\{O, e_1, e_2\}$, fulfil the conditions:

1. $p, q, r, s \in \mathbb{Z}$, are integers where $ps - qr = 1$, moreover $p + s = N = 2 \cosh t^3_3 \geq 3$;

2. $0 \leq p \leq \lfloor \frac{N}{2} \rfloor$ and $0 < q$

is called an $(N, p, q)$ lattice for a given trace $N = p + s$.

Of course, such an $(N, p, q)$ lattice with fixed $N = p + s$ is determined up to an unimodular or $GL_2(\mathbb{Z})$ conjugacy, i.e.

$$
\begin{pmatrix}
    p' \\
    q'
\end{pmatrix} = \begin{pmatrix}
    u & v \\
    w & w
\end{pmatrix}^{-1} \begin{pmatrix}
    p \\
    q
\end{pmatrix} \begin{pmatrix}
    u & v \\
    w & w
\end{pmatrix} u, v, w, w' \in \mathbb{Z}, \quad w' - vw = \pm 1
$$

leads to equivalent regular $\Gamma(N, p', q')$ lattice.

We obtain by the above considerations and by our modifications (due to E. Molnár) the following summary:

Theorem 4.1. Translations $\tau_1, \tau_2, \tau_3$ generate a lattice $\Gamma(\Phi)(\tau_1, \tau_2, \tau_3)$ in the Sol space if and only if the vectors $\tau_1, \tau_2$ in the $[x, y]$ plane generate a lattice, unimodularly equivalent to a $\Gamma(N, p, q)$ lattice; the components $t^1_3, t^2_3 \in \mathbb{R}$ are given modulo this sublattice $\Gamma^0(\tau_1, \tau_2)$, and the parameter $t^3_3$ in $\tau_3$ satisfies the following equations and equivalence (4.4):

$$
2 \cosh t^3_3 = p + s = N,
\begin{pmatrix}
    e^{-t^3_3} & 0 \\
    0 & e^{t^3_3}
\end{pmatrix} \Leftrightarrow \begin{pmatrix}
    \cosh t^3_3 & \sinh t^3_3 \\
    \sinh t^3_3 & \cosh t^3_3
\end{pmatrix} \Rightarrow t^3_3 = \log\left(\frac{1}{2}(N + \sqrt{N^2 - 4})\right). \quad (4.6)
$$

4.1. On the fundamental parallelepiped

If we take integers as coefficients, then we generate the discrete group $\langle \tau_1, \tau_2, \tau_3 \rangle$ denoted by $\Gamma(\Phi)$, as above.

We know that the orbit space Sol/$\Gamma(\Phi)$ is a compact manifold, i.e. a Sol space form.
Let \( \tilde{F} \) be a fundamental domain of \( \Gamma(\Phi) \) with face identifications. The homogeneous coordinates of the vertices of \( \tilde{F} \) can be determined in our affine model by the translations in Definition 3.1 with the parameters \( t_i^j, i \in \{1, 2, 3\}, j \in \{1, 2, 3\} \) as follows (see Figure 5).

\[
P(1, t_1^1, t_1^2, 0), \quad P'(1, t_2^1, t_2^2, 0), \quad P_3(1, t_3^1, t_3^2, t_3^3), \quad Q(1, t_1^1 + t_2^1, t_1^2 + t_2^2, 0),
\]

\[
Q'(1, (t_1^1 + t_2^1)e^{-t_3^3}, (t_1^2 + t_2^2)e^{t_3^3}, 0),
\]

\[
Q^{\tau_3}(1, t_3^1 + (t_1^1 + t_2^1)e^{-t_3^3}, t_3^2 + (t_1^2 + t_2^2)e^{t_3^3}, t_3^3), \quad P''(1, t_2^1 e^{-t_3^3}, t_2^2 e^{t_3^3}, 0),
\]

\[
P''(1, t_3^1 + t_2^1 e^{-t_3^3}, t_3^2 + t_2^2 e^{t_3^3}, t_3^3), \quad P''(1, t_3^1 + t_2^1 e^{-t_3^3}, t_3^2 + t_2^2 e^{t_3^3}, t_3^3).
\]  

(4.7)

The case \( N = p + s = 3 \) and \( t_1^1 = \frac{1}{\sqrt{2}}, t_2^1 = \frac{3 - \sqrt{5}}{2\sqrt{2}}, t_3^1 = t_3^2 = t_3^3 = 0, t_3^3 = \log \frac{3 + \sqrt{5}}{2} \) is illustrated in Figure 5.

![Figure 5. A fundamental parallelepiped for \( N = 3 \)](image)

It is easy to see by (1.4) and by integration that the volume of \( \tilde{F} \) can be determined by the following formula:

**Theorem 4.2.**

\[
Vol(\tilde{F}) = | \det \left( \begin{array}{ccc} t_1^1 & t_1^2 & t_1^3 \\ t_2^1 & t_2^2 & t_2^3 \\ t_3^1 & t_3^2 & t_3^3 \end{array} \right) \cdot t_3^3 | = \left| (t_1^1 t_2^2 - t_1^2 t_2^1) \cdot \log \left( \frac{1}{2} (N + \sqrt{N^2 - 4}) \right) \right|.
\]  

(4.8)

5. Lattice-like translation ball packings

Let \( B'_i(r) \) denote a translation ball packing of \( \text{Sol} \) space with balls \( B'(r) \) of radius \( r \) where their centres give rise to a \( \text{Sol} \) point lattice \( \Gamma_0(\Phi, t_3^1, t_3^2) \). \( \tilde{F}_0 \) is a \( \text{Sol} \) parallelepiped of this lattice (see (4.7) and Figure 5.). The images of \( \tilde{F}_0 \) by our discrete translation group covers the \( \text{Sol} \) space without overlap. For the density of the packing it is sufficient to relate the volume of the optimal ball to that of the solid \( \tilde{F}_0 \). Analogously to the Euclidean case:
Definition 5.1. 
\[
\delta^t(r) := \frac{Vol(B_t^1(r) \cap \tilde{F}_0)}{Vol(\tilde{F}_0)},
\]
(5.1)
is the density of the lattice-like translation ball packing \(B_t^1(r)\) if the balls do not overlap each other.

Remark 5.1. By definition of \(\Gamma(\Phi)\) above, \(\text{Sol}/\Gamma(\Phi)\) is a compact \(\text{Sol}\) manifold, and (see Section 2),
\[
Vol(B_t^1(r) \cap \tilde{F}_0) = Vol(B^t(r)).
\]

In this paper we investigate a large class of translation ball packings in \(\text{Sol}\) space where the lattices in \([x, y]\) plane are the so called fundamental lattices.

Definition 5.2. (see [1]) \(\Gamma(\Phi) = \Gamma(\Phi^1, t^1_3, t^2_3)\) is called a fundamental lattice in \(\text{Sol}\) space if \(t^1_3 = 0 = t^2_3\) and \(\{\tau_1, \tau_2 = \tau_1^T, \tau_3\}\) is a basis of \(\Gamma(\Phi)\).

Remark 5.2. (Theorem 3.1) For fundamental lattices \(\Phi = \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & N \end{pmatrix}\).
The following basis describes the type of fundamental lattices, that means, \(\text{tr}(\Phi) = N\) is the characteristic free parameter, and
\[
\frac{t^1_2}{t^1_1} = \frac{N - \sqrt{N^2 - 4}}{2} = e^{-t^3_3}, \quad \frac{t^2_2}{t^1_1} = \frac{N + \sqrt{N^2 - 4}}{2} = e^{t^3_3}, \quad (5.2)
\]
for \(\tau_1(t^1_1, t^2_1, 0)\).

5.1. The densest translation ball packing by fundamental lattices

We consider an arbitrary fundamental point lattice in \(\text{Sol}\) with starting point \(O\), affinely equivalent to the lattice \(\Gamma_O(N), \quad (N \geq 3)\). We shall determine the densest packing with translation balls arranged under these lattices for varying \(N\).

First we introduce our final optimal arrangement \(B_t^1(r)\) of balls \(B^t(r)\) that will occur for \(N = 4\), (see Figure 6) where the following equations hold:

(a) \(d^t(O, P_3) = 2r\),
(b) the ball \(B^0_0(r)\) touches the balls \(B_p^t(r)\) and \(B_{p'}^t(r)\) (see in formula (4.7)),
(c) \(N = 4, \quad r = \frac{1}{2} \log \left( \frac{1}{2}(N + \sqrt{N^2 - 4}) \right)\),
(d) \(t^2_2 = e^{-t^3_3} \cdot t^1_1 = t^1_1 \cdot \frac{1}{2}(N - \sqrt{N^2 - 4})\), \(t^2_2 = e^{t^3_3} \cdot t^1_1 = t^1_1 \cdot \frac{1}{2}(N + \sqrt{N^2 - 4})\) by Theorem 3.1.

Here \(d^t\) is the translation distance function in \(\text{Sol}\) (see Definition 2.1). By continuity of the distance function, it follows that there is a (unique) solution of the
equation system (5.3) for $t_1^1$ and $t_2^2$. We have denoted by $B^\text{opt}_\Gamma(r^\text{opt})$ this translation ball packing of the balls $B(r^\text{opt})$. We get the following solution by systematic approximation, where the computations were carried out by Maple V Release 10 up to 30 decimals:

$$N = 4, \quad t_1^{1,\text{opt}} = t_2^{2,\text{opt}} \approx 1,321794147, \quad t_1^{2,\text{opt}} = t_2^{1,\text{opt}} \approx 0,3541736744,$$

$$r^\text{opt} = \frac{1}{2} \log\left(\frac{1}{2}(4 + \sqrt{12})\right) \approx 0,65847895, \quad t_3^{3,\text{opt}} = 2r^\text{opt}.$$  (5.4)

This translation ball packing can be realized in \textbf{Sol} because a ball of radius $r^\text{opt}$ is convex in affine sense and this packing can be generated by the translations $\Gamma(t^\text{opt}_1, t^\text{opt}_2, t^\text{opt}_3)$. Thus we obtain the neighbouring balls around an arbitrary ball of the packing $B^\text{opt}_\Gamma$, the kissing number of the balls is 6. Figure 6 shows the typical arrangement of some balls from $B^\text{opt}_\Gamma(r^\text{opt})$ in our model. We get ball “columns” in $z$-direction and a “centred rectangle like” lattice projection in $[x, y]$-plane.

By formulas (2.6), (4.8), (5.4) and by Definition 5.1 we can compute the density of this ball packing:

$$\text{Vol}(\tilde{\mathcal{F}}^\text{opt}_0) \approx 2,13571164, \quad \text{Vol}(B^\text{opt}_\Gamma) \approx 1,20464592, \quad \delta^\text{opt}_\Gamma \approx 0,56405083.$$  (5.5)

**Theorem 5.1.** The ball arrangement $B^\text{opt}_\Gamma$ given in formulas (5.4) provides the densest translation ball packing by fundamental lattices in \textbf{Sol} space.

**Proof.** We prove this theorem through Lemmas 5.2–6.

We consider an arbitrary \textbf{Sol} fundamental lattice class $\Gamma(\Phi) = \Gamma(N), (N \geq 3)$ given by its basis with origin $O$ in our $[x, y, z]$ system (see (5.2)). Let $\Phi \leftrightarrow \Phi^T$ be an above nonidentical hyperbolic rotation with hyperbolic angle $t_3^3 = \log\left(\frac{1}{2}(N + \sqrt{N^2 - 4})\right)$ in $[x, y]$ plane. The fundamental lattice $\Gamma(\Phi)$ is generated by $(\tau_1, \tau_2 = \tau_1\Phi^T, \tau_3)$ (see Theorem 3.1, formula (4.3) and Definitions 4.1, 5.2). By calculations on formulas (4.8) and (4.3), it follows:
Lemma 5.2.

\[ \text{Vol}(\mathcal{F}_{O}(\Phi)) = |t_1^1 \cdot t_1^2 \cdot \sqrt{N^2 - 4} \cdot \log(\frac{1}{2}(N + \sqrt{N^2 - 4}))|. \] \hspace{1cm} (5.6)

Remark 5.3. From (5.6) we can see that the volume of the Sol fundamental parallelepiped depends on three parameters \( t_1^1, t_1^2, N \).

Definition 5.3. 1. A ball packing \( B_{t}(r) \) is called lattice-like simply touching or basic arrangement (denoted by \( B_{N}^{b}(r) \)) if the following statements hold:

(a) \( r = \frac{1}{2} \log(\frac{1}{2}(N + \sqrt{N^2 - 4})) \),

(b) at least one of the balls \( B_{t}(r) \), \( B_{t}^p(r) \) touches the central ball \( B_{O}(r) \) (see formula (4.7), Figure 5 and Figure 7).

2. A lattice-like simply touching ball arrangement where \( t_1^1 = t_2^2 \) and \( t_1^2 = t_2^1 \) is called symmetrical touching arrangement and denoted by \( B_{N}^{(b,S)}(r) \), see Theorem 3.1.

Remark 5.4. These symmetrical touching arrangements can be realized if and only if \( N \geq 4 \) (see Figures 6, 7, 10).

Lemma 5.3. For a given radius \( r = \frac{1}{2} \log(\frac{1}{2}(N + \sqrt{N^2 - 4})) \), \( N \geq 4 \) the density \( \delta^t(B_{N}^{b}(r)) \) of simply touching arrangement is maximal if it is the symmetrical one.

Figure 7. Symmetrical touching ball arrangement and its “volume hyperbola”

Proof. We consider an arbitrary simply touching (i.e. basic) ball arrangement. From Lemma 5.2 we can see, that the volume of a Sol fundamental parallelepiped and the density for a given \( N \) depends only on the product \( t_1^1 \cdot t_1^2 \). Without loss of generality we may assume, that \( B_{P}(r) \) touches the ball \( B_{O}(r) \) in point \( S \) and \( \frac{1}{2}(N - \sqrt{N^2 - 4}) \leq \frac{t_1^2}{t_1^1} \leq 1 \) (see Figure 7 and Figure 8 a–b). Any touching point \( K(k_1, k_2, 0) \) (\( K \in OP_{K} \)) of the balls \( B_{P}(r) \) and \( B_{O}(r) \) lies on the contour curve of ball \( B_{O}(r) \) (see (2.7) and Figure 3) between the special touching points
$S(s_1, s_2, 0)$ and $A(a_1, a^2, 0)$ given by the proportions $s_1^2 = \frac{1}{2}(N - \sqrt{N^2 - 4})$ and $s_2^2 = 1$, respectively (see Figure 7). By careful discussion of the contour curve of $B'_O(r)$ by formula (2.7) and the hyperbola $x \cdot y = s_1 \cdot s_2$ in the plane $[x, y]$ through $S$ it follows (see Figure 7):

$$\frac{r^2}{2} = a_1 \cdot a^2 \geq 1 \geq \frac{1}{4} \cdot t_1^2 \cdot t_2^2 = k^1 \cdot k^2 \geq s_1 \cdot s_2. \quad (5.7)$$

Thus the maximal density belongs to this symmetrical touching ball arrangement in our case.

**Lemma 5.4.** Comparing symmetrical touching ball arrangements, it stands for $N \geq 4$ the inequality $\delta^t(B_{N}^{(b,S)}) \geq \delta^t(B_{4}^{(b,S)})$.

**Proof.** The intersections of balls of a lattice-like ball packing with an arbitrary horizontal plane are congruent. By Lemma 5.2 and by conditions for the translations, it is sufficient to consider the normal projection of the arrangement into the $[x, y]$ plane for estimating the density of this packing. Figure 8 shows the projections of our ball arrangements $B_{N}^{(b,S)}$ in cases $N = 100$ and $N = 4000$.

![Figure 8. Symmetrical ball arrangements and their “empty parallelograms”](image)

We consider an arbitrary symmetrical touching ball arrangement $B_{N}^{(b,S)}$, $(N \geq 4)$ and four neighbouring balls from this with centres $O, P, P', Q \in [x, y]$. Let $A \in B'_O$, $B \in B'_P$, $C \in B'_Q$, $D \in B'_P$, be points lying in $[x, y]$ plane and $A, C \in OQ$, $B, D \in PP'$. We denote the area of the “empty” parallelogram $ABCD$ by $A_N$ and the area of $OPQP'$ by $A_N'$. By the coordinates of the points $A, B, C, D$ and $O, P, P', Q$, it can be calculated that the function $f(N) = \frac{A_N}{A_N'}$ is increasing for $N \geq 4$, and $f(N) > 0.5$ if $N \geq 400$. Hence for $N \geq 400$ the density $\delta^t(B_{N}^{(b,S)}) \leq \frac{1}{2} < \delta^t(B_{4}^{(b,S)})$. These densities can be calculated for $4 \leq N \leq 400$, summarized in Table 1. Thus we obtain $\delta^t(B_{4}^{(b,S)}) \geq \delta^t(B_{N}^{(b,S)}), \ (N \geq 4)$ as desired. □
We consider a fundamental symmetrical point lattice $\Gamma_S(N)$, $(N \geq 4)$ given by its basis $\langle \tau_1, \tau_2, \tau_3 \rangle$ with starting point $O$ in our $[x,y,z]$ system (i.e. $r$ is not fixed yet by $N$ see (5.2)). Let $\Omega(O, \omega)$ ($\omega > 0$) be an arbitrary homothety with centre $O$ in $[x,y]$ plane (see Definition 4.1–2, Remark 4.1–2). Lattice $\Gamma_S^O$ is equivalent to a lattice $\Gamma_S^{O(w)}$ generated by $\langle \tau_1 \Omega = \omega \tau_1, \tau_2 \Omega = \omega \tau_2, \tau_3 \rangle$ (see (5.2) and Theorem 4.1), i.e. $\Gamma_S^O(N) \sim \Gamma_S^{O(w)}(N)$, thus we have to consider ball packings $\mathcal{B}_{N,\omega}^{(b,S)}(r)$, $(N \geq 4)$ derived by homothety from lattice $\Gamma_S^{O}(N)$. Let $r_{N,\omega}$ be the largest radius of the packing belonging to lattice $\Gamma_S^{O(w)}(N)$.

**Lemma 5.5.** $\delta^t(\mathcal{B}_{N,\omega}^{(b,S)}(r_{N,\omega})) \leq \delta^t(\mathcal{B}_4^{(b,S)}(r_4^*))$ where $r_N^* = \frac{1}{2} \log(\frac{1}{2}(N + \sqrt{N^2 - 4}))$ $(N \geq 4)$.

**Proof.**

1. $\omega > 1$: In these cases the density obviously increases by $\omega$

   \[
   r \leq \frac{1}{2} \log(\frac{1}{2}(N + \sqrt{N^2 - 4})) \quad \text{and} \quad \delta^t(\mathcal{B}_{N,\omega}^{(b,S)}(r)) = \frac{\text{Vol}(\mathcal{B}_{N,\omega}^{(b,S)}(r))}{\text{Vol}(\mathcal{F}_N^{(b,S)})} \leq \delta^t(\mathcal{B}_4^{(b,S)}(r_4^*)), \quad (5.8)
   \]

2. $0 < \omega < 1$: Function $\delta^t(\mathcal{B}_{N,\omega}^{(b,S)}(r_{N,\omega}))$ can be analysed first for parameters $N = 4, 5, \ldots 24$ by computer, since we do not have explicit formula for ball volume. Figure 9a. shows density function $\delta^t(\mathcal{B}_{N,\omega}^{(b,S)}(r_{N,\omega}))$ for $N = 4$. We get that Lemma 5.5 is true for these parameters (see Table 1).

We use mathematical induction. Let us now assume that Lemma 5.5 is true for some $N - 1$ $(N \geq 25)$ and prove that it will also be true for $N$.

We can assume that $\omega$ is such a ratio where inequality $r_{N,\omega} \leq r_{N-1}^*$ holds, since the proportion of radii $r_{N-1}^*/r_N^* \leq 0.9892444126$ if $N \geq 25$ and if $\omega$ is such a ratio where $r_{N,\omega} > r_{N-1}^*$ then $\delta^t(\mathcal{B}_{N,\omega}^{(b,S)}(r_{N,\omega})) \leq \delta^t(\mathcal{B}_{N,\omega}^{(b,S)}(r_{N-1}^*)) \leq \delta^t(\mathcal{B}_4^{(b,S)}(r_4^*))$ hold. This follows by discussion of the real function $g(x)$ below:

\[
g(x) = \frac{1}{2} \log(\frac{1}{2}(x + \sqrt{x^2 - 4})) \quad \text{and} \quad g(x) \rightarrow 1,
\]
Figure 10. $B_3^{(b,S)}(r)$ does not exist

if $x \to \infty$. We have illustrated moving of the centre of ball $B_{P_N}$ in Figure 9 b with increasing density. Consider the fundamental parallelepiped, denoted by $F_0^{N,\omega}$ to the parameters $N$ and $r_{N,\omega}$ ($r_1 = OP_N$, see Figure 9 b), moreover the fundamental parallelepiped $F_0^{N-1,\omega}$ with parameters $N - 1$ and $r_{N,\omega}$ ($r_1 = OP_{N'}$ see Figure 9 b). It is clear that $Vol(F_0^{N-1,\omega}) \leq Vol(F_0^{N,\omega})$, thus holds the following inequality:

$$\delta^t(B_N^{(b,S)}(r_{N,\omega})) \leq \delta^t(B_{N-1,\omega}(r_{N,\omega})).$$

By assumption of the induction we obtain

$$\delta^t(B_{N-1,\omega}(r_{N,\omega})) \leq \delta^t(B_{N-1}(r_{N-1})).$$

Hence $\delta^t(B_N^{(b,S)}(r_{N,\omega})) \leq \delta^t(B_{N-1,\omega}(r_{N-1})) \leq \delta^t(B_{N-1}(r_{N-1}))$ (by Lemma 5.4) for any $N \geq 25$ by mathematical induction. □

It remains to examine case $N = 3$ yet. Then a lattice-like symmetrical touching ball packing $B_3^{(b,S)}(r)$ does not exist, as Figure 10 also shows it. Then we consider the following ball arrangement

(a) $2r = d^t(O, P_3)$,
(b) the ball $B_{P}(r)$ touches the balls $B_{O}^t(r)$ and $B_{P'}^t(r)$,
(c) $N = 3$, $r = \frac{1}{2} log \left(\frac{1}{2}(N + \sqrt{N^2 - 4})\right),$
(d) $t_1^1 = e^{t_3^3} \cdot t_2^1 = \frac{1}{2}(N + \sqrt{N^2 - 4}) \cdot t_2^1$,
(e) $t_2^2 = e^{t_3^3} \cdot t_1^2 = \frac{1}{2}(N + \sqrt{N^2 - 4}) \cdot t_1^2$. (5.9)
Table 1

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<th>$Vol(\mathcal{F}_0^S)$</th>
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Again, there is a (unique) solution for $t_1, t_2$ of equation system (5.9). We have denoted by $\mathcal{B}_3^{opt}(r_{(opt,3)})$ this translation ball packing $B(r_{(opt,3)})$ satisfying the above equation system. We get the following solution by systematic approximation, where computations were carried out by Maple V Release 10 up to 30 decimals:

$$\begin{align*}
N = 3, & \quad t_1^{1,(opt,3)} \approx 0.8223112184, \quad t_1^{2,(opt,3)} \approx 0.5009878194, \\
& \quad r_{(opt,3)} = \frac{1}{2} \log \left( \frac{1}{2} (3 + \sqrt{5}) \right) \approx 0.48121182 = \frac{1}{2} t_3^{3,(opt,3)}, \\
& \quad t_2^{1,(opt,3)} \approx 0.3144695818, \quad t_2^{2,(opt,3)} \approx 1.33099680. \quad (5.10)
\end{align*}$$

Thus, we have obtained the following:

**Lemma 5.6.** Ball arrangement $\mathcal{B}_3^{opt}$ given in formulas (5.9), (5.10) provides the densest translation ball packing by fundamental lattices in case $N = 3$.

By formulas (2.6), (4.8) and by Definition 5.1 we can compute the density of this ball packing:

$$\begin{align*}
& \quad Vol(\mathcal{F}_0^{(opt,3)}) \approx 0.90075749, \quad Vol(\mathcal{B}_3^{opt}(r_{(opt,3)})) \approx 0.46857155, \\
& \quad \delta^t(r_{(opt,3)}) \approx 0.52019723 < \delta^t(r_{opt}) \approx 0.56405083. \quad (5.11)
\end{align*}$$

Thus, we have proved Theorem 5.1 by Lemmas 5.2–6. $\square$
Our projective method gives us a way of investigating homogeneous spaces, which suits to study and solve similar problems (see [10], [11]).

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References


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