Compactness and Boundedness of Tangent Spaces to Metric Spaces

O. Dovgoshey  F. Abdullayev  M. Küçükaslan

Institute of Applied Mathematics and Mechanics of NASU,
R. Luxemburg str. 74, Donetsk 83114, Ukraine
e-mail: aleksdov@mail.ru

Mersin University, Faculty of Literature and Science
Department of Mathematics, 33342 Mersin, Turkey
e-mail: fabdul@mersin.edu.tr mkucukaslan@mersin.edu.tr

Abstract. We describe metric spaces with bounded pretangent spaces and characterize proper metric spaces with proper tangent spaces. We also present the necessary and sufficient conditions under which a tangent space is compact and build a compact ultrametric space \( X \) such that some pretangent space to \( X \) has the density \( c \).

MSC 2000: 54E35
Keywords: metric spaces, tangent spaces to metric spaces, compact tangent spaces, bounded tangent spaces, proper tangent spaces

1. Introduction

Analysis on metric spaces with no a priori smooth structure is in need of some generalized differentiations. Important examples of such generalizations and even an axiomatics of so-called “pseudo-gradients” can be found in [1], [4], [3], [9], [15], [24], [16] and, respectively, in [2]. Another natural way to obtain suitable differentiations on metric spaces is to induce some tangents at the points of these space. The Gromov-Hausdorff convergence and the ultra-convergence are, probably, the most widely applied today’s tools for the construction of such tangent spaces (see, for example, [7], [6] and, respectively, [5], [21]). Recently a new approach to the introduction of the tangent spaces at the points of general metric
spaces was proposed in [12]. For convenience we recall the main notions from [12], see also [11].

**Tangent and pretangent spaces.** Let \((X, d)\) be a metric space. Fix a sequence \(\tilde{r}\) of positive real numbers \(r_n\) which tend to zero. In what follows this sequence \(\tilde{r}\) will be called a *normalizing sequence*. Let us denote by \(\tilde{X}\) the set of all sequence of points from \(X\).

**Definition 1.** Two sequences \(\tilde{x}, \tilde{y} \in \tilde{X}\), \(\tilde{x} = \{x_n\}_{n \in \mathbb{N}}\) and \(\tilde{y} = \{y_n\}_{n \in \mathbb{N}}\) are mutually stable with respect to \(\text{w.r.t.} \tilde{r}\) a normalizing sequence \(\tilde{r} = \{r_n\}_{n \in \mathbb{N}}\) if there is a finite limit

\[
\lim_{n \to \infty} \frac{d(x_n, y_n)}{r_n} := \tilde{d}(\tilde{x}, \tilde{y}) = \tilde{d}(\tilde{x}, \tilde{y}).
\]

We shall say that a family \(\tilde{F} \subseteq \tilde{X}\) is *self-stable* \(\text{w.r.t.} \tilde{r}\) if every two \(\tilde{x}, \tilde{y} \in \tilde{F}\) are mutually stable. A family \(\tilde{F} \subseteq \tilde{X}\) is *maximal self-stable* if \(\tilde{F}\) is self-stable and for an arbitrary \(\tilde{z} \in \tilde{X} \setminus \tilde{F}\) there is \(\tilde{x} \in \tilde{F}\) such that \(\tilde{x}\) and \(\tilde{z}\) are not mutually stable.

A standard application of Zorn’s lemma leads to the following

**Proposition 1.** Let \((X, d)\) be a metric space and let \(p \in X\). Then for every normalizing sequence \(\tilde{r} = \{r_n\}_{n \in \mathbb{N}}\) there exists a maximal self-stable family \(\tilde{X}_{a, \tilde{r}}\) such that \(\tilde{a} = \{a, a, \ldots\} \in \tilde{X}_{a, \tilde{r}}\).

Consider a function \(\tilde{d} : \tilde{X}_{a, \tilde{r}} \times \tilde{X}_{a, \tilde{r}} \to \mathbb{R}\) where \(\tilde{d}(\tilde{x}, \tilde{y}) = \tilde{d}(\tilde{x}, \tilde{y})\) is defined by \((1.1)\). Obviously, \(\tilde{d}\) is symmetric and nonnegative. Moreover, the triangle inequality implies

\[
\tilde{d}(\tilde{x}, \tilde{y}) \leq \tilde{d}(\tilde{x}, \tilde{z}) + \tilde{d}(\tilde{z}, \tilde{y})
\]

for all \(\tilde{x}, \tilde{y}, \tilde{z}\) from \(\tilde{X}_{a, \tilde{r}}\). Hence \((\tilde{X}_{a, \tilde{r}}, \tilde{d})\) is a pseudometric space.

Define a relation \(\sim\) on \(\tilde{X}_{a, \tilde{r}}\) by \(\tilde{x} \sim \tilde{y}\) if and only if \(\tilde{d}(\tilde{x}, \tilde{y}) = 0\). Then \(\sim\) is an equivalence relation. Let us denote by \(\Omega^X_{a, \tilde{r}} = \Omega_{a, \tilde{r}}\) the set of equivalence classes in \(\tilde{X}_{a, \tilde{r}}\) under the equivalence relation \(\sim\). If a function \(\rho\) is defined on \(\Omega^X_{a, \tilde{r}} \times \Omega^X_{a, \tilde{r}}\) by

\[
\rho(\alpha, \beta) = \tilde{d}(\tilde{x}, \tilde{y})
\]

for \(\tilde{x} \in \alpha\) and \(\tilde{y} \in \beta\), then \(\rho\) is the well-defined metric on \(\Omega_{a, \tilde{r}}\). Thus the metric space \((\Omega^X_{a, \tilde{r}}, \rho)\) is the *metric identification* of \((\tilde{X}_{a, \tilde{r}}, \tilde{d})\).

**Definition 2.** The space \((\Omega^X_{a, \tilde{r}}, \rho)\) is pretangent to the space \(X\) at the point \(a\) w.r.t. the normalizing sequence \(\tilde{r}\).

Note that \(\Omega_{a, \tilde{r}} \neq \emptyset\) because the constant sequence \(\tilde{a}\) belongs to \(\tilde{X}_{a, \tilde{r}}\), see Proposition 1.

Let \(\{n_k\}_{k \in \mathbb{N}}\) be an infinite, strictly increasing sequence of natural numbers. Let us denote by \(\tilde{r}'\) the subsequence \(\{r_{n_k}\}_{k \in \mathbb{N}}\) of the normalizing sequence \(\tilde{r} = \{r_n\}_{n \in \mathbb{N}}\)
and let $\tilde{x}' := \{x_n\}_{k \in \mathbb{N}}$ for every $\tilde{x} = \{x_n\}_{n \in \mathbb{N}} \in \tilde{X}$. It is clear that if $\tilde{x}$ and $\tilde{y}$ are mutually stable w.r.t. $\tilde{r}$, then $\tilde{x}'$ and $\tilde{y}'$ are mutually stable w.r.t. $\tilde{r}'$ and
\[
\tilde{d}_r(\tilde{x}, \tilde{y}) = \tilde{d}_{r'}(\tilde{x}', \tilde{y}').
\]
If $\tilde{X}_{a,\tilde{r}}$ is a maximal self-stable (w.r.t. $\tilde{r}$) family, then, by Zorn’s lemma, there exists a maximal self-stable (w.r.t. $\tilde{r}'$) family $\tilde{X}_{a,\tilde{r}'}$ such that
\[
\{\tilde{x}' : \tilde{x} \in \tilde{X}_{a,\tilde{r}}\} \subseteq \tilde{X}_{a,\tilde{r}'}.
\]
Denote by $\text{in}_{\tilde{r}'}$ the mapping from $\tilde{X}_{a,\tilde{r}}$ to $\tilde{X}_{a,\tilde{r}'}$ with $\text{in}_{\tilde{r}'}(\tilde{x}) = \tilde{x}'$ for all $\tilde{x} \in \tilde{X}_{a,\tilde{r}}$.

It follows from (1.3) that, after metric identifications, $\text{in}_{\tilde{r}'}$ pass to an isometric embedding $\text{em}': \Omega^X_{a,\tilde{r}} \to \Omega^X_{a,\tilde{r}'}$, under which the diagram
\[
\begin{array}{ccc}
\tilde{X}_{a,\tilde{r}} & \xrightarrow{\text{in}_{\tilde{r}'}} & \tilde{X}_{a,\tilde{r}'} \\
\pi \downarrow & & \downarrow \pi' \\
\Omega^X_{a,\tilde{r}} & \xrightarrow{\text{em}'} & \Omega^X_{a,\tilde{r}'}
\end{array}
\]
is commutative. Here $\pi$ and $\pi'$ are canonical projections $\pi(\tilde{x}) := \{\tilde{y} \in \tilde{X}_{a,\tilde{r}} : \tilde{d}_r(\tilde{x}, \tilde{y}) = 0\}$ and $\pi'(\tilde{x}) := \{\tilde{y} \in \tilde{X}_{a,\tilde{r}'} : \tilde{d}_{r'}(\tilde{x}, \tilde{y}) = 0\}$.

Let $X$ and $Y$ be two metric spaces. Recall that a map $f : X \to Y$ is called an isometry if $f$ is distance-preserving and onto.

**Definition 3.** A pretangent $\Omega^X_{a,\tilde{r}}$ is tangent if $\text{em}': \Omega^X_{a,\tilde{r}} \to \Omega^X_{a,\tilde{r}'}$ is an isometry for every $\tilde{X}_{a,\tilde{r}'}$.

Now we can briefly describe some results of the present paper. Write $O_{t,a,\tilde{r}}$ for the family of open balls in $X$ with the center $a$ and radiuses $tr_1, tr_2, \ldots$ where $t \in ]0, \infty[ $ is fixed.

(i) A separable tangent space $\Omega^X_{a,\tilde{r}}$ is locally compact at the point $\pi(\tilde{a})$ if and only if the blow up of the family $O_{a,t,\tilde{r}}$ is uniformly precompact for some $t > 0$. See Theorem 2.

(ii) A tangent space $\Omega^X_{a,\tilde{r}}$ to the proper metric space $X$ is proper if and only if the blow up of $O_{t,a,\tilde{r}}$ is uniformly precompact for all $t > 0$. See Theorem 3 and Corollary 3.

(iii) All pretangent spaces $\Omega^X_{a,\tilde{r}'}$ are bounded for some $\tilde{r}$ if and only if the right-side porosity of the set $\{d(x,a) : x \in X\}$ equals 1 at zero. See Theorem 4.

(iv) The necessary and sufficient conditions under which a tangent space $\Omega^X_{a,\tilde{r}}$ is compact. The exact formulations are awkward, so see Theorem 6 and Theorem 7.

(v) An example of compact ultrametric space with a pretangent space having the density $c$. For the construction see the proof of Theorem 8.
Note that the many results of the paper are, after some modifications, valid also
for tangent spaces obtained via the ultra-convergence. Roughly speaking, each
ultralimit of a sequence of pointed metric spaces $(X, d, a) \text{ w.r.t. of a nonprincipal}
ultrafilter } \omega \text{ can be obtained, under some additional conditions, as an inductive}
limit of pretangent spaces $\Omega^X_{a, r'}$ where $r' = \{r_{n_k}\}_{k \in \mathbb{N}}$ are the subsequences of $r$
such that $\{n_k : k \in \mathbb{N}\} \in \omega$. The exact description of the connections between
pretangent spaces $\Omega^X_{a, r}$ and such ultralimits is technical enough and the authors
plan to do it at the separate paper.

Let us discuss now some metric characteristics of precompact sets which will be
used in the sequel.

**Covering numbers and packing numbers.** Let $(X, d)$ be a metric space, $W \subseteq X$ and let $\varepsilon > 0$. A set $C \subseteq X$ is an $\varepsilon$-net for $W$ if

$$W \subseteq \bigcup_{c \in C} B(c, \varepsilon)$$

where $B(c, \varepsilon)$ are closed balls with a center $c \in X$ and a radius $\varepsilon$. $W \subseteq X$ is
totally bounded (or precompact) if for every $\varepsilon > 0$ there is a finite $\varepsilon$-net for $W$.
The covering number of a totally bounded set $W \subseteq X$ is the smallest cardinality
of subsets of $W$ which are $\varepsilon$-nets for $W$. A set $A \subseteq X$ is called $\varepsilon$-distinguishable
if $d(x, y) > \varepsilon$ for every distinct points $x, y \in A$, [18]. The packing number of a
precompact set $W \subseteq X$ is the maximal cardinality of the $\varepsilon$-distinguishable sets
$A \subseteq W$.

We denote by $N_\varepsilon(W)$ and by $M_\varepsilon(W)$ the covering number and, respectively,
the packing number of totally bounded sets $W \subseteq X$. Write $N_\varepsilon(W) = \infty$ if all
$\varepsilon$-nets for $W$ are infinite and, similarly, $M_\varepsilon(W) = \infty$ if there is an infinite $\varepsilon$-
distinguishable set $A \subseteq W$. The functions $N_\varepsilon$ and $M_\varepsilon$ have been invented by
Kolmogorov [17] in order to classify compact metric sets. Note that $\log_2 N_\varepsilon(W)$ is
the so-called metric entropy and it has been widely applied in the approximation
theory, geometric functional analysis, probability theory and complexity theory,
see, for example, [18], [20], [8], [14].

A main general fact about packing and covering numbers is the simple double
inequality

$$M_{2\varepsilon}(W) \leq N_\varepsilon(W) \leq M_\varepsilon(W). \quad (1.5)$$

See, for example, [18] or [23, 10.1.1].

The following well known proposition can be easily inferred from (1.5).

**Proposition 2.** A subset $W$ of a metric space $X$ is totally bounded if and only
if the inequality $M_\varepsilon(W) < \infty$ holds for all $\varepsilon > 0$.

The next lemma will be used in the proof of Theorem 3.

**Lemma 1.** Let $X$ be a metric space, $W \subseteq X$, $D$ a dense subset of $W$. If $D$ is
totally bounded then $W$ is also totally bounded and the double inequality

$$N_{k\varepsilon}(W) \geq N_{k\varepsilon}(D) \geq N_{k\varepsilon}(W) \quad (1.6)$$
holds for every \( k \in [0, 1] \) and all \( \varepsilon > 0 \).

**Proof.** Suppose \( D \) is totally bounded and \( 0 < k < 1 \). If \( P = \{ p_i : i \in I \} \) is an \( \varepsilon \)-net for \( D \) with \( \text{card}(P) = \mathcal{N}_\varepsilon(D) \), then for every \( x \in W \) there is \( p_i \in P \) such that
\[
x \in B(p_i, \frac{\varepsilon}{k}).
\]
Hence \( P \) is an \( \varepsilon_k \)-net for \( W \), that implies the second inequality in (1.6) and shows that \( W \) is totally bounded.

If \( C = \{ c_i : i \in I \} \) is \( k\varepsilon \)-net for \( W \) with \( \text{card}(C) = \mathcal{N}_{k\varepsilon}(W) \), then the density of \( D \) in \( W \) implies that for every \( c_i \in C \) there is \( b_i \in D \) such that \( B(b_i, \varepsilon) \supseteq B(c_i, k\varepsilon) \). Hence we have
\[
D \subseteq W \subseteq \bigcup_{i \in I} B(c_i, k\varepsilon) \subseteq \bigcup_{i \in I} B(b_i, \varepsilon),
\]
i.e., \( \{ b_i : i \in I \} \) is an \( \varepsilon \)-net for \( D \), so the first inequality in (1.6) is proved. \( \square \)

2. Precompactness of balls in pretangent spaces and proper tangent spaces

In this section we obtain some estimations of the packing numbers of balls in pretangent spaces and using these estimations describe the conditions under which tangent spaces to proper spaces are also proper.

Let \((X, d)\) be a metric space with a marked point \( a \in X \). The open balls with the center \( a \in X \) and with a radius \( t > 0 \) are denoted by
\[
O_t = O(a, t) := \{ x \in X : d(a, x) < t \},
\]
and write
\[
B_t(\Omega_{a,\tilde{r}}) := \{ \beta \in \Omega_{a,\tilde{r}} : \rho(\alpha, \beta) \leq t \},
\]
\[
\Theta_t(\Omega_{a,\tilde{r}}) = \Theta(\alpha, t) := \{ \beta \in \Omega_{a,\tilde{r}} : \rho(\alpha, \beta) < t \}
\]
for the closed and, respectively, open balls with the center \( \alpha = \pi(\tilde{a}) \) and a radius \( t \) in a pretangent space \( \Omega_{a,\tilde{r}} \).

Let us denote by \( \overline{\mathbb{N}} \) the extended set of natural numbers, \( \overline{\mathbb{N}} = \mathbb{N} \cup \{ \infty \} \). Define the functions \( \nu_{\tilde{r},t} \) and \( \nu_{\tilde{r},t} \) from \( \mathbb{R}^+ \) to \( \overline{\mathbb{N}} \) by the rules
\[
\nu_{\tilde{r},t}(\varepsilon) := \liminf_{n \to \infty} \mathcal{M}_{\varepsilon r_n}(O_{tr_n}), \quad \sigma_{\tilde{r},t}(\varepsilon) := \limsup_{n \to \infty} \mathcal{M}_{\varepsilon r_n}(O_{tr_n}).
\]
(2.1)
The following theorem indicates some useful bounds of the packing numbers of balls in pretangent spaces.

**Theorem 1.** Let \((X, d)\) be a metric space with a marked point \( a \). The following statements are true for all \( t, \varepsilon \in (0, \infty) \).

(i) The inequality
\[
\nu_{\tilde{r},t}(\varepsilon) \geq \mathcal{M}_t(\Theta_t(\Omega_{a,\tilde{r}}))
\]
holds in each pretangent space \( \Omega_{a,\tilde{r}}^X \).
(ii) For each normalizing sequence $\tilde{r}$, there are a subsequence $\tilde{r}' = \tilde{r}'(\varepsilon)$ and a pretangent space $\Omega_{a,\tilde{r}}^X$ such that
\[
\psi_{\tilde{r},t}(\varepsilon) \leq M_\varepsilon\left(B_t(\Omega_{a,\tilde{r}}^X)\right).
\] (2.3)

(iii) If a pretangent space $\Omega_{a,\tilde{r}}^X$ is tangent and separable, then we have the inequality
\[
\psi_{\tilde{r},t}(\varepsilon) \leq M_\varepsilon\left(B_t(\Omega_{a,\tilde{r}}^X)\right).
\] (2.4)

To prove this we start from the following lemma.

**Lemma 2.** Let $\Theta(\alpha, t)$ be an open ball in a pretangent space $\Omega_{a,\tilde{r}}^X$, $\alpha = \pi(a)$. Then for every $\beta \in \Theta(\alpha, t)$ there is $\tilde{x} = \{x_n\}_{n \in \mathbb{N}} \in \tilde{X}_{a,\tilde{r}}$ such that $\pi(\tilde{x}) = \beta$ and
\[
d(x_n, a) < t
\] for all $n \in \mathbb{N}$.

**Proof.** Let $\tilde{x}^* = \{x_n^*\}_{n \in \mathbb{N}} \in \tilde{X}_{a,\tilde{r}}$ and let $\pi(\tilde{x}^*) = \beta \in \Theta(\alpha, t)$ where $\pi$ is the canonical projection, see diagram (1.4). Write
\[
x_n = \begin{cases} x_n^* & \text{if } d(x_n^*, a) < r_n t \\ a & \text{if } d(x_n^*, a) \geq r_n t \end{cases}
\]
for each $n \in \mathbb{N}$ and put $\tilde{x} = \{x_n\}_{n \in \mathbb{N}}$. Then there is $n_0 \in \mathbb{N}$ such that $x_n = x_n^*$ for all $n \geq n_0$. Consequently we have $\tilde{x} \in \tilde{X}_{a,\tilde{r}}$, $\pi(\tilde{x}) = \beta$ and (2.5) holds for all $n \in \mathbb{N}$. \hfill \Box

Lemma 2 has a partial converse.

**Lemma 3.** Let $\tilde{X}_{a,\tilde{r}}$ be a maximal self-stable family, $\tilde{F} \subseteq \tilde{X}_{a,\tilde{r}}$, $\Omega_{a,\tilde{r}}^X$ a pretangent space corresponding to $\tilde{X}_{a,\tilde{r}}$ and $t > 0$. If there is an infinite, strictly increasing sequence $n_1, n_2, \ldots$ of natural numbers such that the inequality
\[
d(x_{n_k}, a) \leq t \frac{r_{n_k}}{r_{n_k}}
\]
holds for every $\tilde{x} = \{x_n\}_{n \in \mathbb{N}} \in \tilde{F}$ and for all $k \in \mathbb{N}$, then $\pi(\tilde{F}) \subseteq B_t(\Omega_{a,\tilde{r}})$ where $\pi$ is the canonical projection, see (1.4).

The proof is simple and we omit it.

The next lemma provides us a condition under which Cauchy sequences in pretangent spaces are convergent.
Lemma 4. Let \( \tilde{X}_{a,\tilde{r}} \) be a maximal self-stable family and let \( \tilde{x} = \{x_n\}_{n \in \mathbb{N}} \in \tilde{X} \). If there are sequences \( \tilde{x}_j = \{x^j_n\}_{n \in \mathbb{N}} \in \tilde{X}_{a,\tilde{r}}, \ j \in \mathbb{N}, \) such that \( \tilde{x} \) and \( \tilde{x}_j \) are mutually stable w.r.t. \( \tilde{r} \) for all \( j \in \mathbb{N} \) and

\[
\lim_{j \to \infty} \lim_{n \to \infty} \frac{d(x_n, x^j_n)}{r_n} = 0, \tag{2.6}
\]

then \( \tilde{x} \in \tilde{X}_{a,\tilde{r}}. \)

Proof. Let \( \tilde{z} = \{z_n\}_{n \in \mathbb{N}} \) be an arbitrary element of \( \tilde{X}_{a,\tilde{r}}. \) We have the following inequalities

\[
\begin{align*}
\limsup_{n \to \infty} \frac{d(z_n, x_n)}{r_n} & \geq \tilde{d}(\tilde{z}, \tilde{x}_j) - \tilde{d}(\tilde{x}_j, \tilde{x}), \\
\liminf_{n \to \infty} \frac{d(z_n, x_n)}{r_n} & \leq \tilde{d}(\tilde{z}, \tilde{x}_j) + \tilde{d}(\tilde{x}_j, \tilde{x})
\end{align*} \tag{2.7}
\]

for all \( j \in \mathbb{N}. \) Hence

\[
\left| \limsup_{n \to \infty} \frac{d(z_n, x_n)}{r_n} - \liminf_{n \to \infty} \frac{d(x_n, z_n)}{r_n} \right| \leq 2\tilde{d}(\tilde{x}, \tilde{x}_j).
\]

The last inequality and (2.6) imply the existence of \( \lim_{n \to \infty} \frac{d(z_n, x_n)}{r_n}. \) This limit is finite by (2.7). Therefore \( \tilde{x} \) and \( \tilde{z} \) are mutually stable for all \( \tilde{z} \in \tilde{X}_{a,\tilde{r}}. \) Since \( \tilde{X}_{a,\tilde{r}} \) is maximal self-stable, \( \tilde{x} \in \tilde{X}_{a,\tilde{r}}. \)

Remark 1. It is easy to see that \( \{\pi(\tilde{x}_j)\}_{j \in \mathbb{N}} \) is a Cauchy sequence in the space \( \Omega_{a,\tilde{r}} = \pi(\tilde{X}_{a,\tilde{r}}). \) If \( \Omega_{a,\tilde{r}} \) is tangent, then this space is complete, see [12], and, consequently, it implies Lemma 4 for tangent spaces. However, the corresponding space \( \Omega_{a,\tilde{r}} \) in the lemma is, in general, pretangent.

Corollary 1. Let \( A \) be a dense subset of \( \tilde{X}_{a,\tilde{r}}, \ a \in A. \) If \( \tilde{b} = \{b_n\}_{n \in \mathbb{N}} \in \tilde{X} \) is a sequence such that \( \tilde{x} \) and \( \tilde{b} \) are mutually stable for all \( \tilde{x} \in A, \) then \( \tilde{b} \in \tilde{X}_{a,\tilde{r}}. \)

Proof. Let \( \tilde{X}^0_{a,\tilde{r}} \) be a maximal self-stable family such that \( \tilde{b} \in \tilde{X}^0_{a,\tilde{r}} \supseteq A. \) By Lemma 4 we have \( \tilde{x} \in \tilde{X}^0_{a,\tilde{r}} \) for every \( \tilde{x} \in \tilde{X}_{a,\tilde{r}}. \) Hence \( \tilde{X}^0_{a,\tilde{r}} \supseteq \tilde{X}_{a,\tilde{r}}. \) This inclusion implies the equality \( \tilde{X}^0_{a,\tilde{r}} = \tilde{X}_{a,\tilde{r}} \) because \( \tilde{X}_{a,\tilde{r}} \) is maximal self-stable.

In the following proposition we use the notations from Lemma 4 and from diagram (1.4).

Proposition 3. Let \( \mathfrak{B} \) be a countable subfamily of \( \tilde{X}, \ \tilde{r} = \{r_n\}_{n \in \mathbb{N}} \) a normalizing sequence and \( \tilde{X}_{a,\tilde{r}} \) a maximal self-stable family. Suppose that the inequality

\[
\limsup_{n \to \infty} \frac{d(b_n, a)}{r_n} < \infty \tag{2.8}
\]
holds for every $b = \{b_n\}_{n \in \mathbb{N}} \in \mathcal{B}$ and that a pretangent space $\Omega_{a, \tilde{r}} = \pi(\tilde{X}_{a, \tilde{r}})$ is separable and tangent. Then there is a strictly increasing, infinite sequence $\tilde{n} = \{n_k\}_{k \in \mathbb{N}}$ of natural numbers such that for every $b = \{b_n\}_{n \in \mathbb{N}} \in \mathcal{B}$ there exists $\tilde{y} = \{y_n\}_{n \in \mathbb{N}} \in \tilde{X}_{a, \tilde{r}}$ with $\tilde{y}' = b'$, that is the equality

$$y_{n_k} = b_{n_k}$$

(2.9)

holds for all $k \in \mathbb{N}$.

**Proof.** Let $\Gamma$ be a countable, dense subset of $\Omega_{a, \tilde{r}}$ and let $A$ be a countable subset of $X_{a, \tilde{r}}$ such that $\Gamma = \{\pi(\tilde{x}) : \tilde{x} \in A\}$. Without loss of generality, we suppose $\tilde{a} = \{a, a, \ldots\} \in A$. Since $\mathcal{B}$ and $A$ are countable, the set of all ordered pairs $(\tilde{b}, \tilde{x})$, $\tilde{b} \in \mathcal{B}$ and $\tilde{x} \in A$, is also countable, so its elements can be enumerated as $(\tilde{b}_1, \tilde{x}_1), (\tilde{b}_2, \tilde{x}_2), \ldots$. The triangle inequality and (2.8) imply that

$$\sup_{n \in \mathbb{N}} \frac{d(b_n, x_n)}{r_n} < \infty$$

for each pair $(\tilde{b}_j, \tilde{x}_j)$, $\tilde{b}_j = \{b_n\}_{n \in \mathbb{N}}$ and $\tilde{x}_j = \{x_n\}_{n \in \mathbb{N}}$, in particular, we have

$$\sup_{n \in \mathbb{N}} \frac{d(b_n, x_n)}{r_n} < \infty.$$ 

Since every bounded, infinite sequence of reals has a subsequence that converges to a real number, there is a strictly increasing, infinite sequence $\tilde{n}_1 = \{n_{k(1)}^1\}_{k \in \mathbb{N}}$ of natural numbers such that

$$\lim_{k \to \infty} \frac{d(b_{n_{k(1)}^1}, x_{n_{k(1)}^1})}{r_{n_{k(1)}^1}}$$

is finite. Hence, the sequences $\{b_{n_{k(1)}^1}\}_{k \in \mathbb{N}}$ and $\{x_{n_{k(1)}^1}\}_{k \in \mathbb{N}}$ are mutually stable w.r.t. $\{r_{n_{k(1)}^1}\}_{k \in \mathbb{N}}$. Analogously, by induction, we can prove that for every integer $i \geq 2$ there is a subsequence $\tilde{n}_i = \{n_{k(i)}^i\}_{k \in \mathbb{N}}$ of sequence $\tilde{n}_{i-1}$ such that $\{b_{n_{k(i)}^i}\}_{k \in \mathbb{N}}$ and $\{x_{n_{k(i)}^i}\}_{k \in \mathbb{N}}$ are mutually stable w.r.t. $\{r_{n_{k(i)}^i}\}_{k \in \mathbb{N}}$. Using Cantor’s diagonal construction, write

$$\tilde{n} := \{n_{k(i)}^i\}_{k \in \mathbb{N}}, \quad \tilde{r}' := \{r_{n_{k(i)}^i}\}_{k \in \mathbb{N}}$$

and, for all $\tilde{b} = \{b_n\}_{n \in \mathbb{N}} \in \mathcal{B}$, $\tilde{x} = \{x_n\}_{n \in \mathbb{N}} \in A$, define $\tilde{b}'$ and $\tilde{x}'$ as

$$\tilde{b}' := \{b_{n_{k(i)}^i}\}_{k \in \mathbb{N}}, \quad \tilde{x}' := \{x_{n_{k(i)}^i}\}_{k \in \mathbb{N}}$$

and put

$$A' := \{\tilde{x}' : \tilde{x} \in A\}, \quad \mathcal{B}' := \{\tilde{b}' : \tilde{b} \in \mathcal{B}\}.$$ 

It is shown that every two $\tilde{b}' \in \mathcal{B}'$ and $\tilde{x}' \in A'$ are mutually stable w.r.t. $\tilde{r}'$.

Since $\Omega_{a, \tilde{r}}$ is tangent the space $\Omega_{a, \tilde{r}'}$ is also tangent, and $\text{em}' : \Omega_{a, \tilde{r}} \to \Omega_{a, \tilde{r}'}$ is an isometry, and, in addition, the density of $\Gamma$ in $\Omega_{a, \tilde{r}}$ implies that $\Gamma' := \text{em}'(\Gamma)$
is a dense subset of \( \Omega_{a,\tilde{r}} \). Note also that we have \( \Gamma' = \{ \pi'(\tilde{x}') : \tilde{x}' \in A' \} \). Hence, by Corollary 1, we obtain the inclusion \( \mathcal{B} \subseteq \tilde{X}_{a,\tilde{r}} \). Since the mapping \( \pi_r \) is surjective, the preimage \( \pi_r^{-1}(b') \) is nonvoid for every \( b' \in \mathcal{B} \). If \( \tilde{y} = \{ y_n \}_{n \in \mathbb{N}} \) belongs to \( \pi_r^{-1}(b') \) for \( b = \{ b_n \}_{n \in \mathbb{N}} \), then (2.9) evidently holds with \( n_k = n_{k}^{(k)} \) for all \( k \in \mathbb{N} \).

Reasoning as in the proof of the last proposition, we can obtain the following

**Lemma 5.** Let \( \mathcal{B} \) be a countable subfamily of \( \tilde{X} \) and let \( \tilde{\rho} = \{ \rho_n \}_{n \in \mathbb{N}} \) be a normalizing sequence. Suppose that the inequality

\[
\limsup_{n \to \infty} \frac{d(b_n, a)}{\rho_n} < \infty
\]

holds for every \( b = \{ b_n \}_{n \in \mathbb{N}} \in \mathcal{B} \). Then there is an infinite subsequence \( \tilde{\rho}' \) of \( \tilde{\rho} \) such that the family \( \mathcal{B}' := \{ b' : b \in \mathcal{B} \} \) is self-stable w.r.t. \( \tilde{\rho}' \).

**Proof of Theorem 1.** (i) Suppose there are \( \varepsilon > 0, \ t > 0 \) and a pretangent space \( \Omega_{a,\tilde{r}} \) such that

\[
\nu_{\tilde{r}, t}(\varepsilon) < \mathcal{M}_\varepsilon(\Theta_t(\Omega_{a,\tilde{r}})).
\]

Then \( \nu^1 := \nu_{\tilde{r}, t}(\varepsilon) \) is finite. Let \( \{ r_{n_k} \}_{k \in \mathbb{N}} \) be a subsequence of \( \tilde{r} \) such that

\[
\nu^1 = \liminf_{n \to \infty} \mathcal{M}_{\varepsilon r_{n_k}}(O_{t r_{n_k}}) = \lim_{k \to \infty} \mathcal{M}_{\varepsilon r_{n_k}}(O_{t r_{n_k}}).
\]

Since all \( \mathcal{M}_{\varepsilon r_{n_k}}(O_{t r_{n_k}}) \) are integer numbers, relations (2.11) imply

\[
\mathcal{M}_{\varepsilon r_{n_k}}(O_{t r_{n_k}}) = \nu^1
\]

for all sufficiently large \( k \). We can find, by (2.10), the points \( \beta_1, \ldots, \beta_{\nu^1+1} \) from \( \Theta_t(\Omega_{a,\tilde{r}}) \) such that

\[
\rho(\beta_i, \beta_j) > \varepsilon
\]

whenever \( 1 \leq i < j \leq \nu^1 + 1 \). By Lemma 2, there are sequences \( \tilde{x}_i = \{ x_n^i \}_{n \in \mathbb{N}}, \ i = 1, \ldots, \nu^1 + 1 \), such that \( \pi(\tilde{x}_i) = \beta_i \) and \( x_n^i \in O_{t r_{n_k}} \) for all \( n \in \mathbb{N} \) and all \( i \in \{ 1, \nu^1 + 1 \} \). It follows from (2.13) that

\[
\lim_{n \to \infty} \frac{d(x_n^i, x_n^j)}{r_n} > \varepsilon
\]

whenever \( 1 \leq i < j \leq \nu^1 + 1 \). Hence there exists \( n_0 \in \mathbb{N} \) such that

\[
\frac{d(x_n^i, x_n^j)}{r_n} > \varepsilon
\]

if \( n \geq n_0 \) and \( 1 \leq i < j \leq \nu^1 + 1 \), so we obtain the inequality \( \mathcal{M}_{\varepsilon r_{n_k}}(O_{t r_{n_k}}) \geq \nu^1 + 1 \) for all \( n \geq n_0 \), contrary to (2.12).

(ii) Let \( \tilde{r} = \{ r_n \}_{n \in \mathbb{N}} \) be a normalizing sequence \( \varepsilon, t \in ]0, \infty[ \), and let \( \{ r_{n_k} \}_{k \in \mathbb{N}} \) be a subsequence of \( \tilde{r} \) such that

\[
\nu^2 := \nu_{\tilde{r}, t}(\varepsilon) = \lim_{k \to \infty} \mathcal{M}_{\varepsilon r_{n_k}}(O_{t r_{n_k}}).
\]
Consider firstly the case where $\nu^2$ is finite. Then there is a natural number $k_0 \in \mathbb{N}$ for which we have the equality

$$\nu^2 = M_{\varepsilon r_{n_k}}(O_{tr_{n_k}})$$

whenever $k \geq k_0$. Hence, for every $k \geq k_0$, there exist some points $b^1_k, \ldots, b^\nu_k \in O_{tr_{n_k}}$ such that

$$d(b^i_k, b^j_k) > \varepsilon r_{n_k}$$

if $1 \leq i < j \leq \nu^2$. For $k < k_0$ put $b^1_k = \cdots = b^\nu_k = a$ and write $\tilde{b}_j = \{b^i_k\}_{k \in \mathbb{N}}$ with $j = 1, \ldots, \nu^2$. The family $\{\tilde{b}_1, \ldots, \tilde{b}_{\nu^2}\} \cup \{\tilde{a}\}$ satisfies the conditions of Lemma 5 with $\tilde{\rho} = \{r_{n_k}\}_{k \in \mathbb{N}}$. Hence there is an infinite subsequence $\tilde{r}' := \{r_{m(m)}\}_{m \in \mathbb{N}}$ of the sequence $\{r_{n_k}\}_{k \in \mathbb{N}}$ such that the family

$$\mathcal{B}' = \{\tilde{b}_1, \ldots, \tilde{b}_{\nu^2}\} \cup \{\tilde{a}\}, \quad \tilde{b}_j := \{b^i_{k(m)}\}_{m \in \mathbb{N}}$$

is self-stable w.r.t. $\tilde{r}'$. Let $\tilde{X}_{a,\tilde{r}'}$ be a maximal self-stable family for which $\mathcal{B}' \subseteq \tilde{X}_{a,\tilde{r}'}$ and let $\pi' : \tilde{X}_{a,\tilde{r}'} \to \Omega_{a,\tilde{r}'}$ be the canonical projection. Conditions $b^1_k, \ldots, b^\nu_k \in O_{tr_{n_k}}$ imply $\pi'(\mathcal{B}') \subseteq B_t(\Omega_{a,\tilde{r}'})$ where $B_t(\Omega_{a,\tilde{r}'})$ is the closed ball with the center $\pi'(\tilde{a})$ and the radius $t$, see Lemma 4. It follows from (2.15) that inequality (2.3) holds. Consider now the case $\nu^2 = \infty$. Passing, if necessary, to a subsequence we may suppose that

$$M_{\varepsilon r_{n_k}}(O_{tr_{n_k}}) \geq k$$

for all elements $r_{n_k}$ of the sequence $\tilde{r}'$. Let $b^1_k, b^2_k, \ldots, b^\nu_k$ be points of $O_{tr_{n_k}}$ such that for every $k \in \mathbb{N}$ we have $d(b^i_k, b^j_k) > \varepsilon r_{n_k}$ whenever $1 \leq i < j \leq k$. Write $\tilde{b}_j$ for the $j$-th column of the following infinite matrix

$$\begin{bmatrix}
    b^1_1 & a & a & a & \\
    b^1_2 & b^2_2 & a & a & \\
    b^1_3 & b^2_3 & b^3_3 & a & \\
    \vdots & \vdots & \vdots & \vdots & \\
\end{bmatrix}$$

The countable family $\{\tilde{b}_1, \tilde{b}_2, \ldots\} \cup \{\tilde{a}\}$ satisfies the conditions of Lemma 5. Reasoning as in the case $\nu^2 < \infty$ we can find a closed ball $B_t(\Omega_{a,\tilde{r}'})$ for which the inequality $M_{\varepsilon}(B_t(\Omega_{a,\tilde{r}'})) \geq m$ holds for all $m \in \mathbb{N}$. Hence $M_{\varepsilon}(B_t(\Omega_{a,\tilde{r}'})) = \infty$. Statement (ii) follows.

(iii) Suppose that $\Omega_{a,\tilde{r}} = \Omega_{a,\tilde{\rho}}$ is tangent and separable. We must show that the inequality

$$M_{\varepsilon}(B_t(\Omega_{a,\tilde{r}})) \geq \mathcal{P}_{\varepsilon,t}(\varepsilon)$$

(2.16)

holds for all $\varepsilon > 0$. To prove it we can repeat the proof of inequality (2.3) using Proposition 3 instead of Lemma 5. Indeed, if inequality (2.16) does not hold for some $\varepsilon > 0$ and $\nu^2 := \mathcal{P}_{\varepsilon,t}(\varepsilon) < \infty$ for the same $\varepsilon$, then (2.14) holds with some $\{r_{n_k}\}_{k \in \mathbb{N}}$. For convenience rename $r_{n_k} = \rho_k$ for $k \in \mathbb{N}$ and set $\tilde{\rho} = \{\rho_k\}_{k \in \mathbb{N}}$. Note that the corresponding $\Omega_{a,\tilde{\rho}}$ is also tangent and separable. Limit relation
(2.14) implies that there is \( k_0 \in \mathbb{N} \) such that for every \( k \geq k_0 \) there are points \( b'_{k_i}, \ldots, b'_{k_j} \in O_{t^{\rho_k}} \) with
\[
d(b'_{k_i}, b'_{k_j}) > \varepsilon \rho_k
\] (2.17)
whenever \( 1 \leq i < j \leq \nu^2 \). The family \( B = \{\hat{b}_1, \ldots, \hat{b}_{\nu^2}\} \) where \( \hat{b}_j = \{b'_{k_i}\}_{k_i \in \mathbb{N}} \) with \( b'_{k_i} := a \) for \( k < k_0 \), \( j = 1, \ldots, \nu^2 \), satisfies the conditions of Proposition 3 if we take \( \Omega_{a,\tilde{r}} = \Omega_{a,\tilde{\rho}} \) in this proposition. Consequently, there exist \( \tilde{y}_1, \ldots, \tilde{y}_{\nu^2} \in \hat{X}_{a,\tilde{\rho}} \) such that the equality
\[
\tilde{y}_j' = \hat{b}_j'
\] (2.18)
holds for an infinite subsequence \( \tilde{\rho}' \) of \( \tilde{\rho} \) and for every \( j = 1, \ldots, \nu^2 \). Lemma 4, the relations \( b'_{k_i}, \ldots, b'_{k_j} \in O_{t^{\rho_k}} \), (2.21) and the definition of \( \Omega_{a,\tilde{\rho}} \) imply
\[
\{\pi(\tilde{y}_1), \ldots, \pi(\tilde{y}_{\nu^2})\} \subseteq B_t(\Omega_{a,\tilde{\rho}})
\]
where \( \pi : \hat{X}_{a,\tilde{\rho}} \to \Omega_{a,\tilde{\rho}} \) is the canonical projection. It follows from the last inclusion, (2.20) and from (2.21) that \( M_\varepsilon(B_t(\Omega_{a,\tilde{\rho}})) \geq \nu^2 \). Since \( \Omega_{a,\tilde{\rho}} \) and \( \Omega_{a,\tilde{\rho}'} \) are isometric, we have also
\[
M_\varepsilon(B_t(\Omega_{a,\tilde{\rho}})) \geq \nu^2,
\]
contrary to the supposition. The case \( \nu^2 = \infty \) can be considered similarly. \( \square \)

Let \( \mathbf{X} \) be a family of precompact metric spaces. We shall call \( \mathbf{X} \) \textit{uniformly precompact} if the inequality
\[
\sup\{\mathcal{N}_\varepsilon(X) : X \in \mathbf{X}\} < \infty
\]
holds for every \( \varepsilon > 0 \).

Let \( (X, d) \) be a metric space with a marked point \( a \) and let \( \tilde{r} \) be a normalizing sequence. Define a family \( \mathbf{B}_{t,\alpha}, \ t > 0, \ \alpha = \pi(\tilde{a}) \), by the rule:
\[
(B \in \mathbf{B}_{t,\alpha}) \iff (\text{there is } \Omega_{a,\tilde{r}'} \text{ such that } B = B_t(\Omega_{a,\tilde{r}'})�),
\]
i.e., elements of \( \mathbf{B}_{t,\alpha} \) are closed balls with the radius \( t \) and the centers \( \pi'(\tilde{a}') \), see diagram (1.4), in all possible pretangent spaces \( \Omega_{a,\tilde{r}'} \).

**Corollary 2.** The following statements are valid for every \( t > 0 \).

(i) If the inequality \( \mathcal{O}_{t,\tilde{r}}(\varepsilon) < \infty \) holds for all \( \varepsilon > 0 \), then the open balls \( \Theta_t(\Omega_{a,\tilde{r}}) \) are precompact in all \( \Omega_{a,\tilde{r}'} \).

(ii) If the family \( \mathbf{B}_{t,\alpha} \) is uniformly precompact, then we have \( \mathcal{O}_{t,\tilde{r}}(\varepsilon) < \infty \) for all \( \varepsilon > 0 \).

(iii) If \( \Omega_{a,\tilde{r}} \) is separable, tangent and if the closed ball \( B_t(\Omega_{a,\tilde{r}}) \) is compact, then the inequality \( \mathcal{O}_{t,\tilde{r}}(\varepsilon) < \infty \) holds for all \( \varepsilon > 0 \).

**Proof.** The first and the third statements directly follow from Proposition 2 and statements (i), (iii) of Theorem 1. To prove statement (ii) note that if \( \mathbf{B}_{t,\alpha} \) is uniformly precompact, then double inequality (1.5) implies
\[
M_\varepsilon(B_t(\Omega_{a,\tilde{r'}})) < \infty
\]
for the balls \( B_t(\Omega_{a,\tilde{r'}}) \) in each \( \Omega_{a,\tilde{r'}} \). The last inequality (2.3) has statement (ii) as a consequence. \( \square \)
Each normalizing sequence $\tilde{r} = \{r_n\}_{n \in \mathbb{N}}$ gives rise to the blow up of metric spaces $(X, d)$. Let us consider the sequence of metrics $d_1, d_2, \ldots$ on the space $X$,

$$
d_n(x, y) := \frac{1}{r_n} d(x, y), \quad x, y \in X. \tag{2.19}
$$

Denote by $O^n(a, t)$ the open ball with the center $a \in X$ and the radius $t > 0$ in the space $(X, d_n)$. It is clear that

$$
O^n(a, t) = \{x \in X : d(a, x) < r_n t\} = O(a, r_n t) \tag{2.20}
$$

and that

$$(d_n(x, y) > \varepsilon) \iff (d(x, y) > r_n \varepsilon). \tag{2.21}$$

Recall that a Hausdorff space $X$ is \textit{locally compact} if each point $x \in X$ has a neighborhood $U(x)$ with the compact closure $\overline{U(x)}$. Similarly $X$ is \textit{locally compact at the point} $x \in X$ if some open neighborhood $V$ of $x$ is a locally compact subspace of $X$.

Now we can formulate the following criterion of the precompactness of the balls in spaces $\Omega_{a, \tilde{r}}$. Write, for $t > 0$,

$$
O_{t, a, \tilde{r}} = O_{t, a, \tilde{r}}(X) := \{O^n(a, t) : n \in \mathbb{N}\},
$$

that is $O_{t, a, \tilde{r}}$ is the family of metric spaces $(O^n(a, t), d_n), n \in \mathbb{N}$, where sets $O^n(a, t)$ are defined by (2.20) and distance functions $d_n$ by (2.19).

**Proposition 4.** Let $(X, d)$ be a locally compact, metric space with a marked point $a \in X$ and let $\tilde{r} = \{r_n\}_{n \in \mathbb{N}}$ be a normalizing sequence. There exists $t_0 > 0$ such that the following statements are true for all $t \in [0, t_0[$.

(i) If the family $O_{t, a, \tilde{r}}$ is uniformly precompact, then the ball $\Theta_t(\Omega_{a, \tilde{r}})$ is totally bounded for each pretangent space $\Omega_{a, \tilde{r}} = \Omega_{a, \tilde{r}}^X$.

(ii) If the balls $B_t(\Omega_{a, \tilde{r}})$ are totally bounded for all $\Omega_{a, \tilde{r}}$, then $O_{t, a, \tilde{r}}$ is uniformly precompact.

(iii) If a pretangent space $\Omega_{a, \tilde{r}}$ is separable and tangent and, for this space, the ball $B_t(\Omega_{a, \tilde{r}})$ is totally bounded, then $O_{t, a, \tilde{r}}$ is uniformly precompact.

**Proof.** Let $t_1$ be a positive real number such that the closure $\overline{O(a, t_1]}$ is compact subspace of the metric space $(X, d)$. Let $c > 1$ be a constant for which the inequality $r_n < c$ holds for all $n \in \mathbb{N}$. We claim that statements (i)–(iii) are true for all $t \in [0, t_1[$.

(i) Using the definition of uniformly precompact families and inequality (1.5) we see that $O_{t, a, \tilde{r}}$ is uniformly precompact if and only if the inequality

$$
\sup_{n \in \mathbb{N}} \mathcal{M}_c(O^n(a, t)) < \infty \tag{2.22}
$$

holds for all $\varepsilon > 0$. It follows from (2.21) that

$$
\mathcal{M}_{c}^{r_n}(O_{tr_n}) = \mathcal{M}_c(O^n(a, t))
$$
for all \( n \in \mathbb{N} \) and every \( \varepsilon > 0 \). Hence

\[
\nu_{\tau,t}(\varepsilon) \leq \nu_{\tilde{\tau},t}(\varepsilon) \leq \sup_n M_\varepsilon(O^n(a,t)),
\]

so (2.22) and (2.2) imply the inequality \( M_\varepsilon(\Theta_t(\Omega_{a,\tilde{\tau}})) < \infty \) for all \( \varepsilon > 0 \) and for each pretangent space \( \Omega_{a,\tilde{\tau}} \). Therefore, by Proposition 2, \( \Theta_t(\Omega_{a,\tilde{\tau}}) \) is a totally bounded subset of \( \Omega_{a,\tilde{\tau}} \) if \( O_{t,a,\tilde{\tau}} \) is uniformly precompact.

(ii) All compact sets are totally bounded, so \( O(a,t_1) \) is totally bounded. Since for every \( A \subseteq X \) the closure \( \overline{A} \) is totally bounded if and only if \( A \) is totally bounded, see, for example, [10, p. 298], the ball \( O(a,t_1) \) is totally bounded. Each ball \( O^n(a,t), 0 < t \leq \frac{t_1}{c} \), is also totally bounded w.r.t. the distance function \( d_n = \frac{1}{r_n} d \) because

\[
((r_n \vee 1) < c) \Rightarrow (O^n(a,t) \subseteq O(a,t_1))
\]

and because the family of totally bounded sets is invariant when we change \( d \) for \( \frac{1}{r_n} d \). Hence, if \( O_{t,a,\tilde{\tau}} \) is not uniformly precompact for some \( t \in ]0, \frac{t_1}{c}[, \) then there are \( \varepsilon_1 > 0 \) and an infinite, strictly increasing sequence of natural numbers \( n_1, n_2, \ldots \) such that

\[
\infty = \sup_n M_{\varepsilon_1}(O^n(a,t)) = \lim_{k \to \infty} M_{\varepsilon_1}(O^{n_k}(a,t))
\]

\[
= \lim_{k \to \infty} M_{\varepsilon_1}(O^{n_k}_{tr_{n_k}}) \leq \limsup_{n \to \infty} M_{\varepsilon_1}r_n(O_{tr_n}) = \nu_{\tilde{\tau},t}(\varepsilon_1),
\]

so we have \( \nu_{\tilde{\tau},t}(\varepsilon_1) = \infty \). The last equality and inequality (2.3) imply that there is a subsequence \( \tilde{\tau}' \) of \( \tilde{\tau} \) such that the equality

\[
M_{\varepsilon_1}(B_t(\Omega_{a,\tilde{\tau}'})) = \infty
\]

holds for some pretangent space \( \Omega_{a,\tilde{\tau}'} \). Consequently the closed ball \( B_t(\Omega_{a,\tilde{\tau}'}) \) is not totally bounded. Statement (ii) follows.

(iii) Suppose that a pretangent space \( \Omega_{a,\tilde{\tau}} \) is separable and tangent and that \( B_t(\Omega_{a,\tilde{\tau}}) \) is totally bounded. Let \( t \) be a number from \( ]0, \frac{t_1}{c}[. \) We must show that \( O_{t,a,\tilde{\tau}} \) is uniformly precompact. It was shown in the proof of statement (ii) that if \( O_{t,a,\tilde{\tau}} \) is not uniformly precompact, then there is \( \varepsilon_1 > 0 \) such that \( \nu_{\tilde{\tau},t}(\varepsilon_1) = \infty \). This equality and (2.4) imply

\[
M_{\varepsilon}(B_t(\Omega_{a,\tilde{\tau}})) = \infty
\]

if \( \varepsilon < \varepsilon_1 \). Consequently, \( B_t(\Omega_{a,\tilde{\tau}}) \) is not totally bounded, contrary to the supposition.

\( \square \)

**Theorem 2.** Let \((X,d)\) be a locally compact, metric space with a marked point \( a \) and let \( \tilde{\tau} = \{r\}_{r \in \mathbb{N}} \) be a normalizing sequence. Suppose that a pretangent space \( \Omega_{a,\tilde{\tau}} \) is separable and tangent. Then \( \Omega_{a,\tilde{\tau}} \) is locally compact at the point \( \alpha = \pi(\tilde{a}) \), if and only if the family \( O_{t,a,\tilde{\tau}} \) is uniformly precompact for some \( t > 0 \).
Proof. Suppose that $\Omega_{a,\tilde{r}}$ is locally compact at the point $\alpha = \pi(\tilde{a})$. Then there is an open neighborhood $U(\alpha)$ of the point $\alpha$ with the compact closure $\overline{U(\alpha)}$. Let $B_t(\Omega_{a,\tilde{r}}) \subseteq \overline{U(\alpha)}$. Then $B_t(\Omega_{a,\tilde{r}})$ is compact and, consequently, totally bounded. Hence, by statement (iii) of Proposition 4, the family $O_{t,a,\tilde{r}}$ is uniformly precompact. Conversely, if $O_{t,a,\tilde{r}}$ is uniformly precompact, then, by statement (i) of Proposition 4, $\Theta_t(\Omega_{a,\tilde{r}})$ is totally bounded subset of $\Omega_{a,\tilde{r}}$. Since $\Omega_{a,\tilde{r}}$ is complete, see [12], the closure of $\Theta_t(\Omega_{a,\tilde{r}})$ is compact as the closure of totally bounded set in a complete metric space. Hence $\Omega_{a,\tilde{r}}$ is locally compact at the point $\alpha$. 

Before stating the following result, we recall a definition. A metric space $X$ is said to be proper (or finitely compact) if every bounded, closed subset of $X$ is compact, see, for example, [22, p. 37].

**Lemma 6.** Let $X$ be a metric space with a marked point $a$. The space $X$ is proper if and only if the closure of every open ball $O(a, t)$, $t > 0$, is compact.

**Proof.** All closed subsets of a compact set are compact and for every bounded, closed set $A \subseteq X$ there is an open ball $O(a, t)$ such that $O(a, t) \supseteq A$. Hence the closures of all open balls $O(a, t)$ are compact if and only if every bounded, closed subset of $X$ is compact. 

**Theorem 3.** Let $X$ be a metric space with a marked point $a$ and let $\tilde{r}$ be a normalizing sequence. Suppose that a pretangent space $\Omega_{a, \tilde{r}}$ is tangent and that every bounded subset of $X$ is totally bounded. Then the space $\Omega_{a, \tilde{r}}$ is proper if and only if the family $O_{t,a,\tilde{r}}(X)$ is uniformly precompact for every $t > 0$.

**Proof.** Let $Y$ be the completion of $X$. We identify $X$ with a corresponding dense subset of $Y$, that is

$$X \subseteq Y \text{ and } \overline{X} = Y.$$ 

It is well known that if $W$ is a subset of a complete metric space $Y$, then the closure of $W$ in $Y$ is isometric with the completion of $W$ and, moreover, $W$ is totally bounded if and only if the completion of $W$ is compact, see, for example, [13, 4.3.30]. Consequently, the closure of every open ball $O(a, t) \subseteq Y$ is compact. Hence, by Lemma 6, $Y$ is a proper metric space. Let $\tilde{X}_{a,\tilde{r}}$ be a maximal self-stable family such that $\Omega_{a,\tilde{r}} = \pi(\tilde{X}_{a,\tilde{r}})$ and let $\tilde{Y}_{a,\tilde{r}} \subseteq \tilde{Y}$ be a maximal self-stable family for which $\tilde{Y}_{a,\tilde{r}} \supseteq \tilde{X}_{a,\tilde{r}}$. Since $X$ is a dense subset of $Y$, the metric identification of $\tilde{Y}_{a,\tilde{r}}$ is isometric to $\Omega_{a,\tilde{r}}$, see Corollary 3.3 in [11]. Furthermore, Lemma 1 implies that $O_{t,a,\tilde{r}}(X)$ is uniformly precompact if and only if the family $O_{t,a,\tilde{r}}(Y)$ is uniformly precompact. Hence, without loss of generality, we assume that $X$ is a proper metric space. Since $X$ is proper, the closure $\overline{O(a, t_1)}$ is compact for every $t_1 > 0$. Consequently statements (i)–(iii) of Proposition 4 are valid for all $t > 0$, see the first four sentences of the proof of this proposition.

Suppose that a tangent space $\Omega_{a,\tilde{r}}$ is proper. Then $\Omega_{a,\tilde{r}}$ is separable and every $B_t(\Omega_{a,\tilde{r}})$ is totally bounded. Hence statement (iii) of Proposition 4 implies that $O_{t,a,\tilde{r}}$ is uniformly precompact for every $t > 0$.
Conversely, if $O_{t,a,\tilde{r}}$ is uniformly precompact for every $t > 0$, then, by statement (i) of Proposition 4, all open balls $\Theta_t(\Omega_{a,\tilde{r}})$ are totally bounded. In a complete metric space a set is totally bounded if and only if the closure of that set is compact. Since, by Theorem 3.6 from [12], the space $\Omega_{a,\tilde{r}}$ is complete, the closure $\Theta_t(\Omega_{a,\tilde{r}})$ is compact for each $t > 0$. Hence, by Lemma 6, $\Omega_{a,\tilde{r}}$ is proper. \hfill \qed

**Corollary 3.** Let $X$ be a proper metric space with marked point $a$, $\tilde{r}$ a normalizing sequence and $\Omega_{a,\tilde{r}}$ a tangent space to $X$. Then $\Omega_{a,\tilde{r}}$ is proper if and only if $O_{t,a,\tilde{r}}$ is uniformly precompact for every $t > 0$.

**Corollary 4.** Let $X$ be a subset of a normed linear space $(\mathbb{R}^n, \|\cdot\|)$, $n \in \mathbb{N}$, let $\tilde{r} = \{r_n\}_{n \in \mathbb{N}}$ be a normalizing sequence and let $a \in X$. Then each tangent space $\Omega_{a,\tilde{r}}^X$ is proper.

**Proof.** To prove note that every ball $O^n(a, t)$, in the space $(X, d_n)$ with

$$d_n(x, y) = \frac{1}{r_n} \|x - y\|,$$

is isometric to some subset of the open ball $\{x \in \mathbb{R}^n : \|x\| < t\}$ which is totally bounded. Using Corollary 3 we obtain the desired result. \hfill \qed

3. Boundedness of pretangent spaces and compactness of tangent ones

In this section we obtain some conditions under which pretangent spaces $\Omega_{a,\tilde{r}}^X$ are bounded for given $\tilde{r}$ and describe compact tangent spaces.

Let $(X, d)$ be a metric space and let $a \in X$ be a marked point. Write

$$R_{X,a} := \{d(x, a) : x \in X\},$$

i.e., a positive real number $t$ belongs to $R_{X,a}$ if and only if the sphere $S(a, t) = \{x \in X : d(x, a) = t\}$ is nonvoid.

The conditions for the boundedness of $\Omega_{a}^X$ will be presented in terms of a porosity of the set $R_{X,a}$, so recall a definition.

**Definition 4.** Let $A \subseteq \mathbb{R}$ and let $x \in A$. The right-side porosity of $A$ at the point $x$ is the quantity

$$p(A) = p(A, x) := \limsup_{h \to 0} \frac{l(x, h, A)}{h}$$

(3.1)

where $l(x, h, A)$ is the length of the longest interval in $[x, x + h] \setminus A$.

We will use the following lemma.
Lemma 7. Let \( A \subseteq \mathbb{R}^+ \) and let \( 0 \in A \). If the inequality
\[
p(A, 0) < p_1
\]
holds with \( p_1 \in [0, 1[ \), then for every infinite, strictly decreasing sequence of real numbers \( r_n \) with \( \lim_{n \to \infty} r_n = 0 \) there is an infinite subsequence \( \{r_{n_k}\}_{k \in \mathbb{N}} \) such that for every \( k \in \mathbb{N} \) there are points \( x_1^{(k)}, \ldots, x_k^{(k)} \in A \) for which
\[
\frac{r_{n_k+1}}{1 - p_1} < \frac{r_{n_k+1}}{(1 - p_1)^2} \leq x_1^{(k)} \leq \frac{r_{n_k+1}}{(1 - p_1)^m} \leq \frac{x_2^{(k)}}{(1 - p_1)^3} \leq \frac{r_{n_k+1}}{(1 - p_1)^{m+1}} \leq x_2^{(k)} \leq \frac{r_{n_k+1}}{(1 - p_1)^{2k-1}} \leq \frac{r_{n_k+1}}{(1 - p_1)^{2k}} \leq x_k^{(k)} \leq r_{n_k}.
\]

Proof. Suppose that inequality (3.2) holds with \( p_1 \in [0, 1[ \). Let \( n_1 \) be a natural number such that \( l((0, h, A) < p_1 h \) for all \( h \in ]0, r_{n_1}[ \), see Definition 4. Suppose that \( r_{n_1}, \ldots, r_{n_k} \) are defined. Define the remaining \( r_{n_{m}} \) inductively. Write \( r_{n_{k+1}} \) for the first \( r_n \) with
\[
r_n \leq (1 - p_1)^{2k+1} r_{n_k}.
\]
Since for all \( m \in \mathbb{N} \) we have
\[
\frac{r_{n_{k+1}}}{(1 - p_1)^{2m}} - \frac{r_{n_{k+1}}}{(1 - p_1)^{2m+1}} = p_1,
\]
Definition 4 and inequality (3.2) imply that
\[
A \cap \left[ \frac{r_{n_{k+1}}}{(1 - p_1)^m}, \frac{r_{n_{k+1}}}{(1 - p_1)^{m+1}} \right] \neq \emptyset
\]
for all \( m = 1, \ldots, k \). Thus we can find points \( x_1^{(k)}, \ldots, x_k^{(k)} \) which satisfy (3.3). \( \square \)

Theorem 4. Let \((X, d)\) be a metric space with a marked point \( a \). The following statements are equivalent.

(i) The right-side porosity of \( R_{X,a} \) at 0 is 1.

(ii) There is a normalizing sequence \( \tilde{r} = \{r_n\}_{n \in \mathbb{N}} \) such that some \( \Omega_{a, \tilde{r}} \) is tangent and one-point.

(iii) There is normalizing sequence \( \tilde{r} \) such that all pretangent spaces \( \Omega_{a, \tilde{r}'} \) are bounded for each subsequence \( \tilde{r}' \) of \( \tilde{r} \).

Proof. Without loss of generality we assume that \( a \) is not an isolated point of \( X \) because, in the opposite case, the theorem is trivially true.

First we prove the implication (i) \( \Rightarrow \) (ii). Suppose that the equality
\[
\limsup_{h \to 0} \frac{l((0, h, R_{X,a})}{h} = 1
\]
holds. Let \( h = \{ h_n \} \) be a strictly decreasing sequence of positive numbers such that
\[
\lim_{n \to \infty} h_n = 0 \quad \text{and} \quad \lim_{n \to \infty} \frac{l(0, h_n, R_{X,a})}{h_n} = 1.
\]
Consider a sequence of intervals \( [c_n, d_n] \subseteq [0, h_n] \setminus R_{X,a} \) for which
\[
\lim_{n \to \infty} \frac{d_n - c_n}{h_n} = 1.
\] (3.4)
Since \( a \) is a limit point of \( X \), we have the inequality \( 0 < c_n \) for every \( n \). Since \( \lim_{n \to \infty} h_n = 0 \), we may suppose, passing, if it is necessary, to a subsequence, that
\[
0 < h_{n+1} \leq c_{n+1} < d_n \leq h_n
\] (3.5)
for all \( n \in \mathbb{N} \). Write
\[
r_n := \frac{d_n}{c_n} \quad \text{for all} \quad n \in \mathbb{N}.
\] (3.6)
Let \( \hat{x} = \{ x_n \} \subseteq \hat{x} \) be a strictly decreasing sequence of positive numbers such that \( \lim_{n \to \infty} h_n = 0 \) and \( \lim_{n \to \infty} \frac{l(0, h_n, R_{X,a})}{h_n} = 1 \). Consider a sequence of intervals \( [c_n, d_n] \subseteq [0, h_n] \setminus R_{X,a} \) for which
\[
\lim_{n \to \infty} \frac{d_n - c_n}{h_n} = 1.
\] (3.4)
Since \( a \) is a limit point of \( X \), we have the inequality \( 0 < c_n \) for every \( n \). Since \( \lim_{n \to \infty} h_n = 0 \), we may suppose, passing, if it is necessary, to a subsequence, that
\[
0 < h_{n+1} \leq c_{n+1} < d_n \leq h_n
\] (3.5)
for all \( n \in \mathbb{N} \). Write
\[
r_n := \frac{d_n}{c_n} \quad \text{for all} \quad n \in \mathbb{N}.
\] (3.6)
Let \( \hat{x} = \{ x_n \} \subseteq \hat{x} \). We claim that if \( \hat{x} \) and \( \hat{a} = (a, a, \ldots) \) are mutually stable w.r.t. the normalizing sequence \( \hat{r} = \{ r_n \} \) where \( r_n \)'s were defined by (3.6), then
\[
\hat{d}_{\hat{r}}(\hat{x}, \hat{a}) = 0.
\] (3.7)
Indeed, (3.4) and (3.5) imply the limit relations
\[
\lim_{n \to \infty} \frac{c_n}{h_n} = 0 \quad \text{and} \quad \lim_{n \to \infty} \frac{d_n}{h_n} = 1.
\]
Hence, we obtain
\[
\lim_{n \to \infty} \frac{d_n}{c_n} = \infty.
\] (3.8)
Moreover, since \( [c_n, d_n] \subseteq [0, h_n] \setminus R_{X,a} \), we have either \( d(x_n, a) \geq d_n \) or \( d(x_n, a) \leq c_n \) for all \( n \in \mathbb{N} \). Consequently we obtain either
\[
\frac{d(x_n, a)}{r_n} \geq \sqrt{\frac{d_n}{c_n}}
\] (3.9)
or
\[
\frac{d(x_n, a)}{r_n} \leq \sqrt{\frac{c_n}{d_n}}
\] (3.10)
for every natural \( n \). Relation (3.8) shows that (3.9) cannot be valid for sufficiently large \( n \) if \( \hat{x} \) and \( \hat{a} \) are mutually stable. Thus (3.7) follows from (3.10) and (3.8).

It is proved that if \( \hat{r} = \{ r_n \} \) is defined by (3.6), then there is a unique pretangent space \( \Omega_{\hat{r},a}^X \) and this space is one-point. Note also that this \( \Omega_{\hat{r},a}^X \) is tangent. To prove it we can suppose that \( \hat{x}' \) and \( \hat{a}' \) are mutually stable w.r.t. a subsequence \( \hat{r}' = \{ r_{n_k} \} \) and repeat the proof of equality (3.7) substituting \( d_{n_k}, c_{n_k}, h_{n_k} \) and \( r_{n_k} \) instead of \( d_n, c_n, h_n \) and, respectively, \( r_n \). The implication (i)\( \Rightarrow \) (ii) is proved.

The implication (ii)\( \Rightarrow \) (iii) follows from the definition of tangent spaces, for details see Proposition 1.2 in [11].
It still remains to prove that (iii) implies (i). Suppose that statement (iii) holds but there is \( p_1 \in [0,1[ \) such that \( p(R_{X,a},0) < p_1 \) where \( p(R_{X,a},0) \) is the right-side porosity of \( R_{X,a} \) at 0. Let \( \tilde{r} = \{ r_n \}_{n \in \mathbb{N}} \) be a normalizing sequence such that all \( \Omega^X_{a,\tilde{r}} \) are bounded for each subsequence \( \tilde{r}' \) of \( \tilde{r} \). Applying Lemma 7 with \( A = R_{X,a} \) we can find a subsequence \( \tilde{r}' = \{ r_{n_k} \}_{k \in \mathbb{N}} \) of the sequence \( \tilde{r} \) such that for every \( k \in \mathbb{N} \) there are points \( x_1^{(k)}, \ldots, x_k^{(k)} \in X \) for which

\[
\begin{align*}
\frac{1}{1 - p_1} & \leq \frac{d(x_1^{(k)}, a)}{r_{n_k + 1}} \leq \frac{1}{(1 - p_1)^2}, \\
\frac{1}{(1 - p_1)^3} & \leq \frac{d(x_2^{(k)}, a)}{r_{n_k + 1}} \leq \frac{1}{(1 - p_1)^4}, \\
& \vdots \\
\frac{1}{(1 - p_1)^{2k-1}} & \leq \frac{d(x_k^{(k)}, a)}{r_{n_k + 1}} \leq \frac{1}{(1 - p_1)^{2k}}.
\end{align*}
\tag{3.11}
\]

For convenience we rename the elements of \( \tilde{r}' \) as \( q_k = r_{n_k + 1}, k \in \mathbb{N} \).

Write \( \tilde{x}_j \) for the \( j \)-th column of the following infinite matrix

\[
\begin{array}{ccccccc}
x_1^{(1)} & a & a & a & a & \ldots \\
x_1^{(2)} & a & a & a & a & \ldots \\
x_2^{(3)} & x_3^{(3)} & a & a & \ldots \\
x_1^{(4)} & x_2^{(4)} & x_3^{(4)} & x_4^{(4)} & a & \ldots \\
& \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{array}
\]

It follows from (3.11) that the inequalities

\[
\frac{1}{(1 - p_1)^{2j-1}} \leq \liminf_{k \to \infty} \frac{d(x_j^{(k)}, a)}{q_k} \leq \limsup_{k \to \infty} \frac{d(x_j^{(k)}, a)}{q_k} \leq \frac{1}{(1 - p_1)^{2j}}
\tag{3.12}
\]

hold for all \( j \in \mathbb{N} \). Hence the family \( \mathfrak{B} := \{ \tilde{x}_1, \tilde{x}_2, \ldots \} \) satisfies the conditions of Lemma 5. Thus there is an infinite subsequence \( \tilde{q}' \) of \( \tilde{q} = \{ q_k \}_{k \in \mathbb{N}} \) such that the corresponding family \( \mathfrak{B}' \) from this lemma is self-stable. Let \( \tilde{X}_{a,\tilde{q}'} \supseteq \mathfrak{B}' \) be a maximal self-stable family. Then the first inequality in (3.12) implies that \( \Omega^X_{a,\tilde{q}'} \), the metric identification of \( \tilde{X}_{a,\tilde{q}'} \), is unbounded, contrary to (iii).

\begin{corollary}
Let \((X,d)\) be a metric space with a marked point \( a \). Then the right-side porosity of \( R_{X,a} \) at 0 is 1 if and only if there is a normalizing sequence \( \tilde{r} \) such that all \( \Omega^X_{a,\tilde{r}} \) are tangent and compact.
\end{corollary}

\begin{proof}
The one-point tangent space in statement (ii) of Theorem 4 is compact and, moreover, if for some normalizing sequence \( \tilde{r} \) all \( \Omega^X_{a,\tilde{r}} \) are tangent and compact, then all \( \Omega^X_{a,\tilde{r}'} \) are bounded. Thus the corollary follows from Theorem 4.
\end{proof}
Remark 2. It should be observed that we cannot guarantee the presence of a compact tangent space $\Omega_{a,\bar{r}}^X$ with $\text{card}(\Omega_{a,\bar{r}}^X) \geq 2$, even if $p(R_{X,a},0) = 1$. As an example, consider a metric space $X$ for which the marked point $a$ is isolated.

Let $\tilde{X}_{a,\bar{r}}$ be a maximal self-stable family and let $\Omega_{a,\bar{r}}^X$ be the corresponding pre-tangent space. Write

$$D(\Omega_{a,\bar{r}}) = D(\Omega_{a,\bar{r}}^X) := \sup \left\{ d_\rho(x, \bar{a}) : x \in \tilde{X}_{a,\bar{r}} \right\} = \sup \left\{ \rho(\alpha, \beta) : \beta \in \Omega_{a,\bar{r}}^X \right\}$$

where $\rho$ is defined by formula (1.2) and $\alpha = \pi(\bar{a})$, see diagram (1.4). It is clear that $D(\Omega_{a,\bar{r}} < \infty$ if and only if $\Omega_{a,\bar{r}}$ is bounded. Moreover, if $0 < D(\Omega_{a,\bar{r}}) < \infty$, then for the normalizing sequence $\tilde{t} = \{t_n\}_{n \in \mathbb{N}}$ with $t_n := r_nD(\Omega_{a,\bar{r}})$, we obtain $D(\Omega_{a,\bar{r}}) = 1$.

Definition 5. Let $X$ be a metric space, let $a$ be a limit point of $X$ and let $p(R_{X,a},0) = 1$. Let us define $\tilde{SI}$ as the set of all sequences $\tilde{s} = \{[g_n, b_n]\}_{n \in \mathbb{N}}$ of open intervals $]a_n, b_n[ \subseteq \mathbb{R}$ which meet the following conditions

(i) For every $\tilde{s} = \{[g_n, b_n]\}_{n \in \mathbb{N}} \in \tilde{SI}$ each $]g_n, b_n[\subset \mathbb{R}$ is a connected component of the set $\text{Int}(\mathbb{R}^+ \setminus R_{X,a})$, that is $]g_n, b_n[ \cap R_{X,a} = \emptyset$ but if $]g, b[ \supseteq ]g_n, b_n[$, then $]g, b[ \cap R_{X,a} \neq \emptyset$.

(ii) For every $\tilde{s} = \{[g_n, b_n]\}_{n \in \mathbb{N}} \in \tilde{SI}$ the sequence $\{g_n\}_{n \in \mathbb{N}}$ is strictly decreasing and

$$\lim_{n \to \infty} \frac{|g_n - b_n|}{|b_n|} = 1.$$

It should be pointed out that conditions (i) and (ii) imply the limit relation $\lim_{n \to \infty} g_n = 0$ for every $\tilde{s} \in \tilde{SI}$. Hence these $\tilde{g} = \{g_n\}_{n \in \mathbb{N}}$ can be regarded as some normalizing sequences. If $\tilde{s} \in \tilde{SI}$ and $\tilde{g}$ is the corresponding normalizing sequence, then it is not difficult to establish the inequality

$$D(\Omega_{a,\tilde{g}}^X) \leq 1$$

for all $\Omega_{a,\tilde{g}}^X$ and to find some $\Omega_{a,\tilde{g}}^X$ such that

$$D(\Omega_{a,\tilde{g}}^X) = 1.$$

Theorem 5. Let $(X, d)$ be a metric space with a marked point $a$ and let $\tilde{g} = \{g_n\}_{n \in \mathbb{N}}$ be a normalizing sequence. Suppose that $a$ is not an isolated point of $X$ but that

$$p(R_{X,a},0) = 1.$$

The following conditions are equivalent:

(i) Inequality (3.14) holds for all $\Omega_{a,\tilde{g}}^X$ and, in addition, there is $\Omega_{a,\tilde{g}}^X$ such that equality (3.15) holds.
(ii) There exists \( \tilde{s} = \{ [d_n, b_n] \}_{n \in \mathbb{N}} \in \tilde{S} \) I with
\[
\lim_{n \to \infty} \frac{g_n}{d_n} = 1. \tag{3.17}
\]

**Proof.** Suppose that condition (ii) is satisfied. Then it is not difficult to prove that (3.17) implies (3.14) for all pretangent spaces \( \Omega_{a,\tilde{g}}^X \). Furthermore, using (3.17) we can find \( \tilde{x} = \{ x_n \}_{n \in \mathbb{N}} \in \tilde{X} \) such that
\[
\lim_{n \to \infty} \frac{d(x_n, a)}{g_n} = 1. \tag{3.18}
\]

If \( \tilde{X}_{a,\tilde{g}} \supseteq \{ \tilde{x}, \tilde{a} \} \) is a maximal self-stable family, then we obtain \( D(\Omega_{a,\tilde{g}}^X) \geq 1 \) for the corresponding \( \Omega_{a,\tilde{g}}^X \). The last inequality and (3.14) imply (3.15), so we obtain the implication \( (ii) \Rightarrow (i) \).

Now suppose that (i) is true. Equality (3.15) implies that there is \( \tilde{x} = \{ x_n \}_{n \in \mathbb{N}} \in \tilde{X} \) for which (3.18) holds. Indeed, by (3.15), for every \( \varepsilon > 0 \) there is \( \tilde{x}(\varepsilon) = \{ x_n(\varepsilon) \}_{n \in \mathbb{N}} \in \tilde{X} \) such that
\[
1 - \varepsilon \leq \lim_{n \to \infty} \frac{d(x_n(\varepsilon), a)}{g_n} \leq 1. \tag{3.19}
\]

Let \( \{ \varepsilon(i) \}_{i \in \mathbb{N}} \) be a strictly decreasing sequence of positive numbers with \( \lim_{i \to \infty} \varepsilon(i) = 0 \). Then there exists an infinite, strictly increasing sequence \( \{ n_i \}_{i \in \mathbb{N}} \) of natural numbers such that
\[
1 - \varepsilon(i) \leq \frac{d(x_n(\varepsilon(i)), a)}{g_n} \leq 1 + \varepsilon(i) \tag{3.20}
\]
for all \( i \in \mathbb{N} \) and \( n = n_i, 1 + n_i, \ldots, n_{i+1} - 1 \). Write
\[
x_n := \begin{cases} x_n(\varepsilon_1) & \text{if } 1 \leq n < n_2 \\ x_n(\varepsilon_2) & \text{if } n_2 \leq n < n_3 \\ \cdots & \cdots \\ x_n(\varepsilon_i) & \text{if } n_i \leq n < n_{i+1} \\ \cdots & \cdots 
\end{cases}
\]
and set \( \tilde{x} := \{ x_n \}_{n \in \mathbb{N}} \). Note that, for this \( \tilde{x} \), (3.18) follows from (3.20). Let us consider the sequence \( \tilde{t} = \{ t_n \}_{n \in \mathbb{N}} \) with \( t_n := d(a, x_n), n \in \mathbb{N} \). Let \( \varepsilon \in [0, 1[ \). We claim that the equality
\[
R_{X,a} \cap \left[ t_n(1 + \varepsilon), t_n(1 + \varepsilon) \right] = \emptyset \tag{3.21}
\]
holds if \( n \) is sufficiently large. Indeed, in the opposite case we can find an infinite, strictly increasing sequence of natural numbers \( n_k, k \in \mathbb{N} \), and sequence \( \tilde{y} = \{ y_k \}_{k \in \mathbb{N}} \in \tilde{X} \) for which
\[
1 + \varepsilon \leq \liminf_{n \to \infty} \frac{d(y_k, a)}{t_{n_k}} \leq \limsup_{n \to \infty} \frac{d(y_k, a)}{t_{n_k}} \leq 1 + \frac{1}{\varepsilon}.
\]
Using Proposition 3 with $\mathcal{B} = \{ \tilde{y}, \tilde{a} \}$ we see that there is a subsequence $t^*$ of the sequence $\{t_n\}_{k \in \mathbb{N}}$ with a self-stable $\mathcal{B}'$. Hence for some $\Omega_{a,\tilde{y}}$ we obtain $D(\Omega_{a,\tilde{y}}) \geq 1 + \varepsilon$. This inequality, the definition of $t_n$ and limit relation (3.18) imply that there is $\tilde{y}'$ such that $D(\Omega_{a,\tilde{y}'}) \geq 1 + \varepsilon$ for some $\Omega_{a,\tilde{y}'}$, contrary to (3.14). Consequently, (3.21) holds. Letting $\varepsilon \to 0$ in (3.21) and taking the connected components of $\mathbb{R}^+ \setminus R_{X,a}$ which contain $t_n(1 + \varepsilon), t_n(1 + \frac{1}{\varepsilon})$, we can find $\tilde{s} = \{ [g_n, b_n] \}_{n \in \mathbb{N}} \in \tilde{SI}$ such that (3.17) holds.

Theorem 6. Let $(X, d)$ be a metric space and $a$ a limit point of $X$ such that $p(R_{X,a}) = 1$, and $\tilde{s} = \{ [g_n, b_n] \}_{n \in \mathbb{N}} \in \tilde{SI}$, $\tilde{y} := \{ g_n \}_{n \in \mathbb{N}}$, and $\Omega_{a,\tilde{y}}^X$ a tangent space. Write

$$B_{g_n} := B(a, g_n) = \{ x \in X : d(a, x) \leq g_n \}.$$ 

Then $\Omega_{a,\tilde{y}}^X$ is compact if and only if the inequality

$$\limsup_{n \to \infty} M_{g_n}(B_{g_n}) < \infty \quad (3.22)$$

holds for every $\varepsilon > 0$.

In the following proof we use the notations from Theorem 1.

Proof. Since $\tilde{s} \in \tilde{SI}$, we have the inequality $D(\Omega_{a,\tilde{y}}) \leq 1$, so

$$B_t(\Omega_{a,\tilde{y}}) \subseteq \Omega_{a,\tilde{y}} = B_1(\Omega_{a,\tilde{y}}) \quad (3.23)$$

for all $t > 0$. It is clear that $B_{g_n} = B(a, g_n) \subseteq O(a, t g_n)$ for every $t > 1$. Hence

$$\limsup_{n \to \infty} M_{g_n}(B_{g_n}) \leq \limsup_{n \to \infty} M_{g_n}(O(a, t g_n)) = \nu_{\tilde{y},t}(\varepsilon) \quad (3.24)$$

for all $\varepsilon > 0$ and all $t > 1$. Assume that $\Omega_{a,\tilde{y}}$ is compact. Then $\Omega_{a,\tilde{y}}$ is separable and, by (3.23), $B_t(\Omega_{a,\tilde{y}})$ is compact, so statement (iii) of Corollary 2 implies the inequality

$$\nu_{\tilde{y},t}(\varepsilon) < \infty \quad (3.25)$$

for all $\varepsilon > 0$. Now, inequality (3.24) follows from (3.24) and (3.25).

To prove that (3.22) implies the compactness of $\Omega_{a,\tilde{y}}$ we are in need of the following lemma.

Lemma 8. Let $(X, d)$ be a metric space, let $a$ be a limit point of $X$ with $p(R_{X,a}) = 1$ and let $\tilde{s} = \{ [g_n, b_n] \}_{n \in \mathbb{N}} \in \tilde{SI}$. Then for every $t > 1$ there exists $n_0 \in \mathbb{N}$ such that equalities

$$O(a, t g_n) = \{ x \in X : d(x, a) < t g_n \} = \{ x \in X : d(x, a) \leq g_n \} = B_{g_n} \quad (3.26)$$

hold for all natural $n \geq n_0$. 
Proof. Let \( t > 1 \). Since \( \tilde{s}i \in \tilde{SI} \), Definition 5 implies

\[
\lim_{n \to \infty} \frac{b_n}{g_n} = \infty.
\]

Hence there is a natural \( n_0 \) such that \( b_n > tg_n \) whenever \( n \geq n_0 \). Consequently, if \( n \geq n_0 \), then the double inequality

\[
g_n < tg_n < b_n \tag{3.27}
\]

holds. Since \( |g_n, b_n| \subseteq \mathbb{R}^+ \setminus R_{X,a} \), the definition of \( R_{X,a} \) and (3.27) imply the equality

\[
\{ x \in X : g_n < d(x, a) < tg_n \} = \emptyset
\]

for all \( n \geq n_0 \). The last equality and (3.26) are equivalent. \( \Box \)

Continuation of the proof of Theorem 6. Suppose now (3.22) holds for all \( \varepsilon > 0 \). Let \( t_1 > 1 \). Then (3.22), (3.26) and (2.1) imply that \( \nu_{\tilde{g}, t_1}(\varepsilon) < \infty \) for all \( \varepsilon > 0 \).

Hence, by statement (i) of Corollary 2, the open ball \( \Theta_{t_1}(\tilde{\Omega}_{a, \tilde{g}}) \) is precompact. The condition \( D(\tilde{\Omega}_{a, \tilde{g}}) \leq 1 \) implies that \( \Theta_{t_1}(\tilde{\Omega}_{a, \tilde{g}}) = \tilde{\Omega}_{a, \tilde{g}} \). Hence \( \tilde{\Omega}_{a, \tilde{g}} \) is precompact and complete as a tangent space. Consequently \( \tilde{\Omega}_{a, \tilde{g}} \) is compact. \( \Box \)

Lemma 9. Let \((X, d)\) be a metric space with a marked point \( a \) and let \( \tilde{r} = \{r_n\}_{n \in \mathbb{N}} \) and \( \tilde{t} = \{t_n\}_{n \in \mathbb{N}} \) be normalizing sequences such that the limit relation

\[
\lim_{n \to \infty} \frac{r_n}{t_n} = c
\]

holds for some \( c > 0 \). If \( \tilde{X}_{a, \tilde{r}}^* \) is maximal self-stable w.r.t. \( \tilde{r} \), then \( \tilde{X}_{a, \tilde{r}}^* \) is maximal self-stable w.r.t. \( \tilde{t} \) and corresponding pretangent spaces \( \tilde{\Omega}_{a, \tilde{r}}^* \) and \( \tilde{\Omega}_{a, \tilde{t}}^* \) are similar.

The simple proof is omitted here.

The following theorem provides us with conditions under which a tangent space is compact and has more than one point, see Remark 2.

Theorem 7. Let \((X, d)\) be a metric space with a marked point \( a \), \( \tilde{r} = \{r_n\}_{n \in \mathbb{N}} \) and let \( \tilde{\Omega}_{a, \tilde{r}}^* \) be a pretangent space to \( X \) at the point \( a \). The space \( \tilde{\Omega}_{a, \tilde{r}}^* \) is compact and

\[
\text{card}(\tilde{\Omega}_{a, \tilde{r}}^*) \geq 2 \tag{3.28}
\]

if and only if the following conditions are satisfied:

(i) The point \( a \) is a limit point of \( X \);

(ii) The equality \( p(R_{X,a}) = 1 \) holds;
(iii) There exists \( \tilde{s}_i = \{ [d_n, b_n] \}_{n \in \mathbb{N}} \in \tilde{SI} \) such that
\[
0 < \lim_{n \to \infty} \frac{r_n}{d_n} = c < \infty, \tag{3.29}
\]
and that the inequality
\[
\limsup_{n \to \infty} M_{\varepsilon d_n}(B_{d_n}) < \infty \tag{3.30}
\]
holds for all \( \varepsilon > 0 \) where \( B_{d_n} = \{ x \in X : d(a, x) \leq d_n \} \).

**Proof.** Suppose that \( \Omega^*_{a, \tilde{r}} \) is compact and (3.28) holds. We will prove that conditions (i)–(iii) are satisfied.

(i) The point \( a \) is an isolated point of \( X \) if and only if \( \text{card}(\Omega^*_{a, \tilde{r}}) = 1 \) for every \( \Omega^*_{a, \tilde{r}} \), see Proposition 1.3 in [11]. Hence (3.28) implies (i).

(ii) To prove (ii) it is sufficient, by Theorem 4, to show that all pretangent spaces \( \Omega^*_{a, \tilde{r}} \) are bounded for every subsequence \( \tilde{r}' \) of \( \tilde{r} \). Assume that there is an unbounded \( \Omega^*_{a, \tilde{r}}, \tilde{r}' = \{ r_{nk} \}_{k \in \mathbb{N}} \). Write, for convenience, \( t_k := r_{nk}, \ k \in \mathbb{N} \) and \( \tilde{t} := \{ t_k \}_{k \in \mathbb{N}} \). Since \( \Omega^*_{a, \tilde{t}} = \Omega^*_{a, \tilde{r}} \) is unbounded, there is a countable family \( \mathfrak{B} = \{ \tilde{b}_1, \tilde{b}_2, \ldots \} \), \( \tilde{b}_m = \{ b^{(m)}_k \}_{k \in \mathbb{N}} \in \tilde{X} \) such that
\[
\lim_{m \to \infty} \left( \lim_{k \to \infty} \frac{d(b^{(m)}_k, a)}{t_k} \right) = \infty \tag{3.31}
\]
but
\[
\tilde{d}(\tilde{a}, \tilde{b}_m) = \lim_{k \to \infty} \frac{d(b^{(m)}_k, a)}{t_k} < \infty \tag{3.32}
\]
for every \( m \in \mathbb{N} \). Consider the space \( \Omega^*_{a, \tilde{t}} = \Omega^*_{a, \tilde{r}} \). Since \( \Omega^*_{a, \tilde{r}} \) is tangent, \( \Omega^*_{a, \tilde{t}} \) is isometric to \( \Omega^*_{a, \tilde{r}} \) and also is tangent. Hence \( \Omega^*_{a, \tilde{t}} \) is separable and bounded. Taking into account (3.32) we see that all conditions of Proposition 3 are satisfied. Using this we see that equality (3.31) implies the unboundedness of \( \Omega^*_{a, \tilde{t}} \). The obtained contradiction shows that all pretangent spaces \( \Omega^*_{a, \tilde{r}} \) are bounded.

(iii) Write
\[
g_n := r_n D(\Omega^*_{a, \tilde{r}}) \text{ and } \tilde{g} := \{ g_n \}_{n \in \mathbb{N}}
\]
where
\[
D(\Omega^*_{a, \tilde{r}}) = \sup \{ \rho(\alpha, \beta) : \rho \in \Omega^*_{a, \tilde{r}}, \ \alpha = \pi(a) \}, \tag{1.2}
\]
see (1.2) and (1.4). Note that \( 0 < D(\Omega^*_{a, \tilde{r}}) < \infty \) because \( \Omega^*_{a, \tilde{r}} \) is compact with \( \text{card}(\Omega^*_{a, \tilde{r}}) \geq 2 \). Since \( \Omega^*_{a, \tilde{r}} \) and \( \Omega^*_{a, \tilde{g}} \) are similar, see Lemma 9, \( \Omega^*_{a, \tilde{g}} \) is compact if and only if \( \Omega^*_{a, \tilde{g}} \) is compact. Moreover, \( D(\Omega^*_{a, \tilde{g}}) = 1 \) and reasoning as in the proof of condition (ii), we can find that \( D(\Omega^*_{a, \tilde{g}}) \leq 1 \) for all pretangent spaces \( \Omega^*_{a, \tilde{g}} \).

Hence, by Theorem 5, there exists \( \tilde{s}_i = \{ [d_n, b_n] \} \in \tilde{SI} \) such that
\[
\lim_{n \to \infty} \frac{g_n}{d_n} = 1.
\]
Consequently (3.29) holds for some $\tilde{s}_i \in \tilde{S}I$ with $c = \frac{1}{D(\Omega_{a,\tilde{r}})}$. Consider the space $\Omega_{a,d}$ where $\tilde{d} = \{d_n\}_{n \in \mathbb{N}}$. It is clear that $\Omega_{a,d}$ is compact and $\text{card}(\Omega_{a,d}) = \text{card}(\Omega_{a,\tilde{r}})$. Hence, by Theorem 6, inequality (3.30) holds for all $\epsilon > 0$.

Suppose now that conditions (i)–(iii) are satisfied. Let $\tilde{d} = \{d_n\}_{n \in \mathbb{N}}$ be a sequence defined in condition (iii). Since $\Omega_{a,\tilde{d}}$ is tangent, $\Omega_{a,d}$ is also tangent. Theorem 6 implies that $\Omega_{a,d}$ is compact, so $\Omega_{a,\tilde{d}}$ is the same. To complete the proof, it suffices to show that $\text{card}(\Omega_{a,d}) \geq 2$ because $\text{card}(\Omega_{a,\tilde{d}}) = \text{card}(\Omega_{a,\tilde{r}})$. Let $\tilde{X}_{a,d}$ be a maximal self-stable family with the metric identification $\Omega_{a,d}^*$. If $\text{card}(\Omega_{a,d}^*) = 1$, then we have

$$\lim_{n \to \infty} \frac{d(x_n, a)}{d_n} = 0 \quad (3.33)$$

for every $\tilde{x} = \{x_n\}_{n \in \mathbb{N}} \in \tilde{X}_{a,d}$. Since, $\{d_n, b_n\}_{n \in \mathbb{N}} \in \tilde{S}I$, there is $\tilde{y} = \{y_n\}_{n \in \mathbb{N}} \in \tilde{X}$ such that

$$\lim_{n \to \infty} \frac{d(y_n, a)}{d_n} = 1 \quad (3.34)$$

Relations (3.34) and (3.33) imply that $\tilde{x}$ and $\tilde{y}$ are mutually stable for every $\tilde{x} \in \tilde{X}_{a,d}$. Hence $\tilde{X}_{a,d}$ is not maximal self-stable, contrary to the supposition. \qed

4. Example of a compact metric space with a pretangent space having the density $c$

Recall that a cardinal number $\alpha$ is the density of a metric space $X$ if

$$\alpha = \min_A \text{card}(A)$$

where the minimum is taken over the family of all dense sets $A \subseteq X$. Write, as usual, $\aleph_0$ for $\text{card}(\mathbb{N})$ and $c = 2^{\aleph_0} = \text{card}(\mathbb{R})$. It is well known that every compact, metric space $X$ is separable. Moreover for every dense subset $Y$ of $X$ pretangent spaces to $X$ and to $Y$ are pairwise isometric at each point $a \in Y$, see Corollary 3.3 in [11]. Hence we have the inequality

$$\text{den}(\Omega_{a,\tilde{r}}) \leq \aleph_0 = c \quad (4.1)$$

for all pretangent spaces $\Omega_{a,\tilde{r}}$ to compact metric spaces. (The equality $\aleph_0 = c$ is well known, see, for example, [19, Chapter V, §6].) The following theorem shows that, in general, no smaller cardinal numbers can be taken in (4.1).

**Theorem 8.** There exists a compact ultrametric space $X$ with a marked point $a$ such that the equality

$$\text{den}(\Omega_{a,\tilde{r}}^X) = c \quad (4.2)$$

holds for some $\Omega_{a,\tilde{r}}^X$. 

Proof. Consider a sequence of finite disjoint sets

\[ X_n = \{a_1^{(n)}, a_2^{(n)}, \ldots, a_{2^n}^{(n)}\}, \quad \text{card}(X_n) = 2^n, \]

\[ n \in \mathbb{N} = \{0, 1, \ldots\}. \]

Write

\[ X = \bigcup_{n=0}^{\infty} X_n \cup \{a\} \quad (4.3) \]

where \( \{a\} \) is a one-point set such that \( a \not\in X_n \) for all \( n \). Let \( \tilde{r} = \{r_n\}_{n \in \mathbb{N}} \) be a strictly decreasing sequence of positive numbers with

\[ \lim_{n \to \infty} \frac{r_n}{r_{n+1}} = \infty. \quad (4.4) \]

Let us define a distance function \( d : X \times X \to \mathbb{R}^+ \) by the rule

\[ d(x, y) = \begin{cases} 
(r_n \lor r_m) & \text{if } x \in X_n \text{ and } y \in X_m \text{ and } x \neq y \\
r_n & \text{if } x \in X_n \text{ and } y = a \text{ or if } y \in X_n \text{ and } x = a \\
0 & \text{if } x = y.
\end{cases} \quad (4.5) \]

The ultra-triangle inequality \( d(x, y) \leq d(x, z) \lor d(z, y) \) can be easily proved by the exhaustion of all possible cases. For example, if \( x \in X_n, y \in X_m, z \in X_k \) and \( n \leq m \leq k \), then

\[ d(x, y) = d(x, z) = r_n \geq r_m = d(z, y). \]

The space \((X, d)\) is compact because for every open ball \( O(a, r) \) we have the inclusion

\[ X \setminus O(a, r) \subseteq \bigcup\{x : r_n \geq r\}, \]

so the set \( X \setminus O(a, r) \) is finite. Note also that for every two mutually stable \( \tilde{x}, \tilde{y} \in \tilde{X} \) we have either \( \tilde{d}(\tilde{x}, \tilde{y}) = 0 \) or \( \tilde{d}(\tilde{x}, \tilde{y}) = 1 \). Indeed, if \( \tilde{x} = \{x_n\}_{n \in \mathbb{N}}, \tilde{y} = \{y_n\}_{n \in \mathbb{N}} \) are mutually stable and if

\[ \tilde{d}(\tilde{x}, \tilde{y}) = \lim_{n \to \infty} \frac{d(x_n, y_n)}{r_n} > 0, \]

then (4.4) and (4.5) imply the equality \( d(x_n, y_n) = r_n \) for all sufficiently large \( n \). For a continuation of the proof we are in need of the following lemma.

**Lemma 10.** Let \( \tilde{F} \) be a self-stable family of sequences of points from the metric space \((X, d)\) which was defined by (4.3), (4.5) and let \( \tilde{r} = \{r_n\}_{n \in \mathbb{N}} \) satisfy (4.4). Suppose \( \tilde{a} = (a, a, \ldots) \in \tilde{F} \), then for every \( \tilde{x} = \{x_n\}_{n \in \mathbb{N}}, \tilde{y} = \{y_n\}_{n \in \mathbb{N}} \) from \( \tilde{F} \) the inequality

\[ \tilde{d}(\tilde{x}, \tilde{y}) \lor \tilde{d}(\tilde{x}, \tilde{a}) \lor \tilde{d}(\tilde{y}, \tilde{a}) > 0 \quad (4.6) \]

implies that

\[ \exists n_0 = n_0(\tilde{x}, \tilde{y}) \in \mathbb{N} \ \forall n \geq n_0 : (x_n \neq y_n \& x_n \in X_n \& y_n \in X_n). \quad (4.7) \]
Conversely, if \( A \) is a subset of the Cartesian product of the sets \( X_n \),

\[
A \subseteq \prod_{n \in \mathbb{N}} X_n,
\]

such that for every two distinct \( \tilde{x}, \tilde{y} \in A \), \( \tilde{x} = \{x_n\}_{n \in \mathbb{N}}, \ \tilde{y} = \{y_n\}_{n \in \mathbb{N}}, \) condition (4.7) holds, then the family \( \tilde{F} = A \cup \{\tilde{a}\} \) is self-stable w.r.t. \( \tilde{r} \) and (4.6) holds for every two distinct \( \tilde{x}, \tilde{y} \in A \).

**Remark 3.** As usual, we understand the Cartesian product \( \prod_{n \in \mathbb{N}} X_n \) as the set of all mappings \( f : \mathbb{N} \to \bigcup_{n \in \mathbb{N}} X_n \) for which each number \( n \in \mathbb{N} \) is mapped on an element of the corresponding set \( X_n \).

**Proof of Lemma 10.** Let \( \tilde{x}, \tilde{y} \) be elements of \( \tilde{F} \) for which (4.6) holds. It was stated above that the inequality \( \tilde{d}(\tilde{x}, \tilde{y}) > 0 \) implies the equality \( d(x_n, y_n) = r_n \) for sufficiently large \( n \), hence \( x_n \neq y_n \) for these \( n \). Moreover, (4.6) also implies \( x_n \neq a \neq y_n \) if \( n \) is taken large enough. Suppose now that for these \( \tilde{x}, \tilde{y} \in \tilde{F} \) there is a strictly increasing, infinite sequence of natural numbers \( n_k \) such that

\[
x_{n_k} \in X_{n_1(k)}, \quad y_{n_k} \in X_{n_2(k)}
\]

with \( n_1(k) \neq n_2(k) \). It follows from the equality \( d(x_{n_k}, y_{n_k}) = r_{n_k} \) and (4.5) that

\[
n_1(k) \land n_2(k) = n_k.
\]

Hence, we have the inequality

\[
n_1(k) \lor n_2(k) \geq n_k + 1.
\]

We may assume, without loss of generality, that there is an infinite, strictly increasing subsequence of natural numbers \( k(i) \) such that

\[
n_1(k(i)) = n_1(k(i)) \lor n_2(k(i)).
\]

Relations (4.8)–(4.10) imply that \( x_{n_k(i)} \in X_{n_1(k(i))} \) and that \( n_1(k(i)) > n_2(k(i)) \), hence, by (4.5) and (4.4), we obtain

\[
\lim_{i \to \infty} \frac{d(x_{n_k(i)}, a)}{r_{n_k(i)}} = 0.
\]

Since \( \tilde{x} \) and \( \tilde{a} \) are mutually stable, the last limit relation means that \( \tilde{d}(\tilde{x}, \tilde{a}) = 0 \), contrary to (4.6).

Suppose now that

\[
A \subseteq \prod_{n \in \mathbb{N}} X_n
\]

and condition (4.7) holds for every two distinct \( \tilde{x}, \tilde{y} \in A \). Then, for each pair of distinct \( \tilde{x}, \tilde{y} \in A \), \( \tilde{x} = \{x_n\}_{n \in \mathbb{N}}, \ \tilde{y} = \{y_n\}_{n \in \mathbb{N}}, \) the equalities

\[
d(x_n, a) = d(y_n, a) = r_n
\]

hold.
hold for all \( n \in \mathbb{N} \), and, moreover, we have

\[
d((x_n, y_n)) = r_n
\]  

(4.12)

if \( n \geq n_0(\bar{x}, \bar{y}) \), see (4.6). Thus, the family \( A \cup \{a\} \) is self-stable. To complete the proof, it suffices to observe that (4.11) and (4.12) imply (4.6). \( \square \)

Continuation of the proof of Theorem 8. To prove (4.2) it is sufficient to construct a set

\[ A \subseteq \prod_{i \in \mathbb{N}} X_i \]

such that \( \text{card}(A) = \mathfrak{c} \) and that condition (4.7) takes place for every two distinct \( \bar{x}, \bar{y} \in A \). Indeed, if \( \bar{X}_{a,r} \) is a maximal self-stable family which contains \( A \), then

\[
\text{card}(\Omega_{a,r}) \geq \mathfrak{c}
\]  

(4.13)

for the corresponding pretangent space \( \Omega_{a,r} \) because (4.6) implies the injectivity of the restriction \( \pi|_A \), see (1.4). Now (4.1) and (4.13) imply (4.2).

To construct the desirable \( A \) consider the following infinite binary tree, see Figure 1.

![Infinite binary tree](image)

Figure 1. Infinite binary tree. All nodes are labelled by the symbols “0” and “1”.

The branches of this tree are infinite sequences of symbols “0” and “1”. The set of all these branches has cardinality \( \mathfrak{c} \). Each node of this tree has two nodes immediately below it, called its children. We can say that only names “0” and “1” are used in all generations of children. Using the elements \( a_1^{(n)}, a_2^{(n)}, \ldots, a_{2^n}^{(n)} \) from \( X_n \), for the names of the \( n \)-th generation of children we can obtain the tree on Figure 2.

There is a “natural” bijection between the set of all branches of the tree from Figure 1 and the corresponding one from Figure 2, see Remark 4 below. Hence, we can take \( A \) as the set of all branches of the tree from Figure 2. Thus, we have \( \text{card}(A) = \mathfrak{c} \). Furthermore, note that every two branches \( \bar{x}, \bar{y} \in A \) are distinct if and only if condition (4.7) is satisfied. \( \square \)
Remark 4. For every infinite sequence $x = (x_1, x_2, \ldots)$ of 0’s and 1’s and for every $n = 1, 2, \ldots$ write
\[ n(x) = 1 + \sum_{i=0}^{n-1} x_{i+1}2^i. \]
Then the “natural” bijection between sets of branches of the trees from Figure 1 and from Figure 2 is the mapping
\[ x = (x_1, x_2, \ldots, x_n, \ldots) \mapsto (a_1^{(1)}(x), a_2^{(2)}(x), \ldots, a_n^{(n)}(x), \ldots). \]

The following problem seems to be interesting.

Problem 1. Let $(X, d)$ be a compact metric space and let $a$ be an accumulation point of $X$. Find conditions under which a pretangent space $\Omega_{a, \tilde{r}}^X$ is separable.

Acknowledgments. This research was supported by the Fellowships For Visiting Scientists Programme from the Scientific and Technological Research Council of Turkey (TÜBİTAK). The first author also was partially supported by the State Foundation for Basic Researches of Ukraine, Grant Φ 25.1/055.

References


Zbl 1063.42001


Zbl 0974.46038

Received September 24, 2009